

GENERALIZED VARIATIONAL MODELS
FOR DATA PROCESSING



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Resum

Al llarg dels darrers quaranta anys, el processament d'imatges ha propiciat el desenvolupament d'una àmplia varietat de problemes orientats a modelitzar processos tècnics. Dins d'aquest àmbit, es poden trobar tasques com ara l'eliminació de soroll, l'augment de la resolució, la compressió, la reconstrucció, la detecció de flux òptic o la segmentació d'informació rellevant, entre moltes altres. Matemàticament, aquests problemes constitueixen un marc conceptual en què es poden aplicar diverses estratègies per abordar-los: anàlisi de Fourier, teoria dels problemes inversos, mètodes estadístics, tècniques de transport òptim o equacions en derivades parcials (vegeu, e.g. [26, 27, 37, 59, 141]), per citar-ne alguns exemples. Junta-ment amb aquest interès acadèmic, el processament d'imatges resulta essencial en nombroses situacions del món real, incloent-hi la medicina, l'agricultura, la vigilància, l'entreteniment, la robòtica i les ciències ambientals, entre d'altres.

Tanmateix, cal aclarir el concepte d'*imatge*. Encara que comunament associem una *imatge* amb una *fotografia*, aquesta definició pot resultar excessivament restrictiva quan abordem situacions en què les imatges representen tipus de dades diverses, com ara les obtingudes mitjançant raigs X, ecografies, radar, telescopis o microscòpies electròniques. En aquests casos, les dades sovint representen estructures complexes—com el cos humà, la Terra o materials específics—capturades en un instant determinat. Per tant, en aquest treball adoptem el concepte d'*imatge* com a sinònim d'*instantània*, entenent-lo com qualsevol representació d'un objecte o sistema en un moment concret. Aquesta perspectiva ens permet considerar el processament d'imatges com un subgènere dins del marc més ampli del processament de dades. Al llarg d'aquesta tesi, ens centrem en dos models variacionals clàssics desenvolupats originalment per al processament de *fotografies*, els quals reinterpretarem des d'una perspectiva més general d'acord amb el concepte d'*instantània*. En concret, estudiem el model de reducció de soroll de Rudin–Osher–Fatemi (ROF) [146] i el model de segmentació Chan–Vese (CV) [55], i els estenem a contextos més generals i complexos.

A la primera part de la tesi, analitzem i generalitzem la relació entre els models ROF i CV, inicialment introduïda a [34]. En el cas clàssic isotròpic, es pot construir un minimitzant parcial del model CV a partir del minimitzant del model ROF mitjançant aquesta connexió. La nostra anàlisi amplia aquest resultat a contextos més generals, identificant condicions sota les quals la correspondència es manté en incorporar anisotropies, perímetres fraccionals, passeigs aleatoris o formulacions en grafs mètrics, entre altres. Aquest marc unificat aclareix com les variacions en els termes de regularització o en el domini influeixen en la interacció entre ambdós models. En particular, el cas anisotròpic és tractat amb detall, oferint una comprensió millorada del vincle entre els models respecte al cas isotròpic. Per altra banda, la resta de la tesi està dedicada a la generalització del model ROF, especialment a una variant vectorial restringida a varietats Riemannianes, on el domini de la imatge és representat per una superfície. Aquesta formulació basada en varietats introdueix nous reptes teòrics motivats per aplicacions pràctiques i, encara que s'han fet avanços notables en la literatura aplicada, els desenvolupaments teòrics rigorosos continuen sent escassos. En aquesta tesi contribuïm a reduir aquesta bretxa demostrant l'existència de minimitzants i establint-ne certes propietats de regularitat, tot ressaltant el paper fonamental de la interacció entre les perspectives extrínseca i intrínseca.

Aquesta tesi aprofundeix en els resultats presentats prèviament a [30, 31, 128, 129, 130].

Abstract

Over the past forty years, image processing has led to the development of a wide variety of problems aimed at modelling technical processes. Within this field, one can encounter tasks such as noise removal, resolution enhancement, compression, reconstruction, optical flow detection, or the segmentation of relevant information, among many others. Mathematically, these problems constitute a framework in which a diverse range of strategies can be applied to address them: Fourier analysis, inverse problem theory, statistical methods, optimal transport techniques, or partial differential equations (e.g., see [26, 27, 37, 59, 141]), to name a few examples. Alongside this academic interest, image processing proves essential in numerous real-world situations, including medicine, agriculture, surveillance, entertainment, robotics, and environmental sciences, among others.

However, it is necessary to clarify the concept of an *image*. Although we commonly associate an *image* with a *photograph*, this definition may be overly restrictive when addressing situations where images represent diverse data types, such as those obtained through X-rays, ultrasounds, radar, telescopes, or electron microscopes. In these cases, the data often represent complex structures—such as the human body, the Earth, or specific materials—captured at a given moment in time. Therefore, in this work, we adopt the concept of an *image* as akin to a *snapshot*, understanding it as any representation of an object or system at a particular instant. This perspective allows us to regard image processing as a subgenre within the broader framework of data processing. Throughout this thesis, we focus on two classical variational models originally developed for processing *photographs*, which we reinterpret from a broader perspective aligned with the concept of *snapshot*. Specifically, we study the Rudin–Osher–Fatemi (ROF) denoising model [146] and the Chan–Vese (CV) segmentation model [55], and we extend them to more general and complex settings.

In the first part of the thesis, we analyze and generalize the relationship between the ROF and CV models, initially introduced in [34]. In the classical isotropic setting, a partial minimizer of the CV model can be constructed from the minimizer of the ROF model via this connection. Our analysis extends this result to broader contexts, identifying conditions under which the correspondence persists when incorporating anisotropies, fractional perimeters, random walks, or metric graph formulations, among others. This unified framework clarifies how variations in the regularization terms or the domain influence the interaction between both models. In particular, the anisotropic case is addressed in detail, offering an enhanced understanding of the linkage between the models compared to the isotropic case. In contrast, the rest of this work is devoted to the generalization of the ROF model, particularly to a vector-valued variant constrained to Riemannian manifolds, where the image domain is represented by a surface. This manifold-based formulation introduces new theoretical challenges motivated by practical applications, and although notable progress has been made in the applied literature, rigorous theoretical developments remain limited. In this thesis, we contribute to bridging this gap by proving the existence of minimizers and establishing their regularity properties, emphasizing the crucial role of the interplay between extrinsic and intrinsic viewpoints.

This thesis elaborates on results previously presented in [30, 31, 128, 129, 130].

Introducció

En aquest capítol es presenta el marc teòric fonamental que sustenta els desenvolupaments exposats al llarg d'aquesta tesi. Una part de la notació i dels conceptes utilitzats en aquesta secció es formalitza al Capítol 2, al qual es pot remetre per a una exposició més detallada.

0.1. Modelització d'imatges

En aquesta secció presentem tres perspectives per a la interpretació d'imatges, les quals representem mitjançant una funció $u_0 : \Omega \rightarrow X$, on $\Omega \subseteq \mathbb{R}^2$, i classifiquem els diferents espais imatge X que poden emprar-se.

En primer lloc, considerem les imatges modelitzades com a funcions de valors escalars (per exemple, imatges en escala de grisos). En aquest cas, per a cada posició $(x, y) \in \Omega$, el valor $u_0(x, y)$ representa la lluminositat de la imatge en eixe punt. A la Figura 0.1 il·lustrem la relació entre aquest tipus d'imatge i la gràfica de la funció u_0 .

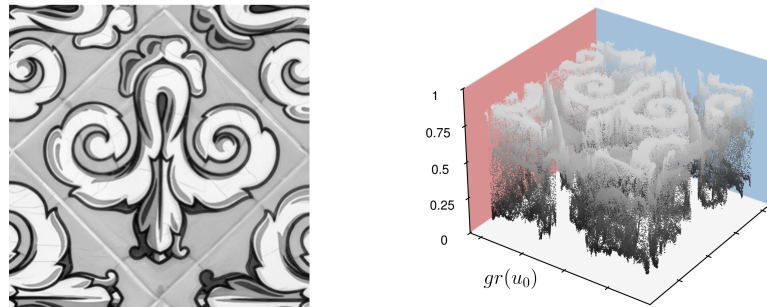


FIGURA 0.1. Exemple de modelització escalar: la imatge en escala de grisos de l'esquerra es pot interpretar com la gràfica d'una funció escalar $u_0 : \Omega \rightarrow [0, 1]$, tal com es mostra en la figura de la dreta.

Tal com s'ha exemplificat a la figura, la modelització escalar es troba clarament restringida, deixant de banda la major part dels escenaris d'aplicació. Per aquest motiu, si considerem $u : \Omega \rightarrow X \subset \mathbb{R}^M$, podem modelitzar una gran varietat de tipus d'imatge: des d'imatges SAR ($M = 2$), imatges RGB ($M = 3$), i imatges CMYK ($M = 4$), fins a formats hiperespectrals amb $M \gg 100$. A més, aquesta representació vectorial és sovint necessària per a l'estudi de alguns fenòmens específics, com ara els gradients de color, l'orientació de contorns i textures, el moviment dins de la imatge o la comparació de canvis entre dues imatges. Com a exemple d'aquest tipus de modelització, mostrem el cas d'una imatge en color RGB (els píxels de la qual viuen en el cub $X = [0, 255]^3$) a la Figura 0.2.

En aquest punt, és important destacar que la modelització d'imatges vectorials pot estar subjecta a restriccions, bé siga per la naturalesa de les dades a representar (per exemple, imatges DT-MRI), o per l'aspecte específic de la imatge que es vol analitzar (com ara la cromaticitat). En el nostre cas, ens centrem en escenaris on els valors de la funció u_0 es troben en varietats diferenciables, generalment com a conseqüència de les propietats intrínseques de la imatge que s'estudia (vegeu, per exemple, [67, 162]). Cal remarcar que l'estudi de senyals

i imatges restringides a varietats no és merament teòric; respon a reptes reals que han sigut explorats en la literatura aplicada (vegeu, e.g., [19]).

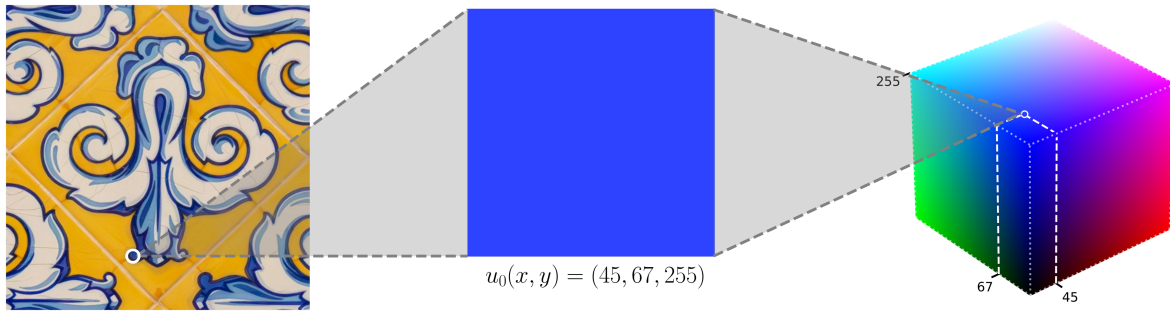


FIGURA 0.2. Exemple de color RGB modelitzat mitjançant una funció vectorial u_0 .

Des de la perspectiva de les aplicacions, un exemple rellevant es troba en la imatge biomèdica, on els models amb restriccions es defineixen sovint sobre varietats de curvatura no positiva (NPC). Un cas destacat és la imatge per ressonància magnètica amb tensor de difusió (DT-MRI) en neurologia, on s'utilitzen sis conjunts de dades diferents, adquirits sota orientacions diverses d'un camp magnètic, per analitzar la difusió de molècules d'aigua en cada píxel. Aquesta informació sobre la difusió es codifica de manera natural en l'espai de matrius simètriques definides positives de dimensió 3×3 . En una altra direcció, l'espai hiperbòlic ha rebut una atenció creixent tant en la segmentació d'imatges [12] com en l'aprenentatge automàtic [125]. En aquests contextos, el procés d'entrenament introdueix típicament soroll, i els algorismes robusts de reducció de soroll poden millorar el reconeixement de patrons. Una explicació natural rau en el fet que les estructures jeràrquiques, com ara arbres o taxonomies, presenten un creixement exponencial en el nombre de nodes, la qual cosa reflecteix l'expansió volumètrica de les boles en espais hiperbòlics. Açò fa que la geometria hiperbòlica siga més adequada que l'entorn euclidià tradicional, on el creixement del volum és només polinòmic.

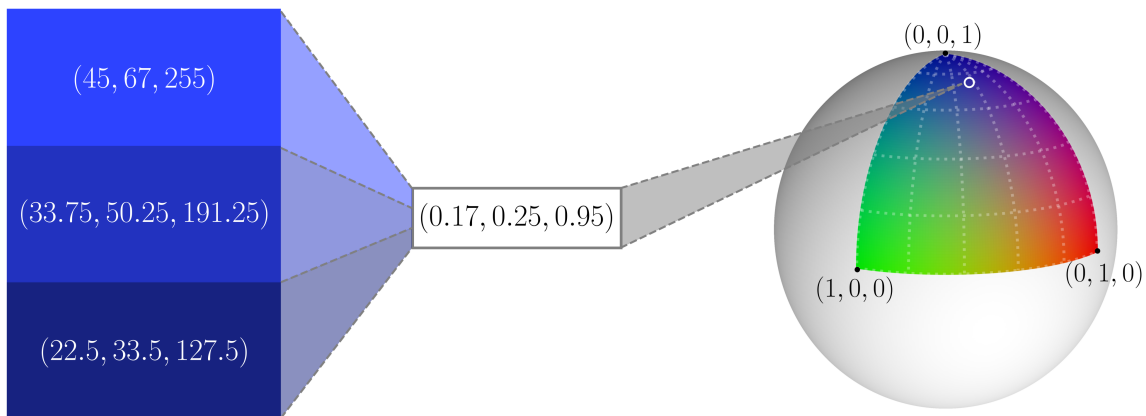


FIGURA 0.3. Exemple de com els colors RGB es representen uniformement en \mathbb{S}^2 mitjançant una normalització.

De manera similar, espais objectiu amb curvatura positiva s'empren en diverses aplicacions. Per exemple, en la cromaticitat RGB, els valors de la imatge es troben en l'octant positiu de l'esfera unitària \mathbb{S}^2 [153], on els colors RGB es tracten de manera normalitzada, excloent la influència de la lluminositat, tal com s'illustra en la Figura 0.3. En ciència de

materials, les tècniques de difracció d'electrons retrodispersats (EBSD) s'utilitzen per estudiar l'estructura microscòpica de materials policristal·lins, amb dades que prenen valors en el grup de rotacions $SO(3)$ [13, 131]. Altres exemples notables inclouen els camps de flux òptic, habituals en l'estimació de moviment i modelitzats en l'esfera \mathbb{S}^2 ; les representacions perceptuals del color, com l'espai de color LCh, definit en l'espai producte $\mathbb{R} \times \mathbb{S}^1$; i les rotacions rígides d'objectes, centrals en diverses tasques de visió per computador, descrites pel grup euclidià especial $SE(2)$.

A més, la modelització d'imatges no es restringeix únicament a generalitzacions del domini. Per exemple, els dominis corbats apareixen de manera natural en contextos com ara pantalles d'ordinador no planes, aplicacions geofísiques i mèdiques (per exemple, la direcció del vent o les superfícies corticals en imatges cerebrals), i fins i tot en la coloració de superfícies 3D amb finalitats artístiques o de disseny [109].

En conclusió, les imatges poden modelitzar-se de diverses maneres segons la naturalesa del problema que es vulga abordar. En les seccions següents, discutirem els models fonamentals considerats en aquest treball i identificarem els tipus d'imatges als quals poden aplicar-se. Açò ens permetrà exposar els punts forts i les limitacions de cada model, tot proporcionant el context necessari per al desenvolupament d'aquesta tesi.

0.2. Model de Rudin–Osher–Fatemi

0.2.1. Motivació. Considerem que la imatge a processar està en escala de grisos i assumim que ha sigut corrompuda per soroll gaussià additiu n amb mitjana zero i desviació estàndard σ . És a dir, podem expressar f com

$$f = u_0 + n,$$

on u_0 és la imatge original, sense alteracions. A la Figura 0.4 il·lustrem aquesta composició.

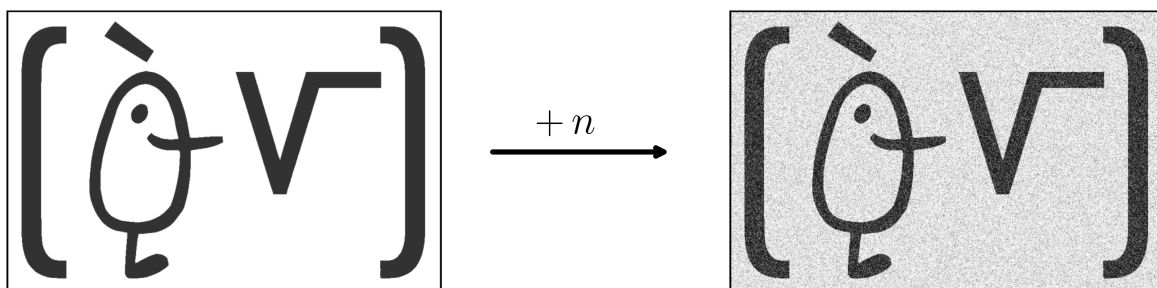


FIGURA 0.4. A l'esquerra es mostra la imatge *neta* u_0 , mentre que a la dreta es mostra la imatge *amb soroll* f , obtinguda afegint soroll gaussià additiu n .

En aquest context, el problema invers que es planteja consisteix a reduir el soroll present en f , és a dir, recuperar u_0 . Des d'una perspectiva variacional, es tracta de trobar una funció u que minimitze un funcional de la forma

$$\phi(u, f) + \gamma \mathcal{R}(u),$$

on ϕ és un terme de fidelitat, \mathcal{R} és un terme de regularització, i $\gamma > 0$ és un paràmetre de regularització. En el nostre cas, el terme ϕ ha de preservar la relació estructural entre la solució u i la imatge sorollosa f , mentre que \mathcal{R} ha de penalitzar les pertorbacions degudes al soroll present en u (per a més detalls, vegeu, e.g., [149, Capítols 3 i 4]). Concretament, ens interessa un problema de la forma següent:

$$(0.1) \quad \min_{u \in L^2(\Omega)} \left\{ J_\gamma(u) := \int_{\Omega} (u - f)^2 dx + \gamma \int_{\Omega} |Qu|^2 dx \right\},$$

on $Q : L^2(\Omega) \rightarrow L^2(\Omega)$ és un operador lineal que actua com a regularitzador de la solució u , i $\gamma > 0$ és el paràmetre de regularització. Calcular la primera variació de $J_\gamma(u)$ dona com a resultat

$$\frac{\delta J_\gamma}{\delta u}(v) = 2 \int_{\Omega} (u - f)v dx + 2\gamma \int_{\Omega} (Qu)(Qv) dx,$$

per a tot $v \in H^1(\Omega)$. Igualant aquesta variació a zero, es dedueix que el minimitzador u de $J_\gamma(u)$ ha de satisfer l'equació següent:

$$(\mathbb{1} + \gamma Q^*Q)u = f,$$

on $\mathbb{1}$ denota l'operador identitat, i Q^* denota l'adjunt de l'operador Q . La rellevància del problema (0.1) rau en la seua connexió amb l'expressió anterior. Concretament, quan l'operador del costat esquerre és invertible, trobar el minimitzador u es redueix a resoldre un sistema lineal d'equacions.

0.2.2. Definició. Suposem ara que u_0 és una funció contínua fora de les vores de la imatge. Per tant, és fonamental triar un operador Q capaç d'eliminar les perturbacions menudes de f (associades al soroll additiu n) i, alhora, preservar les vores. En el seu treball [146], L. Rudin, S. Osher i E. Fatemi van posar de manifest les limitacions de considerar $Qu = Du$, és a dir, l'energia de Dirichlet, en (0.1), ja que les funcions de l'espai $H^1(\Omega)$ no poden presentar discontinuïtats al llarg de corbes rectificables, les quals representen les vores de la imatge u_0 . Per tal de superar aquesta limitació, proposaren prendre $Qu = Du|Du|^{-1}$, és a dir, restringir la minimització a l'espai $BV(\Omega)$ per tal d'assegurar la preservació de les vores de la imatge original u_0 , així com la continuïtat (en el sentit de la teoria de la mesura; vegeu [6]) fora d'aquestes vores. Concretament, en [146] es proposa el següent problema amb restriccions:

$$(0.2) \quad \min \left\{ \int_{\Omega} |Du| : u \in BV(\Omega), \int_{\Omega} (u - f)^2 dx = \sigma \mathcal{L}^2(\Omega) \right\},$$

on es fa ús d'informació prèvia sobre la desviació estàndard del soroll n . Minimitzar (0.2) és equivalent a maximitzar la probabilitat $P(u|f) \propto P(u)P(n)$ de recuperar u a partir d'un f conegut, quan la variació total s'utilitza per a modelitzar $P(u)$. A la pràctica, el problema de ROF es reformula com:

$$(0.3) \quad \min_{u \in BV(\Omega)} \left\{ \mathcal{E}^\lambda(u) := \int_{\Omega} |Du| + \frac{\lambda}{2} \int_{\Omega} (u - f)^2 dx \right\},$$

amb $Qu = Du|Du|^{-1}$ i $\gamma = 2/\lambda$, ja que en [51] s'estableixen tant l'existència d'un minimitzador de (0.3) com l'equivalència entre (0.2) i (0.3), tot relacionant λ i σ . Així, λ actua tant com a paràmetre de regularització, en el sentit de (0.1) com multiplicador de Lagrange que imposa la restricció en (0.2). En les Figures 0.5 i 0.6, mostrem el resultat d'aplicar (0.3), així com l'efecte regularitzador de λ .

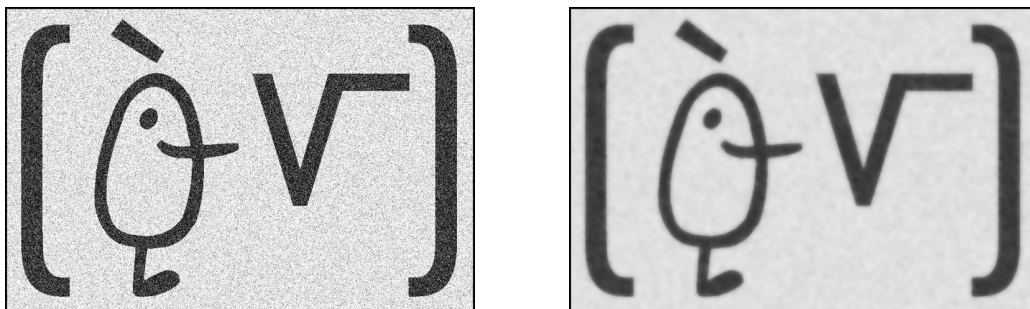


FIGURA 0.5. A l'esquerra es mostra la imatge *sorollosa* f , mentre que a la dreta es mostra la imatge *sense soroll* u , obtinguda mitjançant la minimització de \mathcal{E}^λ .

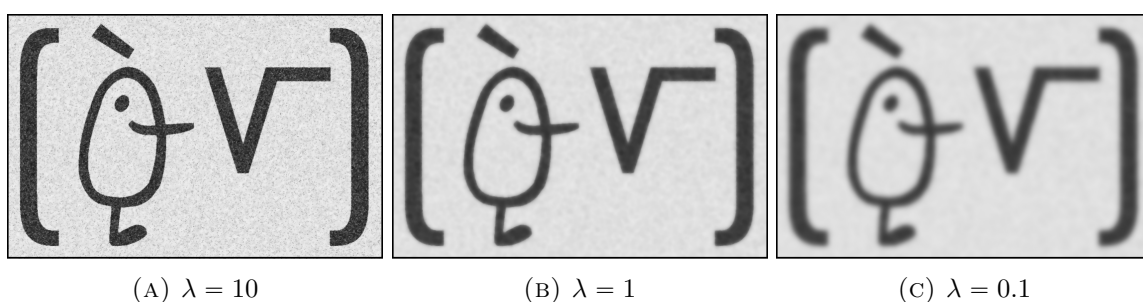
(A) $\lambda = 10$ (B) $\lambda = 1$ (C) $\lambda = 0.1$

FIGURA 0.6. Efecte de λ en la minimització (0.3)

A més, cal destacar que, a causa de la convexitat de \mathcal{E}^λ , es garanteix la unicitat del minimitzador u de (0.3), i aquest ha de satisfer l'equació d'Euler-Lagrange associada

$$\operatorname{div}(z) = \lambda(u - f),$$

on $z \in \partial \operatorname{TV}(u)$ (vegeu la Secció 2.3.3).

0.2.3. Limitacions. Tot i que la variació total pot actuar com un terme de regularització efectiu, cal reconèixer els inconvenients ben coneguts associats al seu ús. Entre aquests, destaquem qüestions com l'aparició de l'efecte d'esglaonat (*staircasing*) [138, 144], la pixelació en determinades regions [44], la pèrdua de textures en la imatge u_0 [92], la dependència del factor d'escala [152], o l'homogeneïtzació cap al valor mitjà de f en Ω si λ és prou menut [7]. S'han proposat diverses estratègies per abordar aquests problemes, com ara la definició de filtres no locals [29], l'establiment d'estimacions estadístiques *a posteriori* per a recuperar u [118], o la reformulació de (0.2) amb restriccions locals [74], per esmentar-ne alguns exemples. Tanmateix, la simplicitat de la variació total i les seues propietats geomètriques han permès una àmplia varietat d'estudis teòrics que han contribuït de manera significativa a la comprensió del comportament de les solucions d'aquest model. Finalment, cal assenyalar que (0.3) ha estat generalitzada al cas vectorial (vegeu, per exemple, [22, 25]), així com al cas restringit a varietats (vegeu, per exemple, [24, 162]), emprant una formulació discreta equivalent a la que es presenta al Capítol 5. A més, (0.3) també és generalitzable en el sentit anisotròpic [94]. Al Capítol 4, presentem detalladament la variant anisotròpica del problema de ROF.

0.2.4. Resultats relacionats. En aquesta secció, introduïm tres temes rellevants per al present treball. En primer lloc, presentem una selecció de resultats referents al conjunt de discontinuïtat J_u (vegeu la Secció 2.3) del minimitzador de (0.3), que emprarem en els Capítols 4 i 5. En segon lloc, resumim diversos resultats sobre la regularitat del minimitzador u de (0.3), els quals són necessaris per contextualitzar el treball del Capítol 5. Finalment, esmentem les tècniques numèriques utilitzades per resoldre (0.3), que s’apliquen a les Seccions 3.4 i 4.4, amb certes modificacions.

0.2.4.1. *Preservació del conjunt de salt.* En la Secció 0.2.2, motivarem l’interés en la selecció de Q en (0.1) de manera que es preserven les vores de la imatge u_0 . Com que $f = u_0 + n$, aquestes regions es corresponen amb punts de J_f . Per aquest motiu, diversos treballs [3, 41, 42, 46] estudien la relació entre J_u i J_f , on u és el minimitzador de (0.3), especialment quan $\Omega \subseteq \mathbb{R}^N$ amb $N \geq 2$. Concretament, aquests treballs exploren la connexió entre (0.3) i el problema de curvatura prescrita

$$(0.4) \quad \min \left\{ \text{Per}(E; \Omega) + \lambda \int_{\Omega} (t - f(x)) \, dx : E \subseteq \Omega \right\}.$$

En [46], es demostra que el conjunt de nivell $E_t(u) := \{x \in \Omega : u(x) > t\}$ és solució de (0.4) per a tot $t \in \mathbb{R}$. En [41], es presenta la següent caracterització dels conjunts de discontinuïtat:

TEOREMA 0.1. [42, Teorema 1] *Siga $f \in BV(\Omega) \cap L^\infty(\Omega)$ i u el minimitzant de \mathcal{E}^λ per a $\lambda > 0$. Aleshores, es compleix que $J_u \subseteq J_f$ excepte en un conjunt \mathcal{H}^{N-1} -nul.*

En aquest resultat, la idea clau és utilitzar la relació entre (0.3) i (0.4), així com analitzar la curvatura dels conjunts de nivell $E_t(u)$ i la impossibilitat d’intersecció entre les fronteres de diferents conjunts de nivell. No obstant això, és important assenyalar que aquesta tècnica es limita a la formulació escalar del problema. Com a resultat més fi, en [62] es proporciona una caracterització del conjunt de discontinuïtat en funció del paràmetre λ , per al cas $\Omega = [0, 1]$, emprant l’equivalència entre el problema de ROF i els problemes de la corda tensa [93].

Pel que fa a la variant vectorial de (0.3), cal esmentar que en [157] s’obté un resultat equivalent al Teorema 0.1 sense necessitat de fer ús dels conjunts de nivell, i que aquest resultat pot generalitzar-se a altres termes de regularització. Com a generalització d’aquest treball, [50] simplifica les tècniques de [157] i elimina la condició $f \in BV(\Omega)$ present en (0.1), tot aplicant una notació relaxada de J_f en el cas que $f \notin BV(\Omega, \mathbb{R}^M)$ [50, §7].

0.2.4.2. *Regularitat del minimitzant.* Un altre aspecte d’interés pel que fa al minimitzador de (0.3) és l’estudi de la seua regularitat en funció de la regularitat de f . Igual que en la secció anterior, els resultats que presentem són vàlids en el cas en què $\Omega \subseteq \mathbb{R}^N$, llevat que s’indique el contrari.

En [43], es demostra que, si $N \leq 7$, la condició $f \in C_{\text{loc}}^{0,\beta}(\Omega)$ implica que $u \in C_{\text{loc}}^{0,\beta}(\Omega)$ per a tot $\beta \in [0, 1]$. A més, si Ω és convex i $f \in UC(\Omega)$, llavors $u \in UC(\Omega)$ sota la mateixa restricció dimensional. Cal destacar que aquesta limitació prové de l’ús, en [43], de la relació entre (0.3) i (0.4). Concretament, l’estratègia de la demostració es basa en una estimació local de la distància entre conjunts de nivell, així com en la regularitat d’aquests conjunts quan $N \leq 7$ (vegeu, per exemple, [76, §5.4.15]).

Aquesta restricció dimensional ha estat eliminada en treballs posteriors. D’una banda, en [124] es demostra que u és contínua si f ho és, sense cap restricció dimensional, tot i que es requereix la convexitat del domini. D’altra banda, no és fins a [143] que s’estudia la regularitat Lipschitz del minimitzador de (0.3) sense restriccions ni de convexitat ni de dimensió, mitjançant un argument de tipus Bernstein. A continuació, introduïm dos resultats que serveixen com a punt de partida per al treball presentat al Capítol 5:

TEOREMA 0.2. [143, Teorema 1.1] *Siga $f \in C_{\text{loc}}^{0,1}(\Omega)$ i siga u el minimitzant de \mathcal{E}^λ . Aleshores, $u \in C_{\text{loc}}^{0,1}(\Omega)$ i existeix una constant $C > 0$ tal que*

$$\|\nabla u\|_{L^\infty(B_r(x_0))} \leq \|\nabla f\|_{L^\infty(B_r(x_0))} + \frac{C}{r^2},$$

per a tot $x_0 \in \Omega$ i tot $r > 0$ tal que f siga lipschitziana en $B_{2r}(x_0) \subseteq \Omega$.

TEOREMA 0.3. [143, Teorema 1.2] *Siga $f \in W^{1,\infty}(\Omega)$ i siga u el minimitzant de \mathcal{E}^λ . Aleshores, $u \in W^{1,\infty}(\Omega)$ i*

$$\|\nabla u\|_{L^\infty(\Omega)} \leq C_1 \|\nabla f\|_{L^\infty(\Omega)} + C_2,$$

on C_1 i C_2 depenen únicament de Ω .

Aquests dos resultats es generalitzen al Capítol 5 per a la formulació de (0.3) per a varietats. Pel que fa a l'estudi de la regularitat en la formulació vectorial de (0.3), cal destacar que [82] obté un resultat equivalent al Teorema 0.3 considerant $\Omega = (a, b)$. A més, en aquest treball s'obté l'estimació següent en el sentit de mesures:

$$a|Du|(A) \leq |Df|(A)$$

per a tot conjunt de Borel $A \subseteq \Omega$. Aquest resultat també serà tractat en el cas on el codomini és una varietat al Capítol 5.

0.2.4.3. *Mètodes numèrics.* Finalment, ens referim a les diferents maneres de resoldre (0.3) en la seua formulació escalar. En l'article original [146], es proposa resoldre (0.2) mitjançant mètodes de projecció del gradient, els quals poden resultar lents a causa del mal condicionament del problema, un aspecte que ha sigut millorat en treballs posteriors, com ara [160]. Algunes formes més eficients de resoldre (0.3) inclouen aquelles basades en el mètode de Newton [54], el preconditionament multigrad [159], o les tècniques de *graph-cut* [23, 63], per citar-ne alguns exemples. Donada la seua popularitat, destaquem especialment el mètode dual de Chambolle [47], el mètode primal-dual [53], i el mètode *Split Bregman* [92].

0.3. Model de Chan–Vese

0.3.1. Motivació. En aquesta secció, suposem que la imatge f consta de n regions distintes, cadascuna caracteritzada per variacions suaus en el seu interior, i el nostre objectiu és identificar aquestes regions. Com que el domini de f és Ω , podem definir la segmentació del domini com

$$\Omega = \left(\bigcup_{i=1}^n \Lambda_i \right) \cup K, \quad K = \bigcup_{i=1}^n \partial\Lambda_i,$$

on el conjunt K representa la frontera definida per aquestes regions. Una manera d'identificar aquestes regions és modelar-les mitjançant una funció $u \in H^1(\Omega \setminus K)$, permetent que u presente discontinuïtats. En resum, la Figura 0.7 il·lustra aquesta composició.

Amb aquesta idea, D. Mumford i J. Shah proposaren resoldre el problema següent [134]:

$$(0.5) \quad \min_{\substack{u \in H^1(\Omega \setminus K) \\ K \subseteq \Omega}} \left\{ \mu \int_{\Omega} (u - f)^2 dx + \int_{\Omega \setminus K} |\nabla u|^2 dx + \nu \mathcal{H}^{m-1}(K) \right\}, \quad \mu, \nu > 0,$$

on el primer terme garanteix la fidelitat entre u i f , el segon penalitza les variacions en les regions presumptament suaus (és a dir, en $\Omega \setminus K$), i el tercer assegura que els minimitzadors de (0.5) no tinguin fronteres excessivament llargues.

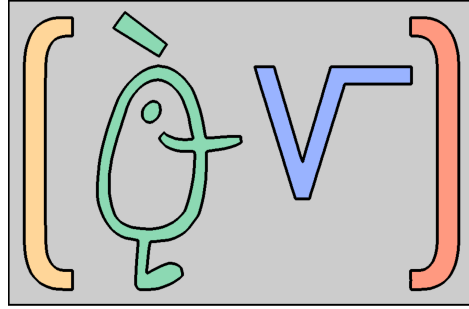


FIGURA 0.7. Descomposició de f en diferents components Λ_i (regions no negres) i el conjunt frontera K (contorns negres), que caracteritzen Ω .

Un cas especialment rellevant és modelar les imatges com la unió de regions amb valor constant més una regió frontera K . En aquest cas, considerem que u és una funció a trossos constant, és a dir, $u(\Lambda_i) = \{c_i\}$ amb $c_i \in \mathbb{R}$ per a tot $1 \leq i \leq n$. En aquest context, cal destacar que (u, K) es pot representar mitjançant $\mathbf{c} = \{c_i\}_{i=1}^n \subseteq \mathbb{R}$ i $\mathbf{\Lambda} = \{\Lambda_i\}_{i=1}^n \in \mathcal{P}_n(\Omega)$, on $\mathcal{P}_n(\Omega)$ denota el conjunt de particions de Ω en n parts. A més, per tal d'emfasitzar la rellevància de cada regió Λ_i dins de la imatge, es pot definir valors diferents de μ en cada zona, denotats per $\mu_i > 0$. Així, sota aquestes noves condicions, el problema de Mumford–Shah es transforma en el següent:

$$(0.6) \quad \min \left\{ \sum_{i=1}^k \mu_i \int_{\Lambda_i} (c_i - f)^2 dx + \nu \mathcal{H}^{m-1} \left(\cup_{i=1}^k \partial \Lambda_i \right) : \{c_i\}_{i=1}^k \subseteq \mathbb{R}, \{\Lambda_i\}_{i=1}^k \in \mathcal{P}_k(\Omega) \right\}.$$

0.3.2. Definició. En el cas $\mu_i = \mu$, $\nu = 1$, $k = 2$ i $m = 2$, l'expressió (0.6) esdevé el problema de Chan–Vese (CV) [55], definit com

$$(0.7) \quad \min_{\substack{\Lambda \subseteq \Omega \\ c_1, c_2 \in [0,1]}} \left\{ \mathcal{F}^\mu(\Lambda, c_1, c_2) := \text{Per}(\Lambda; \Omega) + \mu \int_{\Lambda} (c_1 - f)^2 dx + \mu \int_{\Omega \setminus \Lambda} (c_2 - f)^2 dx \right\}.$$

En aquest context, cal observar que (0.7) proporciona una segmentació de Ω en dues regions segons l'estructura de f . Per tal d'obtindre n regions distintes dins de Ω , podem reformular (0.6) per tal d'obtindre una versió multifase del model de Chan–Vese [56], de la manera següent:

$$(0.8) \quad \min_{\substack{\mathbf{\Lambda} \in \mathcal{P}_n(\Omega) \\ \mathbf{c} \in [0,1]^n}} \left\{ \mathcal{F}^{\mu, n}(\mathbf{\Lambda}, \mathbf{c}) := \sum_{i=1}^n \text{Per}(\Lambda_i; \Omega) + \mu_i \int_{\Lambda_i} (c_i - f)^2 dx \right\},$$

on $\mu \in [0, +\infty)^n$. Per il·lustrar el model de Chan–Vese, en la Figura 0.8 es mostra el minimitzador del problema de Chan–Vese (0.7).

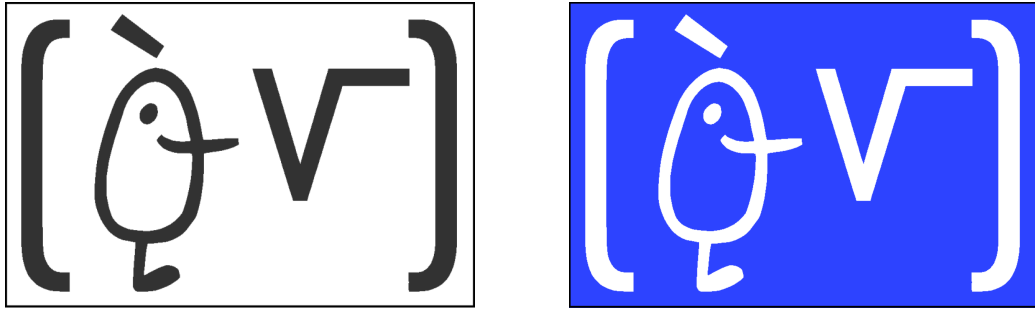


FIGURA 0.8. La imatge de l'esquerra mostra f i la de la dreta representa el conjunt Λ en blau. Per a un valor suficientment gran de μ , un minimitzador de \mathcal{F}^μ és la tupla $(\Lambda, 0, 1)$.

0.3.3. Resultats relacionats. En la secció següent, introduïrem un mètode per a resoldre (0.7) mitjançant la minimització de (0.3), basat en una connexió entre ambdós mètodes que serà fonamental en els Capítols 3 i 4. Tanmateix, volem enumerar diferents enfocaments per a resoldre el problema CV, tant en la seua formulació de dues fases com en la multifàsica.

Pel que fa als algorismes per a resoldre (0.7), en [55] es proposa l'ús d'una tècnica d'*contorn actiu*, que caracteritza $\partial\Lambda$ mitjançant un conjunt de nivells; aquest enfocament fou optimitzat en [166]. Altres mètodes alternatius inclouen algorismes basats en tècniques d'umbrals sobre la relaxació convexa de \mathcal{F}^μ [28], enfocaments basats en derivades topològiques [98], la tècnica *Split-Bregman* [92], mètodes de projecció [68], esquemes multigranulars [14], i tècniques de *graph-cut* [15, 165], per esmentar-ne alguns exemples. D'altra banda, el problema (0.8) es pot resoldre emprant alguns dels mètodes esmentats anteriorment (per exemple, [68, 92, 98]) de manera directa. Dins d'aquestes adaptacions al cas de n fases, destaca la tècnica de contorns actius, que ha propiciat diverses adaptacions [56, 107, 114, 116], així com generalitzacions [154].

0.4. (λ, μ) -connexió

En aquesta secció, presentem la relació entre els models de ROF i de CV introduïts en [34]. Atés el significat d'aquests dos problemes, establir una connexió entre ells ens permet construir vincles entre diferents àrees de recerca i desenvolupar mètodes de resolució més eficients. Al llarg d'aquesta tesi, explorem com la relació entre aquests dos models, que anomenem la (λ, μ) -connexió, esdevé una eina versàtil per a generalitzar aquests models. Per a introduir aquesta connexió, definim primer el concepte de *minimitzant parcial* del problema de CV, és a dir, de \mathcal{F}^μ :

DEFINICIÓ 0.4. *Siga $(\Lambda^*, c_1^*, c_2^*)$ una terna tal que $(c_1^*, c_2^*) \in [0, 1]^2$ i $\Lambda \subseteq \Omega$. Direm que és un minimitzant parcial de \mathcal{F}^μ si es compleixen les condicions següents:*

$$\begin{aligned} \mathcal{F}^\mu(\Lambda^*, c_1^*, c_2^*) &\leq \min \{ \mathcal{F}^\mu(\Lambda^*, c_1, c_2) : (c_1, c_2) \in [0, 1]^2 \}, \\ \mathcal{F}^\mu(\Lambda^*, c_1^*, c_2^*) &\leq \min \{ \mathcal{F}^\mu(\Lambda, c_1^*, c_2^*) : \Lambda \subseteq \Omega \}. \end{aligned}$$

Fent ús d'aquest concepte, els autors de [34] introdueixen la (λ, μ) -connexió:

TEOREMA 0.5. [34, Teorema 3.4] *Siga u el minimitzant de \mathcal{E}^λ , $0 \leq c_1 \leq c_2 \leq 1$ i $\mu = \frac{\lambda}{2(c_1 - c_2)}$. Si el conjunt de nivell*

$$\Lambda = \left\{ x \in \Omega : u(x) > \frac{c_1 + c_2}{2} \right\}$$

compleix $0 < \mathcal{L}^2(\Lambda) < \mathcal{L}^2(\Omega)$, aleshores Λ és un minimitzant de $\mathcal{F}^\mu(\cdot, c_1, c_2)$. A més, si

$$(0.9) \quad c_1 = \int_{\Lambda} f \, dx, \quad c_2 = \int_{\Omega \setminus \Lambda} f \, dx,$$

aleshores (Λ, c_1, c_2) és un minimitzant parcial de \mathcal{F}^μ .

En [34], els autors proporcionen un argument per a establir un mètode de resolució del problema de CV mitjançant la (λ, μ) -connexió. Aquesta tècnica es basa en el fet que els minimitzants de $\mathcal{F}^\mu(\Lambda, \cdot, \cdot)$ estan determinats per l'expressió (0.9). Juntament amb el Teorema 0.5, es proposa un algoritme que convergeix a un minimitzant parcial de \mathcal{F}^μ [34, §4]. Com que es presentaran diverses variants d'aquest algoritme al llarg d'aquest treball, el descrivim a continuació:

Algorithm 0.1 Minimitzant parcial de \mathcal{F}^μ

Require: $k = 0$, $\mu > 0$, $(c_1, c_2) \in [0, 1]$ tals que $c_1 > c_2$.

$w_0 \leftarrow$ Minimitzant de \mathcal{E}^λ amb $\lambda = 2\mu(c_1 - c_2)$.

$\Lambda \leftarrow$ Minimitzant de $\mathcal{F}^\mu(\cdot, c_1, c_2)$ usant el Teorema 0.5.

while $(|\Lambda_k \Delta \Lambda_{k-1}|^2 > \varepsilon_{tol}) \wedge (k < n_{max})$ **do**

$c_1 \leftarrow \int_{\Lambda_k} f \, dx$, $c_2 \leftarrow \int_{\Omega \setminus \Lambda_k} f \, dx$, $k \leftarrow k + 1$.

$w_k \leftarrow$ Minimitzant de \mathcal{E}^λ amb $\lambda = 2\mu(c_1 - c_2)$.

$\Lambda \leftarrow$ Minimitzant de $\mathcal{F}^\mu(\cdot, c_1, c_2)$ usant el Teorema 0.5.

end while

return (Λ, c_1, c_2)

Per concloure aquesta introducció, la Figura 0.9 mostra un diagrama de la (λ, μ) -connexió, que il·lustra la relació entre els minimitzants de \mathcal{E}^λ i \mathcal{F}^μ .

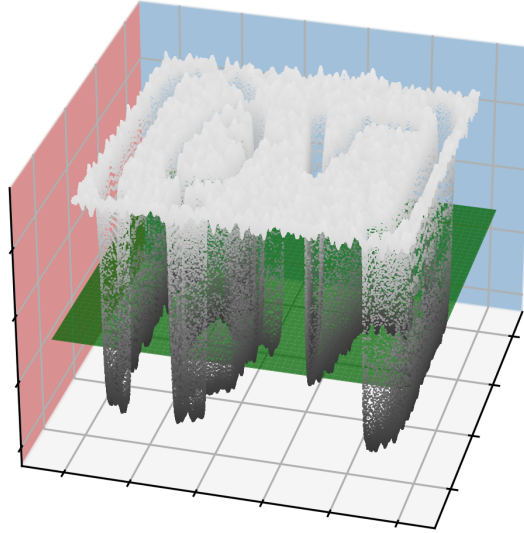


FIGURA 0.9. La imatge mostra la gràfica del minimitzador u de \mathcal{E}^λ . Gràcies a la (λ, μ) -connexió, el conjunt de nivell corresponent a $\frac{c_1+c_2}{2}$, representat en verd, defineix un minimitzador de $\mathcal{F}^\mu(\cdot, c_1, c_2)$ per a $\mu = \frac{\lambda}{2(c_1-c_2)}$.

En [34] es presenta també una extensió multifàsica de la connexió per a (0.8). Per tal de presentar aquest resultat, introduïm la següent modificació de $\mathcal{F}^{\mu, n}$:

$$\widehat{\mathcal{F}}^{\mu,n}(\mathbf{\Lambda}, \mathbf{c}) := \sum_{i=1}^n \text{Per}(\cup_{j=1}^i \Lambda_j; \Omega) + \mu_i \int_{\Lambda_i} (c_i - f)^2 dx, \quad (\mathbf{\Lambda}, \mathbf{c}) \in \mathcal{P}^n(\Omega) \times [0, 1]^n,$$

la qual difereix de $\mathcal{F}^{\mu,n}$ a causa de la sobrepenalització aplicada a les fronteres compartides per més de dues regions Λ_i . Juntament amb aquest funcional, definim

$$\mathcal{E}^{\lambda,n}(\mathbf{\Sigma}, \boldsymbol{\tau}) := \sum_{i=1}^{n-1} \left(\text{Per}(\Sigma_i; \Omega) + \lambda \int_{\Sigma_i} (\tau_i - f) dx \right), \quad (\mathbf{\Sigma}, \boldsymbol{\tau}) \in \mathcal{P}_n^*(\Omega) \times \mathcal{T}^{n-1}([0, 1]),$$

on $\mathcal{T}^n([0, 1])$ i $\mathcal{P}_n^*(\Omega)$ es definixen com

$$\mathcal{T}^n([0, 1]) := \{ \{\tau_i\}_{i=1}^n \in [0, 1]^n : \tau_i < \tau_{i+1} \},$$

$$\mathcal{P}_n^*(\Omega) := \{ \{\Lambda_i\}_{i=0}^n \in \Omega^n : \emptyset = \Lambda_0 \subseteq \Lambda_1 \subseteq \dots \subseteq \Lambda_{n-1} \subseteq \Lambda_n = \Omega \}.$$

Pel que fa a $\mathcal{E}^{\lambda,n}$, observem que cada sumand representa un funcional aïllat. Així, si fixem $\boldsymbol{\tau}$, segons [48, Proposition 2.6], sabem que el conjunt

$$\mathbf{\Sigma} := \{ \Sigma_i \}_{i=1}^{n-1}, \quad \text{on } \Sigma_i = \{ x \in \Omega : u(x) > \tau_i \},$$

on u és el minimitzant de \mathcal{E}^{λ} , és un minimitzant de $\mathcal{E}^{\lambda,n}(\cdot, \boldsymbol{\tau})$. Per aquest motiu, establir una connexió entre $\widehat{\mathcal{F}}^{\mu,n}$ i $\mathcal{E}^{\lambda,n}$ representa, en cert sentit, una extensió de la (λ, μ) -connexió. En [34] es presenta el resultat següent:

TEOREMA 0.6. [34, Teorema 3.7] *Siga $(\mathbf{\Sigma}, \boldsymbol{\tau}) \in \mathcal{P}_n^*(\Omega) \times \mathcal{T}^n([0, 1])$ tal que*

$$\mathcal{E}^{\lambda,n}(\mathbf{\Sigma}, \boldsymbol{\tau}) \leq \mathcal{E}^{\lambda,n}(\mathbf{E}, \boldsymbol{\tau})$$

per a tot $\mathbf{E} \in \mathcal{P}_n^*(\Omega)$. *Suposem que $\mathbf{\Lambda} := \{\Lambda_i\}_{i=1}^n = \{\Sigma_i \setminus \Sigma_{i-1}\}_{i=1}^n$ satisfà*

$$\tau_i = \frac{c_{i+1} + c_i}{2}, \quad \text{on } c_i := \int_{\Lambda_i} f dx,$$

per a tot $i \in \{1, \dots, n-1\}$. *Si $c_i > c_{i+1}$, aleshores $\mathbf{\Lambda}$ és un minimitzant de $\widehat{\mathcal{F}}^{\mu,n}(\cdot, \mathbf{c})$ per als valors següents de $\boldsymbol{\mu} := \{\mu_i\}_{i=1}^n$:*

$$\mu_1 = \frac{\lambda}{2(c_1 - c_2)}, \quad \mu_n = \frac{\lambda}{2(c_{n-1} - c_n)}, \quad \mu_i = \frac{\lambda(c_{i-1} - c_{i+1})}{2(c_{i-1} - c_i)(c_i - c_{i+1})} \quad \text{per a } 1 < i < n.$$

Al Capítol 4, presentem un contraexemple general que demostra que el resultat anterior no es compleix per a cap anisotropia (vegeu la Secció 2.3.2), incloent-hi el cas isotròpic tractat en el Teorema 0.6. Per aquest motiu, adoptem la notació (λ, μ) -connexió per a referir-nos exclusivament a la connexió bifàsica entre els models de ROF i de CV, que sí que és vàlida en el cas isotròpic.

0.5. Contribucions

Aquesta tesi presenta diversos treballs originals en l'estudi dels models de ROF i de CV, així com de la seua relació, els quals es resumeixen a continuació. Els resultats han sigut desenvolupats durant el període de recerca doctoral i han sigut obtinguts per l'autor conjuntament amb els seus coautors. Quan és pertinent, s'indiquen les publicacions corresponents on aquestes contribucions han sigut publicades o es troben actualment en procés de revisió.

0.5.1. Generalitzacions de la (λ, μ) -connexió. Al Capítol 3, analitzem els components essencials dels funcionals \mathcal{E}^λ i \mathcal{F}^μ , que permeten definir la (λ, μ) -connexió. Amb aquest objectiu, introduïm la noció d'Espai de Mesura de Perímetre (PMS, per les seues sigles en anglès), on un espai de mesura $(X, \mathcal{B}, \mathbf{m})$ és dotat amb un funcional P que exerceix el paper de perímetre sota certes hipòtesis. En concret, exigim que el següent funcional de variació total:

$$\mathrm{TV}_P(u) := \begin{cases} \int_{-\infty}^{\infty} P(E_t(u)) dt & \text{si } t \mapsto E_t(u) \text{ és } \mathbf{m}\text{-mesurable,} \\ +\infty & \text{altrament} \end{cases}$$

sigu semicontinu inferiorment respecte a la convergència dèbil en $L^2(X, \mathbf{m})$. Si es compleix aquesta condició, considerem que $(X, \mathcal{B}, \mathbf{m})$ equipat amb P és un PMS, i el denotem com (X, \mathbf{m}, P) .

Dins d'aquest marc, també demostrem que el funcional de variació total TV és convex en $L^2(X, \mathbf{m})$, i adaptem els problemes de ROF i CV, tal com es presenten en (0.3) i (0.7), a aquest entorn general. D'això se'n dedueix directament que tant l'existència com la unicitat del minimitzant del problema de ROF estan garantides. Gràcies a la formulació de tipus coàrea de la variació total, podem estendre els resultats de [48] al context PMS, generalitzant així la (λ, μ) -connexió a aquests espais.

Cal destacar que una gran varietat d'espais rellevants s'inclouen dins del marc PMS, cosa que demostra l'àmplia aplicabilitat de la (λ, μ) -connexió tant en l'àmbit matemàtic com en el de les aplicacions. Entre els exemples més significatius es troben els casos euclidians isotròpic i anisotròpic clàssics, així com entorns en què el perímetre és fraccional. Altres exemples de PMS inclouen els espais de passeigs aleatoris, que engloben estructures discretes i probabilístiques com ara cadenes de Markov i grafs amb pesos, i els grafs mètrics, que permeten l'anàlisi de problemes variacionals en configuracions de tipus xarxa. En particular, proporcionem dos exemples que il·lustren l'aplicabilitat de la (λ, μ) -connexió en el context dels grafs, demostrant de manera concreta els conceptes teòrics exposats.

Un cas particularment rellevant d'un PMS apareix quan $X = \Omega \subseteq \mathbb{R}^2$ és un conjunt obert i fitat, $\mathbf{m} = \mathcal{L}^2$, i $P = \mathrm{Per}_\Phi$ denota el perímetre anisotròpic (vegeu (2.7)) associat a l'anisotropia $\Phi(\xi) := |\xi|_1$. Al Capítol 4, millorem la (λ, μ) -connexió isotròpica sota hipòtesis estructurals sobre f , que, en aquest cas, permeten construir un minimitzant del problema de CV anisotròpic a partir del minimitzant del problema de ROF anisòtrop, i no sols un minimitzant parcial. Aquest entorn anisotròpic és estudiat amb més profunditat al Capítol 4. Més concretament, establim la (λ, μ) -connexió millorada, juntament amb altres resultats que tenen interès per si mateixos.

D'una banda, completem l'estudi anisotròpic de la (λ, μ) -connexió mitjançant l'anàlisi de la generalització del Teorema 0.6, i proporcionem un contraexemple general que invalida aquest resultat per a qualsevol anisotropia Φ . No obstant això, introduïm una connexió alternativa entre models que permet la segmentació amb n fases mitjançant una tècnica d'estratificació per llindars. Açò ens porta a presentar una comparació entre els entorns anisotròpic i isotròpic de les connexions, tant en el cas bifàsic com en el multifàsic, ressaltant així l'aplicabilitat del nostre marc anisotròpic.

D'altra banda, estudiem la propietat de salt equivalent introduïda en la Secció 0.2.4 per als minimitzants dels models de CV, en relació amb la propietat de salt presentada en el Teorema 1.1. En aquest context, observem que per a un minimitzant (Λ, c_1, c_2) de \mathcal{F}^μ , la propietat de salt natural seria la inclusió $\partial\Lambda \subseteq J_f$, la qual ha sigut obtinguda en el cas unidimensional en [128]. En aquesta línia, demostrem que aquesta inclusió no es verifica ni per al funcional isotròpic \mathcal{F}^μ ni per a la seua versió anisòtropa amb $\Phi(\xi) = |\xi|_1$. Per aquest motiu, proposem una propietat de salt més feble, que provem que es compleix en el cas anisotròpic amb $\Phi(\xi) = |\xi|_1$, tant en el cas bifàsic com en el multifàsic.

0.5.2. Generalització del model de ROF. Al Capítol 5, estudiem la regularitat del minimitzant del model de ROF restringint a varietats. Siguen (Σ, g) i (\mathcal{N}, h) , respectivament, una superfície compacta, possiblement amb frontera suau $\partial\Sigma \neq \emptyset$, i una varietat Riemanniana completa, connexa i suau de dimensió n . Aquesta última serà tractada com una subvarietat isomètrica immersa $\mathcal{N} \hookrightarrow \mathbb{R}^N$ en algun espai euclidià, segons el teorema de Nash [137]. Sovint escrivim $\bar{\Sigma} = \Sigma \cup \partial\Sigma$ per remarcar que una propietat és vàlida fins a la frontera. Donat $f \in L^2(\Sigma, \mathcal{N})$ i $\lambda > 0$, estudiem la regularitat dels minimitzants del funcional d'energia

$$(0.10) \quad \mathcal{E}_{\mathcal{N}}^{\lambda, \sigma}(u) := \int_{\Sigma} |du| \, d\mathbf{m}_g + \frac{\lambda}{2} \int_{\Sigma} d_g^2(u, f) \, d\mathbf{m}_g + \frac{\sigma}{2} \int_{\Sigma} |du|^2 \, d\mathbf{m}_g,$$

on $\lambda > 0$, $\sigma \geq 0$, d_g denota la distància geodèsica en \mathcal{N} , i $d\mathbf{m}_g$ és l'element de volum associat a g . En aquesta formulació, observem que (0.10) amb $\sigma = 0$ defineix el funcional de ROF en un entorn amb restricció de varietat. D'ara endavant, denotarem $\mathcal{E}_{\mathcal{N}}^{\lambda, 0}$ com $\mathcal{E}_{\mathcal{N}}^{\lambda}$ per tal d'unificar la notació.

En aquest sentit, aquests resultats poden considerar-se com l'anàleg amb restricció de varietat dels de [43, 143], amb el repte tècnic afegit de considerar superfícies no planes Σ (varietats abstractes de dimensió 2) com a dominis. Tal com s'ha esmentat a la Secció 0.2.4, la transmissió de la regularitat de f al minimitzant de $\mathcal{E}_{\mathcal{N}}^{\lambda}$ es pot establir sota certes condicions. En el nostre cas, aquest resultat pot deduir-se sota restriccions naturals de curvatura sobre \mathcal{N} i una condició topològica sobre el rang de f . Més concretament, considerem el següent fita del radi de convexitat per a $p \in \mathcal{N}$:

$$R_{\kappa} := \begin{cases} \frac{1}{2} \min \left\{ \text{inj}_p \mathcal{N}, \frac{\pi}{\sqrt{\kappa}} \right\} & \text{if } \kappa > 0 \\ \frac{\text{inj}_p \mathcal{N}}{2} & \text{if } \kappa \leq 0 \end{cases}$$

on κ és una fita superior per a totes les curvatures seccionals de la bola geodèsica $B_h(p, R)$ amb $R < R_{\kappa}$, i $\text{inj}_p(\mathcal{N})$ denota el radi d'injectivitat de la bola al seu centre p .

En aquest context, utilitzant tècniques de [104], deduïm principis d'invariància per als minimitzants u de (0.10), és a dir, si $f(\bar{\Sigma}) \subseteq B_g(p, R)$ per a algun $R < R_{\kappa}$, llavors $u(\bar{\Sigma}) \subseteq B_g(p, R)$. Açò ens permet establir l'existència mitjançant el mètode directe del Càlcul de Variacions. Pel que fa a la unicitat, el primer obstacle apareix pel fet que la convexitat (geodèsica) de la variació total no es verifica per a $\kappa > 0$ arbitrari (vegeu, per exemple, [83]). Així, imposant la restricció de curvatura $\kappa \leq 0$ i la limitació de la imatge de f , podem demostrar la unicitat per a $\sigma \geq 0$.

Per a estudiar la regularitat, la nostra estratègia consisteix a estendre els Teoremes 0.2 i 0.3 al nostre entorn generalitzat, amb l'objectiu específic d'establir la transmissió de la regularitat Lipschitz, sota les restriccions abans esmentades. Atés el caràcter degenerat del sistema d'Euler–Lagrange associat a $\mathcal{E}_{\mathcal{N}}^{\lambda, \sigma}$, considerem el sistema de Neumann relaxat:

$$(S_{\varepsilon, \sigma}^f) \quad \text{div}_g \left(\frac{du}{\sqrt{|du|^2 + \varepsilon^2}} + \sigma du \right) = -\lambda \exp_u^{-1} f \quad \text{en } \Sigma, \quad du(\nu) = 0 \quad \text{en } \partial\Sigma,$$

on ν representa la normal unitaria exterior en $\partial\Sigma$. Per tal d'aplicar un argument de tipus Bernstein com el de [143], és necessari diferenciar $(S_{\varepsilon, \sigma}^f)$ i demostrar que $|du|^2$ és una sub-solució d'una EDP el·líptica, cosa que permet emprar el principi del màxim de Hopf per a obtenir estimacions puntuals. Açò introdueix una altra dificultat, ja que, a diferència del cas escalar estudiat en [143], les solucions del sistema relaxat no són conegudes com a suaus

segons la teoria estàndard de regularitat. A més, fins i tot si f fóra suau, aconseguir regularitat almenys $C^3(\bar{\Sigma}, \mathcal{N})$ representa un desafiament, ja que la teoria dels sistemes el·líptics de Neumann no proporciona la informació necessària; en efecte, els resultats de regularitat interior estan vinculats a una condició de coercivitat de la forma $\sigma > C_R R$ (cf. [86, 99]), mentre que el nostre interès és considerar el límit $\sigma \rightarrow 0$ per tal de recuperar el problema original de ROF.

D'una banda, superem la manca de regularitat en el terme del segon membre mitjançant la construcció d'una suavització de f , que denotem per f_δ amb $\delta > 0$ (vegeu la Secció 2.4.4). D'altra banda, l'absència d'una teoria de regularitat ens porta a adaptar detalladament [11], la qual cosa ens permet obtenir regularitat $C^\infty(\bar{\Sigma}, \mathcal{N})$ per al sistema $(S_{\varepsilon, \sigma}^f)$. Aquesta adaptació de [11] és necessària a causa de les diferents condicions estructurals del nostre cas i es basa en resultats auxiliars que hem desenvolupat per separat, entre els quals destaquen una desigualtat de tipus Caccioppoli intrínseca i una integrabilitat superior de les solucions. Aquests resultats tenen un interès propi per la seua aplicació a una àmplia classe de generalitzacions de les aplicacions harmòniques. A més, cal tindre en compte que el fet de considerar Σ com una superfície afegeix dificultats addicionals degudes a la dependència en x de l'operador el·líptic que governa el problema. Aquest fet fa que les estimacions de regularitat Hölder en aquesta tesi siguin tècnicament molt més complexes que en el cas planar.

En aquest punt, cal remarcar que la regularitat obtinguda fa referència al sistema relaxat. Per a continuar, necessitem establir una fita uniforme per a $|du|$, independent de ε , σ i δ . Aquest fet és fonamental per a aplicar un argument de convergència que permeti obtenir regularitat Lipschitz del minimitzant de $\mathcal{E}_{\mathcal{N}}^{\lambda, 0}$ quan $\kappa \leq 0$. Més concretament, per al minimitzant u de $(S_{\varepsilon, \sigma}^f)$, deduïm la fita uniforme següent:

$$|du| \leq C(\text{Lip}(f)^2 + 1),$$

sota les restriccions de curvatura i de imatge establides anteriorment, i sense dependència de ε , δ ni σ . A més, observem que per a $\kappa > 0$ s'obtenen resultats similars en el cas $\sigma > 0$. Un aspecte destacable d'aquest estudi és que l'estratègia només pot dur-se a terme amb èxit des d'un punt de vista intrínsec, atès el paper crucial que juga la geometria de l'espai objectiu al llarg de tot el raonament.

De manera complementària, adaptem els resultats anteriors al cas en què el domini es defineix com $\Sigma = [0, 1]$ o bé $\Sigma = \mathbb{S}^1$, aprofitant les simplificacions derivades de la naturalesa unidimensional del domini.

Introduction

In this chapter, we present the essential background that sets the foundation for the developments discussed throughout this thesis. Some of the notation and concepts used here are treated formally in Chapter 2, to which we may refer for further details.

1.1. Image modeling

In this section we present three perspectives for interpreting images, which we represent by means of a function $u_0 : \Omega \rightarrow X$, where $\Omega \subseteq \mathbb{R}^2$, and classify the various target spaces X that may be employed.

First, we consider images modeled as scalar-valued functions (e.g, gray-scale images). In this case, for each position $(x, y) \in \Omega$, the value $u_0(x, y)$ represents the image brightness at that location. Figure 1.1 illustrates the relationship between such an image and the graph of the function u_0 .

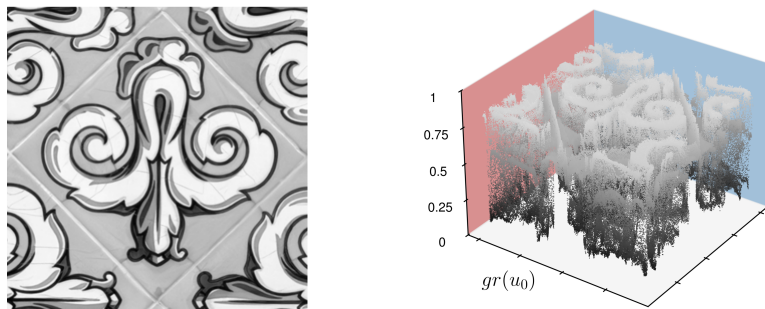


FIGURE 1.1. Example of scalar modeling: the gray-scale tile image on the left can be interpreted as the graph of a scalar function $u_0 : \Omega \rightarrow [0, 1]$, as shown in the right-hand figure.

As it is exemplified in the above example, the scalar modeling is clearly restricted, leaving aside most of application scenarios. For this reason, if we consider $u : \Omega \rightarrow X \subset \mathbb{R}^M$, we can model a wide variety of image types: from SAR images ($M = 2$), RGB images ($M = 3$), and CMYK images ($M = 4$), to hyperspectral formats with $M \gg 100$. Moreover, such a vector-valued representation is often necessary to study some specific phenomena, such as color gradients, contour and texture orientation, motion within the image, or the registration of changes between two images. As an example of this type of modeling, we show the case of a RGB-color image (whose pixels live in the cube $X = [0, 255]^3$) in Figure 1.2.

At this point, it is important to highlight that vector-valued image modeling may be subject to constraints, either due to the nature of the data to be represented (e.g., DT-MRI images) or the specific aspect of the image under study (e.g., chromaticity). In our case, we focus on scenarios where the values of the function u_0 lie on differentiable manifolds, typically as a consequence of the intrinsic properties of the image being analyzed (see, e.g., [67, 162]). It is worth emphasizing that the study of manifold-constrained signals and images is not merely theoretical; it is driven by real-world challenges that have been explored in the applied literature (see, e.g., [19]).

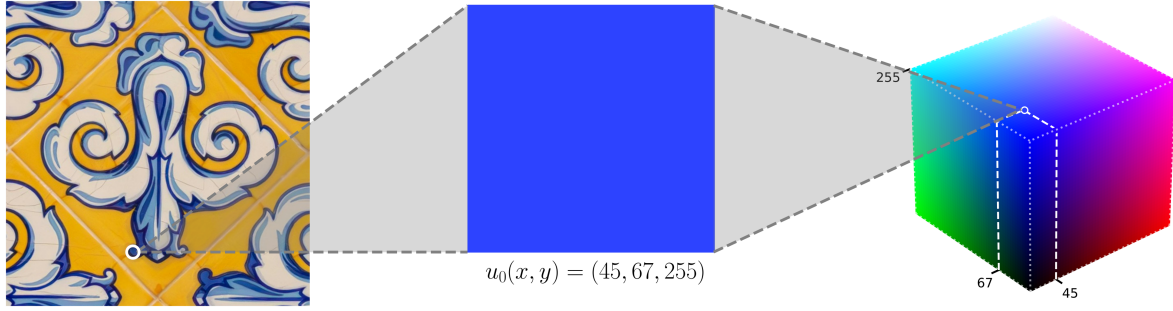


FIGURE 1.2. Example of RGB color modeled by vector function u_0 .

From an application-oriented perspective, a relevant example arises in biomedical imaging, where constrained models are often defined on non-positively curved (NPC) manifolds. A prominent case is diffusion tensor magnetic resonance imaging (DT-MRI) in neurology, where six different datasets, acquired under varying magnetic field orientations, are used to analyze the diffusion of water molecules at each pixel. This diffusion information is naturally encoded in the space of symmetric positive definite 3×3 matrices. In another direction, hyperbolic space has gained increasing attention in both image segmentation [12] and machine learning [125]. In such contexts, the training process typically introduces noise, and robust denoising algorithms can improve pattern recognition. A natural explanation lies in the fact that hierarchical structures, such as trees or taxonomies, exhibit exponential growth in the number of nodes—mirroring the volume expansion of balls in hyperbolic spaces. This makes hyperbolic geometry more appropriate than the traditional Euclidean setting, where volume growth is merely polynomial.

Similarly, positively curved target spaces are employed in various applications. For instance, in RGB chromaticity, image values lie on the positive octant of the unit sphere \mathbb{S}^2 [153], where RGB colors are treated in a normalized manner, excluding the influence of lightness, as illustrated in Figure 1.3. In materials science, electron backscatter diffraction (EBSD) techniques are used to study the microscopic structure of polycrystalline materials, with data taking values in the rotation group $SO(3)$ [13, 131]. Other notable examples include optical flow fields, commonly used in motion estimation and modeled on the sphere \mathbb{S}^2 ; perceptual color representations, such as the LCh color space, defined on the product space $\mathbb{R} \times \mathbb{S}^1$; and rigid object rotations, central to various vision tasks, described by the special Euclidean group $SE(2)$.

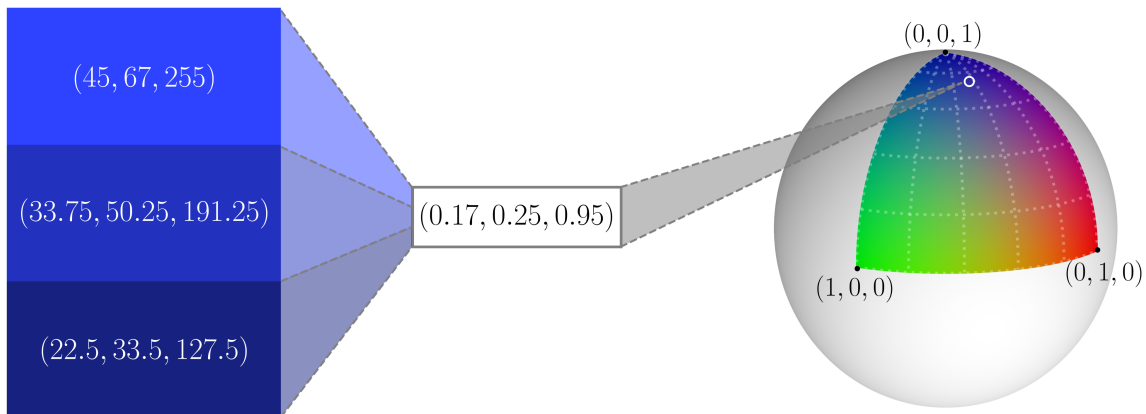


FIGURE 1.3. Example of how RGB colors are uniformly represented on \mathbb{S}^2 through normalization.

Furthermore, image modeling is not restricted to codomain generalizations. For example, curved domains naturally arise in contexts such as non-flat computer screens, geophysical and medical applications (e.g., wind directions or cortical surfaces in brain imaging), and even in 3D surface coloring for artistic and design purposes [109].

In conclusion, images may be modeled in various ways depending on the nature of the problem at hand. In the following sections, we will discuss the core models considered in this work and identify the types of images to which they may be applied. This will allow us to outline the strengths and limitations of each model, thus providing the necessary context for the development of this thesis.

1.2. Rudin–Osher–Fatemi model

1.2.1. Motivation. We consider that the image to be processed is in grayscale and we assume that the image has been corrupted by additive Gaussian noise n with zero mean and standard deviation σ . That is, we can express f as

$$f = u_0 + n,$$

where u_0 is the original, uncorrupted image. In Figure 1.4 we illustrate this composition.

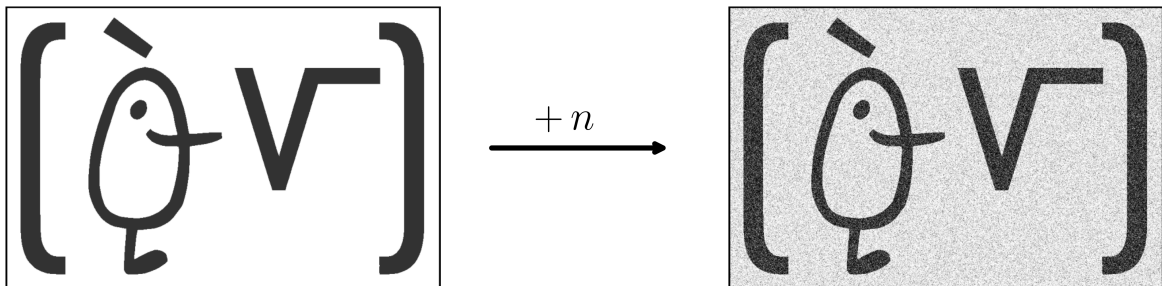


FIGURE 1.4. The left picture shows the *clean* image u_0 , while the right picture displays the *noisy* image f , obtained by adding additive Gaussian noise n .

In this context, the inverse problem at hand is to reduce the noise present in f , that is, to recover u_0 . From a variational perspective, we aim to find a function u that minimizes a functional of the form

$$\phi(u, f) + \gamma \mathcal{R}(u),$$

where ϕ is a fidelity term, \mathcal{R} is a regularization term, and $\gamma > 0$ is a regularization parameter. In our case, ϕ should preserve the structural relationship between the solution u and the noisy image f , while \mathcal{R} should penalize perturbations due to noise present in u (for further details, see, e.g., [149, Chapters 3 & 4]). Specifically, we are interested in a problem of the following form:

$$(1.1) \quad \min_{u \in L^2(\Omega)} \left\{ J_\gamma(u) := \int_{\Omega} (u - f)^2 dx + \gamma \int_{\Omega} |Qu|^2 dx \right\},$$

where $Q : L^2(\Omega) \rightarrow L^2(\Omega)$ is a linear operator acting as a regularizer of the solution u and $\gamma > 0$ is the regularization parameter. Computing the first variation of $J_\gamma(u)$ yields

$$\frac{\delta J_\gamma}{\delta u}(v) = 2 \int_{\Omega} (u - f)v dx + 2\gamma \int_{\Omega} (Qu)(Qv) dx,$$

for all $v \in H^1(\Omega)$. Setting this variation to zero, we deduce that the minimizer u of $J_\gamma(u)$ must satisfy the following equation:

$$(\mathbb{1} + \gamma Q^*Q)u = f,$$

where $\mathbb{1}$ denotes the identity operator, and Q^* denotes the adjoint operator of Q . The relevance of problem (1.1) lies in its connection with the above expression. Specifically, when the operator on the left-hand side is invertible, finding the minimizer u reduces to solving a linear system of equations.

1.2.2. Definition. Now, suppose that u_0 is a continuous function away from the edges of the image. Thus, it is essential to choose an operator Q capable of removing small perturbations of f (associated with the additive noise n) while preserving the edges. In their work [146], L. Rudin, S. Osher, and E. Fatemi pointed out the limitations of considering $Qu = Du$, i.e., the Dirichlet energy, in (1.1), since functions in the space $H^1(\Omega)$ cannot exhibit discontinuities along rectifiable curves, which represent the edges of the image u_0 . To overcome this limitation, they proposed to take $Qu = Du|Du|^{-1}$, i.e., restricting the minimization to the space $BV(\Omega)$ to ensure the preservation of the edges of the original image u_0 , as well as continuity (in the measure-theoretic sense; see [6]) away from these edges. Specifically, in [146] the following constrained problem is proposed:

$$(1.2) \quad \min \left\{ \int_{\Omega} |Du| : u \in BV(\Omega), \int_{\Omega} (u - f)^2 dx = \sigma \mathcal{L}^2(\Omega) \right\},$$

where prior information on the standard deviation of the noise n is used. Minimizing (1.2) is equivalent to maximizing the probability $P(u|f) \propto P(u)P(n)$ of recovering u from a known f , when the total variation is chosen to model $P(u)$. In practice, the ROF problem is reformulated as:

$$(1.3) \quad \min_{u \in BV(\Omega)} \left\{ \mathcal{E}^\lambda(u) := \int_{\Omega} |Du| + \frac{\lambda}{2} \int_{\Omega} (u - f)^2 dx \right\},$$

with $Qu = Du|Du|^{-1}$ and $\gamma = 2/\lambda$, since in [51] the existence of a minimizer of (1.3) and the equivalence between (1.2) and (1.3) are established, relating λ and σ . Thus, λ serves both as the regularization parameter, in the sense of (1.1), and as the Lagrange multiplier enforcing the constraint in (1.2). In Figures 1.5 and 1.6, we show the result of applying (1.3), as well as the regularizing effect of λ .

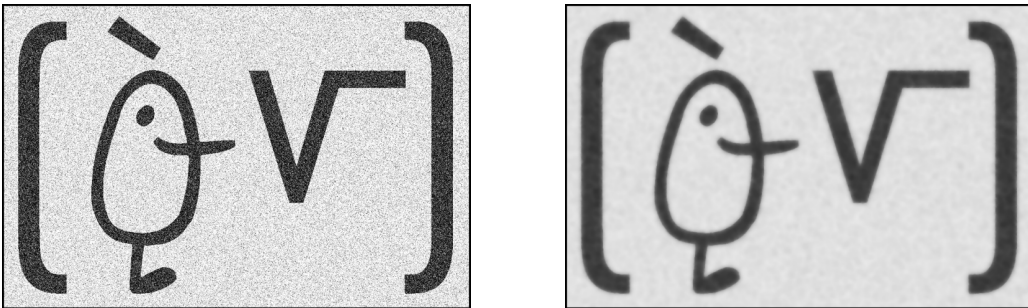
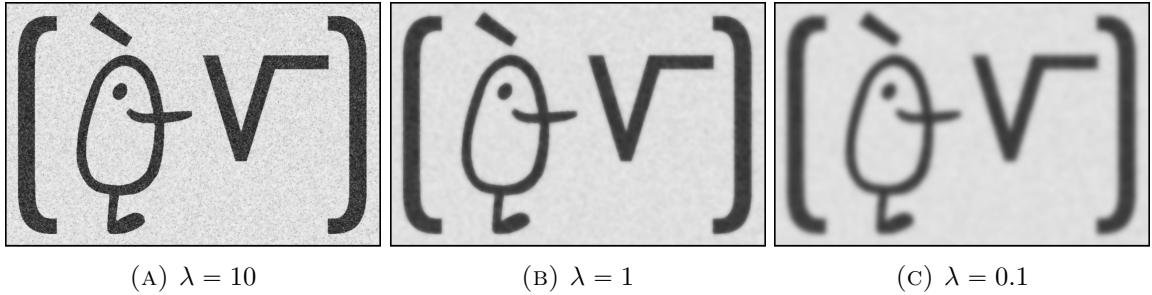


FIGURE 1.5. The left picture shows the *noisy* image f , while the right picture displays the *denoised* image u , obtained by minimizing \mathcal{E}^λ .

FIGURE 1.6. Effect of λ in the minimization (1.3)

Moreover, we note that, due to the convexity of \mathcal{E}^λ , uniqueness of the minimizer u of (1.3) is guaranteed, and it must satisfy the associated Euler-Lagrange equation:

$$\operatorname{div}(z) = \lambda(u - f)$$

where $z \in \partial\operatorname{TV}(u)$ (see Section 2.3.3).

1.2.3. Limitations. While total variation can serve as an effective regularization term, it is essential to acknowledge the well-known drawbacks associated with its use. Among these, we highlight issues such as the presence of the staircasing effect [138, 144], pixelation in certain regions [44], loss of textures in the image u_0 [92], dependence on the scaling factor [152], or homogenization to the mean value of f in Ω if λ is sufficiently small [7]. Several approaches have been proposed to address these problems, including the definition of non-local filters [29], the establishment of a posteriori statistical estimations for recovering u [118], or reformulating (1.2) with local constraints [74], to name a few examples. Nevertheless, the simplicity of total variation and its geometric properties have enabled a wide range of theoretical studies that have significantly contributed to understanding the behavior of the solutions to this model. Finally, we note that (1.3) has been generalized to the vectorial case (e.g., [22, 25]), as well as to the case restricted to manifolds (e.g., [24, 162]), employing a discrete formulation equivalent to the one presented in Chapter 5. Furthermore, (1.3) is also generalizable in the anisotropic sense [94]. In Chapter 4, we provide a detailed presentation of the anisotropic variant of the ROF problem.

1.2.4. Related Results. In this section, we introduce three topics relevant to this work. First, we present a selection of results regarding the jump set J_u (see Section 2.3) of the minimizer of (1.3), which we use in Chapters 4 and 5. Secondly, we summarize various results on the regularity of the minimizer u of (1.3), which are necessary to contextualize the work in Chapter 5. Finally, we mention the numerical techniques employed to solve (1.3), which are used in Sections 3.4 and 4.4, with certain modifications.

1.2.4.1. Preservation of the Jump Set. In Section 1.2.2, we motivated the interest in selecting Q in (1.1) so that the edges of the image u_0 are preserved. Since $f = u_0 + n$, these regions are represented by the points of J_f . For this reason, a series of works [3, 41, 42, 46] study the relationship between J_u and J_f , where u is the minimizer of (1.3), particularly when $\Omega \subseteq \mathbb{R}^N$ with $N \geq 2$. Specifically, these works explore the connection between (1.3) and the prescribed curvature problem:

$$(1.4) \quad \min \left\{ \operatorname{Per}(E; \Omega) + \lambda \int_{\Omega} (t - f(x)) \, dx : E \subseteq \Omega \right\}.$$

In [46], it is shown that the level set $E_t(u) := \{x \in \Omega : u(x) > t\}$ is a solution to (1.4) for all $t \in \mathbb{R}$. In [41], the following characterization of the jump sets is presented:

THEOREM 1.1. [42, Theorem 1] *Let $f \in BV(\Omega) \cap L^\infty(\Omega)$ and u be the minimizer of \mathcal{E}^λ for $\lambda > 0$. Then, it holds that $J_u \subseteq J_f$ up to an \mathcal{H}^{N-1} -null set.*

In this result, the key idea is to use the relationship between (1.3) and (1.4), as well as to analyze the curvature of the level sets $E_t(u)$ and the impossibility of intersection between the boundaries of different level sets. However, it is important to note that this technique is limited to the scalar formulation of the problem. As a finer result, in [62] is provided a characterization of the jump set depending on the parameter λ for $\Omega = [0, 1]$, using the equivalence between ROF problem and taut string problems [93].

In the vectorial variant of (1.3), it is worth mentioning that [157] obtains an equivalent result of Theorem 1.1 without the need to use level sets, and that this result can be generalized to other regularization terms. As a generalization of this work, [50] simplifies the techniques of [157] and removes the condition $f \in BV(\Omega)$ present in (1.1), applying a relaxed notation of J_f if $f \notin BV(\Omega, \mathbb{R}^M)$ [50, §7].

1.2.4.2. *Regularity of the Minimizer.* Another aspect of interest regarding the minimizer of (1.3) is the study of its regularity depending on the regularity of f . As in the previous section, the results we present apply in the case where $\Omega \subseteq \mathbb{R}^N$, unless otherwise indicated.

In [43], it is shown that if $N \leq 7$, the condition $f \in C_{\text{loc}}^{0,\beta}(\Omega)$ implies that $u \in C_{\text{loc}}^{0,\beta}(\Omega)$ for any $\beta \in [0, 1]$. Moreover, if Ω is convex and $f \in UC(\Omega)$, then $u \in UC(\Omega)$ under the same dimensional restriction. It is important to highlight that this limitation arises from the use in [43] of the relationship between (1.3) and (1.4). Specifically, the proof strategy is based on a local estimate of the distance between the level sets, as well as on the regularity of these sets when $N \leq 7$ (see, for example, [76, §5.4.15]).

This dimensional restriction has been removed in subsequent works. On the one hand, in [124] is proven that u is continuous if f is continuous, without any dimensional restriction, although the convexity of the domain is required. On the other hand, it is not until [143] that the Lipschitz regularity of the minimizer of (1.3) is studied without convexity or dimensional restrictions, using a Bernstein-type argument. Now we introduce two results that serve as the starting point for the work presented in Chapter 5:

THEOREM 1.2. [143, Theorem 1.1] *Let $f \in C_{\text{loc}}^{0,1}(\Omega)$ and let u be the minimizer of \mathcal{E}^λ . Then, $u \in C_{\text{loc}}^{0,1}(\Omega)$ and there exists a constant $C > 0$ such that*

$$\|\nabla u\|_{L^\infty(B_r(x_0))} \leq \|\nabla f\|_{L^\infty(B_r(x_0))} + \frac{C}{r^2},$$

for any $x_0 \in \Omega$ and any $r > 0$ such that f is Lipschitz in $B_{2r}(x_0) \subseteq \Omega$.

THEOREM 1.3. [143, Theorem 1.2] *Let $f \in W^{1,\infty}(\Omega)$ and let u be the minimizer of \mathcal{E}^λ . Then, $u \in W^{1,\infty}(\Omega)$ and*

$$\|\nabla u\|_{L^\infty(\Omega)} \leq C_1 \|\nabla f\|_{L^\infty(\Omega)} + C_2,$$

where C_1 and C_2 depend only on Ω .

These two results are generalized in Chapter 5 for the formulation of (1.3) constrained to a manifold. Regarding the study of regularity for the vectorial formulation of (1.3), we highlight that [82] achieves an equivalent result to Theorem 1.3 considering $\Omega = (a, b)$. Moreover, in this work, they obtain the following estimate in the sense of measures:

$$|Du|(A) \leq |Df|(A)$$

for any Borel set $A \subseteq \Omega$. This result will also be addressed in the manifold codomain case in Chapter 5.

1.2.4.3. *Numerical Methods.* Finally, we refer to the different ways to solve (1.3) in the scalar formulation. In the original paper [146], it is proposed to solve (1.2) using gradient projection methods, which may be slow due to the problem's conditioning, an aspect improved in subsequent works, such as [160]. More efficient ways to solve (1.3) include those based on Newton's method [54], multigrid preconditioning [159], or *graph-cut* techniques [23, 63], to name a few examples. Due to their popularity, we specifically highlight Chambolle's dual method [47], the primal-dual method [53], and the *split Bregman* method [92].

1.3. Chan–Vese model

1.3.1. Motivation. In this section, we assume that the image f consists of n distinct regions, each characterized by smooth variations within them, and our goal is to identify these regions. Since the domain of f is Ω , we can define the segmentation of the domain as

$$\Omega = \left(\bigcup_{i=1}^n \Lambda_i \right) \cup K, \quad K = \bigcup_{i=1}^n \partial\Lambda_i,$$

where the set K represents the boundary defined by these regions. One way to identify these regions is to model them via a function $u \in H^1(\Omega \setminus K)$, allowing discontinuities to be present in u . To summarize, Figure 1.7 illustrates this composition.

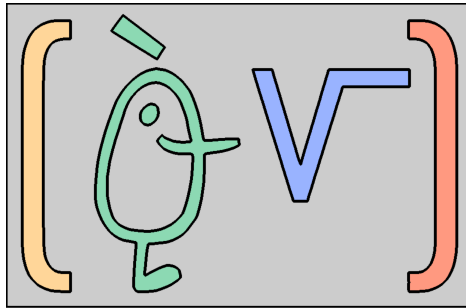


FIGURE 1.7. Decomposition of f into different components Λ_i (non-black regions) and the boundary set K (black contours), characterizing Ω .

With this idea, D. Mumford and J. Shah proposed solving the following problem [134]:

$$(1.5) \quad \min_{\substack{u \in H^1(\Omega \setminus K) \\ K \subseteq \Omega}} \left\{ \mu \int_{\Omega} (u - f)^2 dx + \int_{\Omega \setminus K} |\nabla u|^2 dx + \nu \mathcal{H}^{m-1}(K) \right\}, \quad \mu, \nu > 0,$$

where the first term ensures fidelity between u and f , the second penalizes variations in the presumed smooth regions (i.e., in $\Omega \setminus K$), and the third one ensures that the minimizers of (1.5) do not have excessively long boundaries.

A particularly relevant case is to model images as a union of piecewise constant regions plus a boundary region K . In this case, we consider u to be a piecewise constant function, that is, $u(\Lambda_i) = \{c_i\}$ with $c_i \in \mathbb{R}$ for all $1 \leq i \leq n$. In this context, it is worth noting that (u, K) can be represented by $\mathbf{c} = \{c_i\}_{i=1}^n \subseteq \mathbb{R}$ and $\mathbf{\Lambda} = \{\Lambda_i\}_{i=1}^n \in \mathcal{P}_n(\Omega)$, where $\mathcal{P}_n(\Omega)$ denotes the set of n -partitions of Ω . Additionally, to emphasize the relevance of each region Λ_i in the image, we may define different values of μ in each zone, denoted $\mu_i > 0$. Thus, under these new conditions, the Mumford–Shah problem transforms into the following:

$$(1.6) \quad \min \left\{ \sum_{i=1}^k \mu_i \int_{\Lambda_i} (c_i - f)^2 dx + \nu \mathcal{H}^{m-1} \left(\bigcup_{i=1}^k \partial \Lambda_i \right) : \{c_i\}_{i=1}^k \subseteq \mathbb{R}, \{\Lambda_i\}_{i=1}^k \in \mathcal{P}_k(\Omega) \right\}.$$

1.3.2. Definition. For the case $\mu_i = \mu$, $\nu = 1$, $k = 2$ and $m = 2$, (1.6) becomes the Chan–Vese (CV) problem [55], defined as

$$(1.7) \quad \min_{\substack{\Lambda \subseteq \Omega \\ c_1, c_2 \in [0,1]}} \left\{ \mathcal{F}^\mu(\Lambda, c_1, c_2) := \text{Per}(\Lambda; \Omega) + \mu \int_{\Lambda} (c_1 - f)^2 dx + \mu \int_{\Omega \setminus \Lambda} (c_2 - f)^2 dx \right\}.$$

In this setting, we note that (1.7) provides a segmentation of Ω into two regions based on the structure of f . To obtain n distinct regions in Ω , we can reformulate (1.6) to derive a multiphase version of the Chan–Vese model [56] as follows:

$$(1.8) \quad \min_{\substack{\Lambda \in \mathcal{P}_n(\Omega) \\ \mathbf{c} \in [0,1]^n}} \left\{ \mathcal{F}^{\mu, n}(\Lambda, \mathbf{c}) := \sum_{i=1}^n \text{Per}(\Lambda_i; \Omega) + \mu_i \int_{\Lambda_i} (c_i - f)^2 dx \right\},$$

where $\mu \in [0, +\infty)^n$. To exemplify the CV model, in Figure 1.8 we show the minimizer of Chan–Vese problem (1.7).

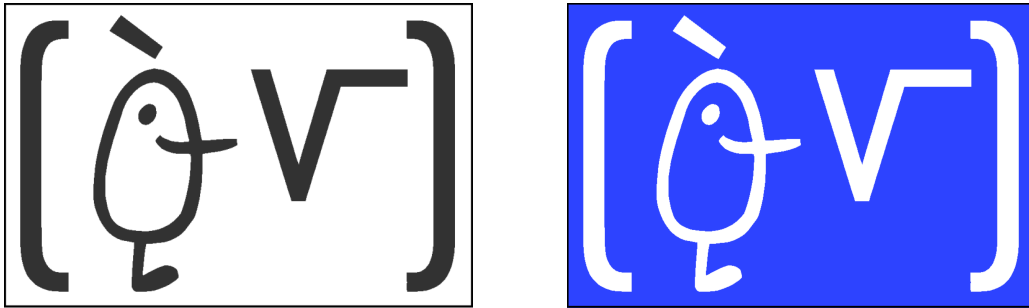


FIGURE 1.8. The left image shows f and the right one displays the set Λ in blue. For a sufficiently large value of μ , a minimizer of \mathcal{F}^μ is the tuple $(\Lambda, 0, 1)$.

1.3.3. Related Results. In the following section, we will introduce a method to solve (1.7) by minimizing (1.3), based on a connection between both methods that will be fundamental in Chapters 3 and 4. Nevertheless, we wish to list different approaches to solve the CV problem, both in its two-phase and n -phase formulations.

Regarding algorithms for solving (1.7), [55] proposes the use of an *active contour* technique, characterizing $\partial \Lambda$ via a level set; this approach was optimized in [166]. Alternative methods include algorithms based on thresholding techniques over the convex relaxation of \mathcal{F}^μ [28], topological derivative approaches [98], the *Split-Bregman* technique [92], projection methods [68], multigrid frameworks [14], and *graph-cut* techniques [15, 165], to name a few examples. On the other hand, the problem (1.8) can be solved using some of the aforementioned methods (e.g., [68, 92, 98]) directly. Within these adaptations to the n -phase case, the active contour technique stands out, which has fostered several adaptations [56, 107, 114, 116], as well as generalizations [154].

1.4. (λ, μ) -linkage

In this section, we present the relationship between the ROF and CV models introduced in [34]. Given the significance of these two problems, establishing a connection between them allows us to build links between different research areas and develop more efficient solution

methods. Throughout this thesis, we explore how the relationship between these two models, which we refer to as the (λ, μ) -connection, becomes a versatile tool for generalizing these models. To introduce this connection, we first define the concept of a *partial minimizer* of the CV problem, i.e., of \mathcal{F}^μ :

DEFINITION 1.4. *Let $(\Lambda^*, c_1^*, c_2^*)$ be a tuple such that $(c_1^*, c_2^*) \in [0, 1]^2$ and $\Lambda \subseteq \Omega$. We say that it is a partial minimizer of \mathcal{F}^μ if the following conditions hold:*

$$\begin{aligned}\mathcal{F}^\mu(\Lambda^*, c_1^*, c_2^*) &\leq \min \{ \mathcal{F}^\mu(\Lambda^*, c_1, c_2) : (c_1, c_2) \in [0, 1]^2 \}, \\ \mathcal{F}^\mu(\Lambda^*, c_1^*, c_2^*) &\leq \min \{ \mathcal{F}^\mu(\Lambda, c_1^*, c_2^*) : \Lambda \subseteq \Omega \}.\end{aligned}$$

Using this concept, the authors in [34] introduce the (λ, μ) -connection:

THEOREM 1.5. [34, Theorem 3.4] *Let u be the minimizer of \mathcal{E}^λ , $0 \leq c_2 \leq c_1 \leq 1$ and $\mu = \frac{\lambda}{2(c_1 - c_2)}$. If the level set*

$$\Lambda = \left\{ x \in \Omega : u(x) > \frac{c_1 + c_2}{2} \right\}$$

satisfies $0 < \mathcal{L}^2(\Lambda) < \mathcal{L}^2(\Omega)$, then Λ is a minimizer of $\mathcal{F}^\mu(\cdot, c_1, c_2)$. Moreover, if

$$(1.9) \quad c_1 = \int_{\Lambda} f \, dx, \quad c_2 = \int_{\Omega \setminus \Lambda} f \, dx,$$

then (Λ, c_1, c_2) is a partial minimizer of \mathcal{F}^μ .

In [34], the authors provide an argument to establish a solution method for the CV problem via the (λ, μ) -connection. This technique relies on the fact that the minimizers of $\mathcal{F}^\mu(\Lambda, \cdot, \cdot)$ are determined by expression (1.9). Together with Theorem 1.5, they propose an algorithm converging to a partial minimizer of \mathcal{F}^μ [34, §4]. Since various variants of this algorithm will be presented throughout this work, we describe it below:

Algorithm 1.1 Partial minimizer of \mathcal{F}^μ

Require: $k = 0$, $\mu > 0$, $(c_1, c_2) \in [0, 1]$ such that $c_1 > c_2$.

$w_0 \leftarrow$ Minimizer of \mathcal{E}^λ with $\lambda = 2\mu(c_1 - c_2)$.

$\Lambda \leftarrow$ Minimizer of $\mathcal{F}^\mu(\cdot, c_1, c_2)$ using Theorem 1.5.

while $(|\Lambda_k \Delta \Lambda_{k-1}|^2 > \varepsilon_{tol}) \wedge (k < n_{max})$ **do**

$c_1 \leftarrow \int_{\Lambda_k} f \, dx$, $c_2 \leftarrow \int_{\Omega \setminus \Lambda_k} f \, dx$, $k \leftarrow k + 1$.

$w_k \leftarrow$ Minimizer of \mathcal{E}^λ with $\lambda = 2\mu(c_1 - c_2)$.

$\Lambda \leftarrow$ Minimizer of $\mathcal{F}^\mu(\cdot, c_1, c_2)$ using Theorem 1.5.

end while

return (Λ, c_1, c_2)

To conclude this introduction, Figure 1.9 shows a diagram of the (λ, μ) -connection, illustrating the relationship between the minimizers of \mathcal{E}^λ and \mathcal{F}^μ .

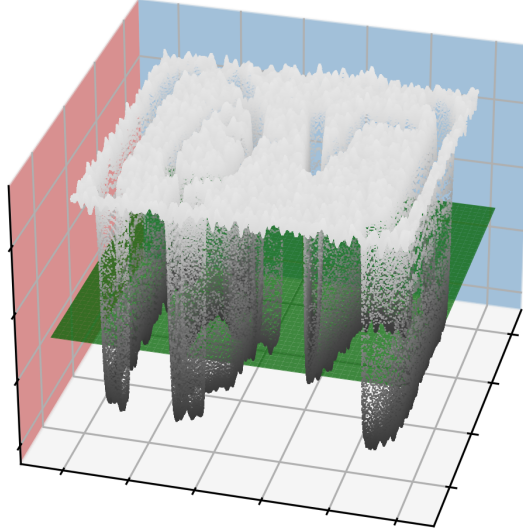


FIGURE 1.9. The image shows the graph of the minimizer u of \mathcal{E}^λ . Thanks to the (λ, μ) -linkage, the level set at $(c_1 + c_2)/2$, shown in green, defines a minimizer of $\mathcal{F}^\mu(\cdot, c_1, c_2)$ for $\mu = \frac{\lambda}{2(c_1 - c_2)}$.

In [34], a multiphase extension of the connection for (1.8) is provided. To present this result, we introduce the next modification of $\mathcal{F}^{\mu, n}$:

$$(1.10) \quad \widehat{\mathcal{F}}^{\mu, n}(\mathbf{\Lambda}, \mathbf{c}) := \sum_{i=1}^n \text{Per}(\cup_{j=1}^i \Lambda_j; \Omega) + \mu_i \int_{\Lambda_i} (c_i - f)^2 dx, \quad (\mathbf{\Lambda}, \mathbf{c}) \in \mathcal{P}^n(\Omega) \times [0, 1]^n,$$

which differs from $\mathcal{F}^{\mu, n}$ due to the over-penalization applied to boundaries shared by more than two regions Λ_i . Alongside this functional, we define

$$\mathcal{E}^{\lambda, n}(\mathbf{\Sigma}, \boldsymbol{\tau}) := \sum_{i=1}^{n-1} \left(\text{Per}(\Sigma_i; \Omega) + \lambda \int_{\Sigma_i} (\tau_i - f) dx \right), \quad (\mathbf{\Sigma}, \boldsymbol{\tau}) \in \mathcal{P}_n^*(\Omega) \times \mathbb{T}^{n-1}([0, 1]),$$

where $\mathbb{T}^n([0, 1])$ and $\mathcal{P}_n^*(\Omega)$ are defined as

$$\mathbb{T}^n([0, 1]) := \{ \{\tau_i\}_{i=1}^n \in [0, 1]^n : \tau_i < \tau_{i+1} \},$$

$$\mathcal{P}_n^*(\Omega) := \{ \{\Lambda_i\}_{i=0}^n \in \Omega^n : \emptyset = \Lambda_0 \subseteq \Lambda_1 \subseteq \dots \subseteq \Lambda_{n-1} \subseteq \Lambda_n = \Omega \}.$$

Regarding $\mathcal{E}^{\lambda, n}$, we note that each summand represents an isolated functional. Thus, if we fix $\boldsymbol{\tau}$, by [48, Proposition 2.6], we know that the set

$$\mathbf{\Sigma} := \{ \Sigma_i \}_{i=1}^{n-1}, \quad \text{such that} \quad \Sigma_i = \{x \in \Omega : u(x) > \tau_i\},$$

where u is the minimizer of \mathcal{E}^λ , is a minimizer of $\mathcal{E}^{\lambda, n}(\cdot, \boldsymbol{\tau})$. For this reason, connecting $\widehat{\mathcal{F}}^{\mu, n}$ and $\mathcal{E}^{\lambda, n}$ represents, in a sense, an extension of the (λ, μ) -connection. In [34] is presented the next result:

THEOREM 1.6. [34, Theorem 3.7] *Let $(\mathbf{\Sigma}, \boldsymbol{\tau}) \in \mathcal{P}_n^*(\Omega) \times \mathbb{T}^n([0, 1])$ such that*

$$\mathcal{E}^{\lambda, n}(\mathbf{\Sigma}, \boldsymbol{\tau}) \leq \mathcal{E}^{\lambda, n}(\mathbf{E}, \boldsymbol{\tau})$$

for all $\mathbf{E} \in \mathcal{P}_n^(\Omega)$. Assume that $\mathbf{\Lambda} := \{\Lambda_i\}_{i=1}^n = \{\Sigma_i \setminus \Sigma_{i-1}\}_{i=1}^n$ satisfies*

$$\tau_i = \frac{c_{i+1} + c_i}{2}, \quad \text{with } c_i := \int_{\Lambda_i} f \, dx,$$

for each $i \in \{1, \dots, n-1\}$. If $c_i > c_{i+1}$, then $\mathbf{\Lambda}$ is a minimizer of $\widehat{\mathcal{F}}^{\mu, n}(\cdot, \mathbf{c})$ for the following values of $\boldsymbol{\mu} := \{\mu_i\}_{i=1}^n$:

$$\mu_1 = \frac{\lambda}{2(c_1 - c_2)}, \quad \mu_n = \frac{\lambda}{2(c_{n-1} - c_n)}, \quad \mu_i = \frac{\lambda(c_{i-1} - c_{i+1})}{2(c_{i-1} - c_i)(c_i - c_{i+1})} \quad 1 < i < n.$$

In Chapter 4, we present a general counterexample demonstrating that the previous result does not hold for any anisotropies (see Section 2.3.2), including the isotropic case addressed in Theorem 1.6. For this reason, we adopt the notation (λ, μ) -linkage to denote exclusively the 2-phase linkage between the ROF and CV models, which does hold in the isotropic case.

1.5. Contributions

This thesis presents several original works to the study of ROF and CV models and its relationship, which are summarised below. The results have been developed in the course of the doctoral research and have been obtained by the author together his coauthors. When applicable, the corresponding publications where these contributions have appeared or are currently submitted are indicated.

1.5.1. Generalizations of (λ, μ) -linkage. In Chapter 3, we analyze the essential components of the functionals \mathcal{E}^λ and \mathcal{F}^μ , which enable the definition of the (λ, μ) -linkage. With this aim, we introduce the notion of a Perimeter Measure Space (PMS, in short), where a measure space $(X, \mathcal{B}, \mathbf{m})$ is endowed with a functional P that plays the role of a perimeter under certain assumptions. Specifically, we require that the following total variation functional:

$$\text{TV}_P(u) := \begin{cases} \int_{-\infty}^{\infty} P(E_t(u)) dt & \text{if } t \mapsto E_t(u) \text{ is } \mathbf{m}\text{-measurable,} \\ +\infty & \text{otherwise} \end{cases}$$

be lower semi-continuous with respect to weak convergence in $L^2(X, \mathbf{m})$. If this is the case, $(X, \mathcal{B}, \mathbf{m})$ equipped with P be qualified as a PMS, which we denote as (X, \mathbf{m}, P) .

Within this framework, we further prove that the total variation functional TV is convex in $L^2(X, \mathbf{m})$, and we adapt the ROF and CV problems, as given in (1.3) and (1.7), to this general setting. It then follows directly that both existence and uniqueness of the ROF minimizer are ensured. Thanks to the coarea-type formulation of the total variation, we are able to extend the results from [48] to the PMS context, thereby generalizing the (λ, μ) -linkage to these spaces.

It is worth emphasizing that a wide variety of significant spaces fall within the PMS framework, which demonstrates the broad applicability of the (λ, μ) -linkage across both mathematical and applied domains. Notable instances include the classical isotropic and anisotropic Euclidean cases, as well as settings in which the perimeter is fractional. Further examples of PMS include random walk spaces, which encompass discrete and probabilistic structures such as Markov chains and weighted graphs, and metric graphs, which enable the analysis of variational problems on network-like configurations. In particular, we provide two examples that illustrate the applicability of the (λ, μ) -linkage in the context of graphs, thereby concretely demonstrating the theoretical concepts.

A particularly noteworthy case of a PMS arises when $X = \Omega \subseteq \mathbb{R}^2$ is an open, bounded set, $\mathbf{m} = \mathcal{L}^2$, and $P = \text{Per}_\Phi$ denotes the anisotropic perimeter (see (2.7)) associated with the anisotropy $\Phi(\xi) := |\xi|_1$. In Chapter 4, we improve upon the isotropic (λ, μ) -linkage under structural assumptions on f , which, in this case, allows the construction of a minimizer of

the anisotropic CV problem via the minimizer of the anisotropic ROF problem, rather than merely a partial minimizer. This anisotropic setting is further studied in Chapter 4. More precisely, we establish the enhanced (λ, μ) -linkage, along with other results that are notable by themselves.

On the one hand, we complete the anisotropic study of the (λ, μ) -linkage by analyzing the generalization of Theorem 1.6, providing a general counterexample that invalidates the result for any anisotropy Φ . However, we introduce an alternative linkage between models that enables n -phase segmentation via a thresholding technique. This leads us to present a comparison between the anisotropic and isotropic settings of the linkages in both the biphasic and multiphasic cases, highlighting the applicability of our anisotropic framework.

On the other hand, we examine the equivalent jump property introduced in Section 1.2.4 for minimizers of CV models, in relation to the jump property presented in Theorem 1.1. In this context, we observe that for a minimizer (Λ, c_1, c_2) of \mathcal{F}^μ , the natural jump property would be the inclusion $\partial\Lambda \subseteq J_f$, which has been obtained in the one-dimensional case in [128]. In this direction, we show that this inclusion does not hold for either the isotropic \mathcal{F}^μ or its anisotropic counterpart with $\Phi(\xi) = |\xi|_1$. For this reason, we propose a weaker jump property, which we prove that it holds in the anisotropic case when $\Phi(\xi) = |\xi|_1$, in both the biphasic and the n -phasic cases.

1.5.2. Generalization of ROF model. In Chapter 5, we study the regularity of the minimizer of the ROF model in a manifold-constrained setting. Let (Σ, g) , (\mathcal{N}, h) be respectively a compact surface, possibly with smooth boundary $\partial\Sigma \neq \emptyset$, and a complete connected smooth n -dimensional Riemannian manifold. The latter will be treated as an isometrically embedded submanifold $\mathcal{N} \hookrightarrow \mathbb{R}^N$ in some Euclidean space, by Nash theorem [137]. Usually we write $\bar{\Sigma} = \Sigma \cup \partial\Sigma$ to stress that a claim is valid up to the boundary. Given $f \in L^2(\Sigma, \mathcal{N})$ and $\lambda > 0$, we study the regularity of minimizers of the energy functional

$$(1.11) \quad \mathcal{E}_{\mathcal{N}}^{\lambda, \sigma}(u) := \int_{\Sigma} |du| \, d\mathbf{m}_g + \frac{\lambda}{2} \int_{\Sigma} d_g^2(u, f) \, d\mathbf{m}_g + \frac{\sigma}{2} \int_{\Sigma} |du|^2 \, d\mathbf{m}_g,$$

where $\lambda > 0$, $\sigma \geq 0$, d_g denotes the geodesic distance on \mathcal{N} , and $d\mathbf{m}_g$ is the volume element corresponding to g . In this formulation, we note that (1.11) for $\sigma = 0$ defines the ROF functional in a manifold-constrained setting. Hereinafter, we denote $\mathcal{E}_{\mathcal{N}}^{\lambda, 0}$ as $\mathcal{E}_{\mathcal{N}}^{\lambda}$ to unify the notation.

In this sense, these results can be regarded as the manifold-constrained counterpart of [43, 143], with the additional technical challenge of considering non-flat surfaces Σ (abstract 2-manifolds) as domains. As mentioned in the Section 1.2.4, the inheritance of the regularity of f by a minimizer of $\mathcal{E}_{\mathcal{N}}^{\lambda}$ can be established under certain constraints. In our case, this result can be derived under natural curvature restrictions on \mathcal{N} and a topological condition on the range of f . More precisely, we consider the following bound of the convexity radius for $p \in \mathcal{N}$:

$$R_{\kappa} := \begin{cases} \frac{1}{2} \min \left\{ \text{inj}_p \mathcal{N}, \frac{\pi}{\sqrt{\kappa}} \right\} & \text{if } \kappa > 0 \\ \frac{\text{inj}_p \mathcal{N}}{2} & \text{if } \kappa \leq 0 \end{cases}$$

where κ is an upper bound for all sectional curvatures of the geodesic ball $B_h(p, R)$ for $R < R_{\kappa}$, and $\text{inj}_p(\mathcal{N})$ denotes the injectivity radius of the ball at its center p .

In this setting, using techniques from [104], we derive invariance principles for the minimizers u of (1.11), i.e. if $f(\bar{\Sigma}) \subseteq B_g(p, R)$ for some $R < R_{\kappa}$, then $u(\bar{\Sigma}) \subseteq B_g(p, R)$. This allows us to establish existence, via the direct method of the Calculus of Variations. Regarding uniqueness, the first challenge arises from the fact that the (geodesic) convexity of the total

variation does not hold for arbitrary $\kappa > 0$ (e.g., see [83]). Thus, by imposing the curvature constraint $\kappa \leq 0$ and the range restriction, we can prove uniqueness for $\sigma \geq 0$.

To study regularity, the strategy is to extend Theorems 1.2 and 1.3 to our generalized setting, specifically to establish the inheritance of Lipschitz regularity, under the aforementioned restrictions. Given the degeneracy of the Euler-Lagrange system associated with $\mathcal{E}_{\mathcal{N}}^{\lambda, \sigma}$, we consider the relaxed Neumann system:

$$(S_{\varepsilon, \sigma}^f) \quad \operatorname{div}_g \left(\frac{du}{\sqrt{|du|^2 + \varepsilon^2}} + \sigma du \right) = -\lambda \exp_u^{-1} f \quad \text{in } \Sigma, \quad du(\nu) = 0 \quad \text{on } \partial\Sigma,$$

where ν represents the outer unit normal in $\partial\Sigma$. To apply a Bernstein-type argument as in [143], it is necessary to differentiate $(S_{\varepsilon, \sigma}^f)$ and prove that $|du|^2$ is a subsolution of an elliptic PDE, enabling the use of Hopf's maximum principle to obtain pointwise estimates. This introduces another complication as, unlike the scalar case studied in [143], the solutions of the relaxed system are not known to be smooth by standard regularity theory. Furthermore, even if f were smooth, achieving regularity at least $C^3(\bar{\Sigma}, \mathcal{N})$ poses a challenge, as the theory of Neumann elliptic systems does not provide the necessary information; indeed, interior regularity results are tied to a coercivity condition of the form $\sigma > C_R R$ (cf. [86, 99]), while we are interested in letting $\sigma \rightarrow 0$ to recover the original ROF problem.

On one hand, we overcome the lack of smoothness in the right-hand term by constructing a mollification of f , denoted f_δ with $\delta > 0$ (see Section 2.4.4). On the other hand, the absence of a regularity theory leads us to adapt in detail [11], which allows us to obtain $C^\infty(\bar{\Sigma}, \mathcal{N})$ regularity for the system $(S_{\varepsilon, \sigma}^f)$. The adaptation of [11] is required due to the different structural conditions of our case, and feeds on auxiliary results, that we developed separately, providing an intrinsic Caccioppoli-type inequality and higher integrability of the solutions. These results are of independent interest for their application to a broad class of generalizations of harmonic maps. Moreover, note that considering Σ as a surface yields further complications due to the x -dependence of the elliptic operator which controls the problem. This fact makes the Hölder regularity estimates in this thesis technically much more challenging than its planar counterpart.

At this point, we emphasize that the regularity obtained concerns the relaxed system. To proceed, we need to establish a uniform bound for $|du|$, independent of ε , σ and δ . This fact is crucial for applying a convergence argument to achieve Lipschitz regularity for the minimizer of $\mathcal{E}_{\mathcal{N}}^{\lambda, 0}$ when $\kappa \leq 0$. More precisely, for the minimizer u of $(S_{\varepsilon, \sigma}^f)$, we derive the following uniform bound:

$$|du| \leq C(\operatorname{Lip}(f)^2 + 1),$$

under the previously stated curvature and range restrictions, without dependence on ε , δ and σ . Additionally, we find that for $\kappa > 0$ we achieve similar results in the case of $\sigma > 0$. A remarkable aspect of this study is that the strategy can only be performed successfully from an intrinsic viewpoint, given the critical role played by the geometry of the target space throughout the reasoning.

Complementarily, we adapt the previous results to the case where the domain is defined as $\Sigma = [0, 1]$ or $\Sigma = \mathbb{S}^1$, taking advantage of the simplifications afforded by the one-dimensional nature of the domain.

CHAPTER 2

Preliminaries

2.1. Notation and conventions

Concerning dimensions, we use capital letters N and M to denote the dimensions of the domain and codomain spaces. The symbols $|\cdot|$ and $\langle \cdot, \cdot \rangle$ refer to the Euclidean norm and the Euclidean inner product in \mathbb{R}^N , respectively; a subscript indicating the dimension of the space is added when necessary. In scalar-valued function spaces, the codomain is omitted in the notation to simplify expressions. Given a space S , we use S_{loc} to denote the space of functions that satisfy the defining properties locally, and $[S]^n$ to indicate the Cartesian product of n copies of S .

We also establish our notation for constants: indexed letters C , such as C_0, C_1 , etc., will retain a fixed meaning throughout the thesis, while the letter C denotes a generic constant that may vary from line to line, depending on the quantities specified in each statement. Moreover, if a constant depends on a specific variable (e.g., ε), this dependence will be indicated with an appropriate subscript, such as C_ε .

2.2. Basic definitions

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with boundary $\partial\Omega$. We denote by $\nu^\Omega(x)$ the exterior unit normal at $x \in \partial\Omega$, when it is well-defined. Moreover, we write $A \Subset \Omega$ to indicate that A is compactly contained in Ω . Now, we provide a concise list of measure and function spaces which will be used throughout the thesis.

2.2.1. Measure spaces. For measure spaces, we denote by $\mathcal{M}(\Omega)^N$ the space of finite N -dimensional Radon measures on Ω , by \mathcal{L}^N the N -dimensional Lebesgue measure, and by \mathcal{H}^{N-1} the $(N-1)$ -dimensional Hausdorff measure.

2.2.2. Function spaces. For function spaces, we denote by $L^p(\Omega)$, for $1 \leq p < +\infty$, the space of Lebesgue measurable functions $f : \Omega \rightarrow \mathbb{R}$ such that the p -th power of their absolute value is integrable over Ω in the Lebesgue sense. Similarly, $L^\infty(\Omega)$ denotes the space of essentially bounded measurable functions on Ω . The norms in these spaces are denoted by $\|\cdot\|_p$ and $\|\cdot\|_\infty$, respectively. In the vector-valued case, we write $f \in L^p(\Omega, \mathbb{R}^M)$ (respectively, $L^\infty(\Omega, \mathbb{R}^M)$) if each component $f^i \in L^p(\Omega)$ (respectively, $f^i \in L^\infty(\Omega)$) for every $1 \leq i \leq M$; the corresponding norms are extended in the natural way.

For $k, p \geq 1$, the Sobolev space $W^{k,p}(\Omega, \mathbb{R}^M)$ consists of functions in $L^p(\Omega, \mathbb{R}^M)$ whose partial derivatives up to order k have finite L^p norm. In particular, we denote by $H^1(\Omega, \mathbb{R}^M)$ the space $W^{1,2}(\Omega, \mathbb{R}^M)$. Now we introduce an inequality for Sobolev functions. Let $\Omega \subseteq \mathbb{R}^N$ be an open bounded set with Lipschitz boundary $\partial\Omega$ and $\Gamma \subseteq \partial\Omega$. For $1 \leq p < N$, $W_\Gamma^{1,p}(\Omega)$ is the subset of $W^{1,p}(\Omega)$ consisting of functions $u \in W^{1,p}(\Omega)$ with $Tu = 0$ on Γ , where T is the trace operator on $\partial\Omega$. In particular, when $\Gamma = \partial\Omega$, $W_\Gamma^{1,p}(\Omega)$ is denoted as $W_0^{1,p}(\Omega)$. We will use the following variant of the Poincaré inequality. We omit the proof since it is verbatim that of the standard $W_0^{1,p}(\Omega)$ case (e.g., [73, Theorem 3, p.279]).

LEMMA 2.1. *Let $u \in W_\Gamma^{1,p}(\Omega)$ for some $1 \leq p < N$. Then, the inequality*

$$\|u\|_{L^q(\Omega)} \leq C \|Du\|_{L^p(\Omega)}$$

holds for each $q \in [1, \frac{Np}{N-p}]$, where $C > 0$ is a constant depending only on p, q, N and Ω .

Additionally, we denote by $UC(\Omega, \mathbb{R}^M)$ the space of uniformly continuous functions, by $C^k(\Omega, \mathbb{R}^M)$ the space of k -times continuously differentiable functions with values in \mathbb{R}^M , and by $C_0^k(\Omega, \mathbb{R}^M)$ the subspace consisting of functions with compact support in Ω . In fact, we recall that $W^{1,p}(\Omega, \mathbb{R}^M)$ is the closure of $C_c^\infty(\Omega, \mathbb{R}^M)$ functions in the standard $W^{1,p}$ -norm. Moreover, $C^{k,\alpha}(\Omega, \mathbb{R}^M)$ denotes the space of Hölder continuous functions of class (k, α) , i.e., functions $u \in C^k(\Omega, \mathbb{R}^M)$ such that the k -th derivative $D_k u$ satisfies

$$|D_k u(x) - D_k u(y)| \leq C|x - y|^\alpha \quad \text{for all } x, y \in \Omega,$$

for some constant $C > 0$. In particular, $C^{0,1}(\Omega, \mathbb{R}^M)$ denotes the space of Lipschitz continuous functions. Finally, we denote by $\mathcal{L}^{(p,\alpha)}(\Omega, \mathbb{R}^M)$ the Campanato space, which consists of functions $u \in L^p(\Omega, \mathbb{R}^M)$ satisfying

$$\sup_{\substack{x \in \Omega, \\ r \leq \text{diam}(\Omega)}} \left(r^{-\alpha} \int_{\Omega \cap B_r(x)} |u - u_{x,r}|^p dx \right) < \infty, \quad \text{where } u_{x,r} := \int_{\Omega \cap B_r(x)} u dx.$$

2.3. Notions on functions of bounded variation

Given its role throughout the thesis, we introduce the BV space separately. Let $u \in L^1(\Omega, \mathbb{R}^M)$. We say that u is a *function of bounded variation* if its distributional partial derivatives are measures with finite total variation in Ω . The set of such functions is denoted by $BV(\Omega, \mathbb{R}^M)$. That is, $u \in BV(\Omega, \mathbb{R}^M)$ if and only if Du is an $M \times N$ matrix of measures $D_i u^\alpha \in \mathcal{M}(\Omega)$ such that

$$\int_{\Omega} u^\alpha \frac{\partial \phi}{\partial x_i} dx = - \int_{\Omega} \phi dD_i u^\alpha \quad \forall \phi \in C_0^\infty(\Omega), \quad 1 \leq i \leq N, \quad 1 \leq \alpha \leq M.$$

Accordingly, the derivative of u , denoted by Du , is a vector-valued measure with finite total variation, defined as

$$(2.1) \quad |Du|(\Omega) := \sup \left\{ \sum_{i=1}^M \int_{\Omega} u^i \operatorname{div}(z^i) dx : z \in [C_0^\infty(\Omega, \mathbb{R}^N)]^M, \|z\|_\infty \leq 1 \text{ a.e. in } \Omega \right\}.$$

Note that this total variation is often referred to as *isotropic*, due to its relation with the Euclidean norm (see Section 2.3.2). Observe that $BV(\Omega, \mathbb{R}^M)$ can be characterized as

$$BV(\Omega, \mathbb{R}^M) := \{u \in L^1(\Omega, \mathbb{R}^M) : |Du|(\Omega) < +\infty\}.$$

Additionally, we may define the local space $BV_{loc}(\Omega)$ as

$$BV_{loc}(\Omega, \mathbb{R}^M) := \{u \in L^1(\Omega, \mathbb{R}^M) : u \in BV(U, \mathbb{R}^M) \quad \forall U \Subset \Omega\}.$$

It is important to highlight that this notion of *bounded variation* allows for the definition of *sets of finite perimeter*. We say that a measurable set E has finite perimeter with respect to Ω if

$$(2.2) \quad \operatorname{Per}(E; \Omega) := |D\chi_E|(\Omega) < +\infty.$$

Moreover, if $u \in BV(\Omega)$, its upper level set $E_t(u) := \{x \in \Omega : u(x) > t\}$ has finite perimeter for \mathcal{L}^1 -almost every $t \in \mathbb{R}$, and satisfies the coarea formula (see [6, Theorem 3.40]), which reads

$$(2.3) \quad |Du|(\Omega) = \int_{\mathbb{R}} \text{Per}(E_t(u); \Omega) dt.$$

Now assume that E is a set of locally finite perimeter, that is, $\chi_E \in BV_{loc}(\Omega)$. We denote by ∂^*E the *reduced boundary* of E , defined as the set of all points $x \in \mathbb{R}^N$ such that $|D\chi_E|(B_r(x)) > 0$ for every $r > 0$, $|\nu^E(x)| = 1$, and

$$\lim_{r \rightarrow 0} \int_{B_r(x)} \nu^E d|D\chi_E| = \nu^E(x).$$

One of the key properties of a function $u \in BV(\Omega)$ is the possibility of decomposing Du . This aspect is particularly relevant, for instance, when employing the space $BV(\Omega)$ in modeling problems, as it enables the study of geometric properties associated with specific parts of Du , as we have shown in Section 1.2.4.

Let $u \in BV(\Omega, \mathbb{R}^M)$. The *approximate limit* at $x \in \mathbb{R}^N$, denoted by $\tilde{u}(x) \in \mathbb{R}^M$, is defined as the value satisfying

$$\lim_{r \rightarrow 0} \int_{B_r(x)} |u(y) - \tilde{u}(x)| dy = 0.$$

Using this concept, we define the set S_u as the set of points of approximate discontinuity, consisting of all points in Ω where the above equality fails. Since $Du \in \mathcal{M}(\Omega)^{N \cdot M}$, by the Calderón–Zygmund theorem [6, Theorem 3.83] and the Lebesgue decomposition theorem [71, Theorem 1.31], we can express Du as

$$Du = \nabla u \mathcal{L}^N + D^s u,$$

where ∇u denotes the Radon–Nikodym derivative of Du with respect to \mathcal{L}^N , and $D^s u$ is the singular part of Du . In particular, note that

$$Du = \nabla u \mathcal{L}^N$$

in $\Omega \setminus S_u$. To further decompose $D^s u$, we introduce the concept of *approximate jump points*, which form a subset of S_u . A point $x \in \Omega$ is said to be an approximate jump point of u if there exist $u^+(x), u^-(x) \in \mathbb{R}^M$ and a vector $\nu^u(x) \in \mathbb{S}^{N-1}$ such that $u^+(x) \neq u^-(x)$ and

$$\lim_{r \rightarrow 0^+} \int_{B_r^\pm(x, \nu^u(x))} |u(y) - u^\pm(x)| dy = 0,$$

where $B_r^\pm(x, \nu^u(x))$ is defined as

$$B_r^\pm(x, \nu^u(x)) := \{y \in B_r(x) : \langle y - x, \nu^u(x) \rangle \gtrless 0\}.$$

The tuple $(u^+(x), u^-(x), \nu^u(x))$ is determined up to a permutation of $(u^+(x), u^-(x))$ and a sign change of $\nu^u(x)$. We denote by J_u the set of all approximate jump points of Ω . The vector ν^u can be identified as the Radon–Nikodym derivative of Du with respect to $|Du|$. Consequently, $D^s u$ can be decomposed into a *jump part* and a *Cantor part*, yielding

$$D^s u = D^j u + D^c u = (u^+ - u^-) \nu^u \mathcal{H}^{N-1}|_{J_u} + D^s u|_{S_u \setminus J_u}.$$

In this context, it is worth mentioning that this decomposition allows the definition of special functions of bounded variation, denoted by $SBV(\Omega, \mathbb{R}^M)$, a subset of $BV(\Omega, \mathbb{R}^M)$ consisting of functions u such that $S_u = J_u$ \mathcal{H}^{N-1} -a.e.

In the remainder of this section, we focus on the scalar case, that is, when $M = 1$, in order to introduce the following concepts.

2.3.1. Traces of vector fields with L^2 -divergence. Let Ω be a bounded Lipschitz domain. We define the space X_Ω as

$$X_\Omega = \{z \in L^\infty(\Omega, \mathbb{R}^N) : \operatorname{div}(z) \in L^2(\Omega)\}.$$

In [9, Theorem 1.2], the weak trace on $\partial\Omega$ of the normal component of $z \in X_\Omega$ is defined as the linear operator $[\cdot, \nu^\Omega] : X_\Omega \rightarrow L^\infty(\partial\Omega)$ such that

$$\begin{aligned} \| [z, \nu^\Omega] \|_{L^\infty(\partial\Omega)} &\leq \| z \|_{L^\infty(\Omega)} \quad \text{for all } z \in X_\Omega, \\ [z, \nu^\Omega](x) &= \langle z(x), \nu^\Omega \rangle \quad \text{for all } x \in \partial\Omega \text{ if } z \in C^1(\bar{\Omega}; \mathbb{R}^N). \end{aligned}$$

If $z \in X_\Omega$ and $u \in BV(\Omega)$, we define the operator $(z, Du) : C_0^\infty(\Omega) \rightarrow \mathbb{R}$ as

$$(2.4) \quad (z, Du)(\psi) := - \int_\Omega u \psi \operatorname{div}(z) \, dx - \int_\Omega u \langle z, \nabla \psi \rangle \, dx,$$

which is known as the Anzellotti pairing [9]. It can be proven that $(z, Du) \in \mathcal{M}(\Omega)$ and the following Green-type formula establishes a relation between $[z, \nu^\Omega]$ and (z, Du) :

$$(2.5) \quad \int_\Omega u \operatorname{div}(z) \, dx + \int_\Omega (z, Du) = \int_{\partial\Omega} [z, \nu^\Omega] u \, d\mathcal{H}^{N-1}.$$

2.3.2. Anisotropic total variation. We begin with the next definition:

DEFINITION 2.2. Let be $\Phi : \mathbb{R}^N \rightarrow [0, +\infty)$. We say that Φ is an anisotropy if the following conditions are satisfied:

- Φ is convex and lower semicontinuous;
- Φ is positively 1-homogeneous, i.e.

$$\Phi(t\xi) = |t|\Phi(\xi)$$

for all $\xi \in \mathbb{R}^N$ and for all $t \in \mathbb{R}$;

- Φ is coercive with sublinear growth, i.e., there exists $C \geq 1$ such that

$$\frac{1}{C} \|\xi\| \leq \Phi(\xi) \leq C \|\xi\|$$

for all $\xi \in \mathbb{R}^N$.

Along with the notion of Φ , we define the dual of Φ as

$$\Phi^0(\xi) := \sup \{ \langle \xi, y \rangle : y \in \mathbb{R}^N, \Phi(y) \leq 1 \}.$$

Thus, if $u \in L^1(\Omega)$, we define the anisotropic total variation of Du as:

$$(2.6) \quad |Du|_\Phi(\Omega) := \sup \left\{ \int_\Omega u \operatorname{div}(z) \, dx : z \in \mathcal{H}_\Phi(\Omega) \right\},$$

where $\mathcal{H}_\Phi(\Omega)$ is defined as

$$\mathcal{H}_\Phi(\Omega) := \{ z \in X_\Omega : \Phi^0(z(x)) \leq 1 \text{ a.e. } x \in \Omega \}.$$

As a direct consequence of the definition, we have that the anisotropic total variation of u is L^1 -lower semicontinuous in Ω . In particular, if $\Phi(\xi) := |\xi|$, we obtain that $|Du|_\Phi = |Du|$, which leads us to denote (2.1) as the *isotropic* total variation of u . Analogously to the isotropic case, we define the Φ -perimeter as

$$(2.7) \quad \text{Per}_\Phi(E; \Omega) := |D\chi_E|_\Phi(\Omega).$$

Moreover, as stated in [4, Remark 4.4], the coarea formula holds for $u \in BV(\Omega)$, namely

$$|Du|_\Phi(\Omega) = \int_{\mathbb{R}} \text{Per}_\Phi(\{x \in \Omega : u(x) > t\}; \Omega) dt.$$

2.3.3. Subdifferential of the total variation. Finally, we define the notion of the subdifferential of a convex functional $F : L^2(\Omega) \rightarrow (-\infty, \infty]$ as $\partial F : L^2(\Omega) \rightarrow 2^{L^2(\Omega)}$:

$$\partial F(u) := \left\{ w \in L^2(\Omega) : F(v) - F(u) \geq \int_{\Omega} w(v - u) dx \text{ for all } v \in L^2(\Omega) \right\}.$$

Now we consider the anisotropic total variation functional $\text{TV}_\Phi : L^2(\Omega) \rightarrow [0, \infty]$ as

$$\text{TV}_\Phi(u) := \begin{cases} |Du|_\Phi(\Omega) & \text{if } u \in L^2(\Omega) \cap BV(\Omega), \\ +\infty & \text{if } u \in L^2(\Omega) \setminus BV(\Omega). \end{cases}$$

The subdifferential of the anisotropic total variation, denoted as $\partial \text{TV}_\Phi(u)$, has the following characterization provided in [127, Theorem 12]:

THEOREM 2.3. *Let Ω be a bounded set with Lipschitz boundary and $u \in L^2(\Omega) \cap BV(\Omega)$. Then $v \in \partial \text{TV}_\Phi(u)$ if and only if $v \in L^2(\Omega)$ and there exists $z \in \mathcal{H}_\Phi(\Omega)$ such that $v = -\text{div}(z)$, $[z, \nu^\Omega] = 0$ and*

$$|Du|_\Phi(\Omega) = \int_{\Omega} u \text{div}(z) dx = - \int_{\Omega} (z, Du),$$

where the second equality is deduced by (2.5).

In particular, we can define the anisotropic variant of the ROF functional (1.3) as

$$(2.8) \quad \mathcal{E}_\Phi^\lambda(u) := \text{TV}_\Phi(u) + \frac{\lambda}{2} \int_{\Omega} (u - f)^2 dx,$$

where \mathcal{E}^λ is recovered if $\Phi(\xi) = |\xi|_2$. If $f \in L^2(\Omega)$, we note that \mathcal{E}_Φ^λ has a unique minimizer u such that there exists $z \in \partial \text{TV}_\Phi(u)$ which satisfies the following Euler-Lagrange equation

$$(2.9) \quad \text{div}(z) = \lambda(u - f).$$

Specially, if $\Phi(\xi) = |\xi|_1$ and $f \in \text{PCR}(\Omega)$ (see Section 2.3.4), we know that in [110, Theorem 5] is provided a constructive method to find the minimizer u of \mathcal{E}_Φ^λ .

2.3.4. Piecewise constant on rectangles functions. Finally, we introduce a family of functions which allow us to characterize the subdifferential of TV_Φ in the case of $\Phi(\xi) = |\xi|_1$:

DEFINITION 2.4. *Let Ω be a rectangle. We say that a function $0 \leq w \leq 1$ is piecewise constant on rectangles and we write $w \in \text{PCR}(\Omega)$ if w has a finite number of level sets of positive \mathcal{L}^2 measure, and each one is a rectilinear polygon up to an \mathcal{L}^2 -null set.*

In order to work with the results introduced in [110], we present some notation:

DEFINITION 2.5. Let Ω be a bounded rectangle in \mathbb{R}^2 . We denote by G any finite set of horizontal and vertical lines on Ω , which we call a grid on Ω , and any of the rectangles in the partition generated by the grid we call it cell. Let F be a rectangular polygon contained in Ω and let $f \in \text{PCR}(\Omega)$. We denote

- by $R(G)$ the set of all rectangular polygons which are a union of cells of G ,
- by $G(F)$ the minimal grid such that each side of F is contained in one line of G , by \mathcal{Q}_f the partition of Ω provided by the level-sets of f and by G_f the grid $\cup_{\Sigma \in \mathcal{Q}_f} G(\Sigma)$.

In particular, we observe that $\text{PCR}(\Omega) \subseteq \text{BV}(\Omega)$. Moreover, if $u \in \text{PCR}(\Omega)$, we have

$$J_u = \bigcup_{F \in \mathcal{Q}_u} \partial F.$$

In Figure 2.1 we present a sketch to illustrate the $\text{PCR}(\Omega)$ functions.

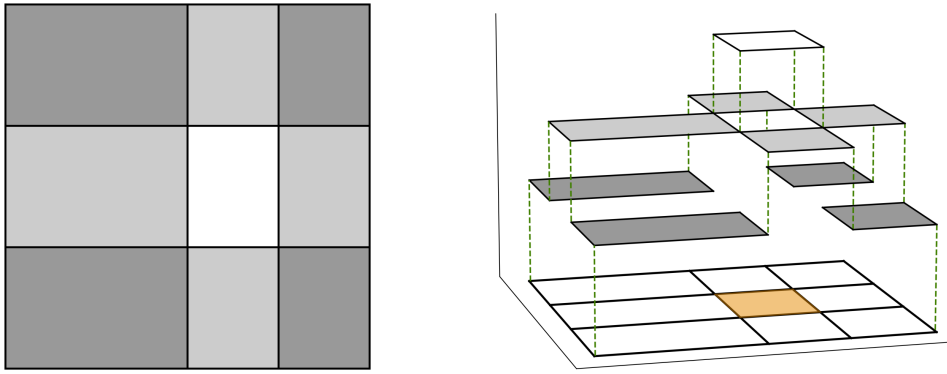


FIGURE 2.1. The image on the left can be represented by a function f , whose graph is shown in the figure on the right, together with the grid G_f below. Note that $G_f = G(F)$ where F is the orange rectangle.

For $\Phi(\xi) = |\xi|_1$, the subdifferential of TV_Φ on PCR functions was described in [110]. Combined with Theorem 2.3, we obtain the next result:

LEMMA 2.6. Let $f \in \text{PCR}(\Omega)$ and $\Phi(\xi) = |\xi|_1$. Then $v \in \partial \text{TV}_\Phi$ if and only if $v \in L^2(\Omega)$ and there exists $z \in \mathcal{H}_\Phi(\Omega)$ such that $v = -\text{div } z$, $[z, \nu^\Omega] = 0$ and

$$[z, \nu^F](x) = \begin{cases} -1 & \text{if } f|_F < f|_{F'} \text{ and } x \in \partial F \cap \partial F', \\ 1 & \text{if } f|_F > f|_{F'} \text{ and } x \in \partial F \cap \partial F', \end{cases} \quad \mathcal{H}^1\text{-a.e. on } \partial F,$$

with $F \neq F' \in \mathcal{Q}_f$.

Now we recall the anisotropic ROF functional (2.8) with $\Phi(\xi) = |\xi|_1$. Moreover, it is shown that $u|_{F_k} > u|_{F_{k+1}}$ and

$$\text{Per}(\{x \in \Omega : u(x) > \tau\}; \Omega) + \lambda \int_{\{u > \tau\}} (u - f) dx = 0, \quad \text{for any } 0 < \tau < 1.$$

By (2.9) and Lemma 2.6, it follows that there exists $z_u \in \partial \text{TV}_\Phi(u)$ such that

$$\begin{cases} \text{div}(z_u) = \lambda(u - f), \\ [z_u, \nu^{\{u > \tau\}}] = -1 \end{cases} \quad \text{for any } 0 < \tau < 1.$$

Finally, observe that, thanks to Lemma 2.6, we have the next inclusion

$$(2.10) \quad \partial \text{TV}_\Phi(u) \subseteq \partial \text{TV}_\Phi(\chi_{\{u>\tau\}}), \quad \text{for any } 0 < \tau < 1.$$

2.4. Notions on Riemannian manifolds

Now, we introduce some concepts about p -energies, Sobolev functions and mollifiers on Riemannian manifolds. Furthermore, we would like to highlight that the generalization of total variation to the manifolds context is provided in Section 5.2, where some of its properties are studied in detail. Let (\mathcal{N}, h) be a complete, connected, smooth Riemannian manifold of dimension n , isometrically embedded in some Euclidean space \mathbb{R}^N according to Nash's embedding theorem [137]. Let (\mathcal{M}, g) be a compact Riemannian manifold of dimension m , with smooth, non-empty boundary, that is, $\partial\mathcal{M} \neq \emptyset$.¹

2.4.1. p -energies. Recall that for each point $x \in \mathcal{M}$ the differential of u at x is a linear map from $T_x\mathcal{M}$ to $T_{u(x)}\mathcal{N}$, namely, it holds $du(x) \in \text{Hom}(T_x\mathcal{M}, T_{u(x)}\mathcal{N})$, and the latter space is in turn isomorphic to the tensor product $T_x^*\mathcal{M} \otimes T_{u(x)}\mathcal{N}$. We take (x^1, \dots, x^m) and (y^1, \dots, y^n) local coordinates in \mathcal{M} and \mathcal{N} , respectively. Then du as a section of the bundle $\mathcal{B} := T^*\mathcal{M} \otimes u^*T\mathcal{N}$ can be locally expressed as

$$du = \partial_i u^\alpha dx^i \otimes \frac{\partial}{\partial y^\alpha} \Big|_u, \quad \text{with } \partial_i := \frac{\partial}{\partial x^i}.$$

For convenience and brevity, we will combine du with the notation u_* as both are quite usual in the literature. From an intrinsic viewpoint, given $p \geq 1$, the p -Dirichlet energy (or total variation functional for $p = 1$) is given by

$$(2.11) \quad \mathcal{E}_p(u) = \frac{1}{p} \int_{\mathcal{M}} |du|^p d\mathbf{m}_g,$$

where $d\mathbf{m}_g$ denotes the Riemannian volume element and the energy density is written as

$$|du|^2 = \langle du, du \rangle_{g \boxtimes h} := \text{tr}_g(u^*h) = g^{ij}(h_{\alpha\beta} \circ u) \partial_i u^\alpha \partial_j u^\beta,$$

which follows from the definition of the pull-back metric $(u^*h)_{ij} := h(\partial_i u, \partial_j u)$.

Given ν the unit outward normal to $\partial\mathcal{M}$, we interpret $\nu \cdot du$ as the inner product with respect to the induced metric on the boundary. Moreover, notice that $\nu \cdot du$ is actually an abuse of notation for $du(\nu) = u_*\nu \in T_u\mathcal{N}$; for convenience, sometimes we will also use the notations $\nabla_\nu u$ or $\frac{\partial u}{\partial \nu}$.

On the other hand, let ∇ be the Levi-Civita connection of (\mathcal{M}, g) and ${}^h\nabla$ the one of (\mathcal{N}, h) . Set $\tilde{\nabla}$ the induced connection on $u^*T\mathcal{N}$, which is defined as follows: for $X \in T\mathcal{M}$, $V \in u^*T\mathcal{N}$,

$$(2.12) \quad \tilde{\nabla}_X V := {}^h\nabla_{u_*X} V \in C^\infty(u^*T\mathcal{N}).$$

In turn, if ∇^* denotes the connection on $T^*\mathcal{M}$ dual to ∇ , we will work with $\nabla^{g \boxtimes h} := \nabla^* \otimes \tilde{\nabla}$, which is the unique linear connection of \mathcal{B} so that

$$\nabla^{g \boxtimes h}(A \otimes B) = (\nabla^* A) \otimes B + A \otimes \tilde{\nabla}$$

for $A \in C^\infty(T^*\mathcal{M})$ and $B \in C^\infty(u^*T\mathcal{N})$. Additionally, such a connection fulfils the compatibility condition

¹When we work on Riemannian manifolds, we swap the letter M and N to denote dimensions by clarity.

$$v \langle X, Y \rangle_{g \boxtimes h} = \langle \nabla_v^{g \boxtimes h} X, Y \rangle_{g \boxtimes h} + \langle X, \nabla_v^{g \boxtimes h} Y \rangle_{g \boxtimes h}$$

for all sections $X, Y \in C^\infty(\mathcal{B})$ and $v \in T\mathcal{M}$.

The critical points of \mathcal{E}_p are known as p -harmonic maps, and have vanishing p -tension or rough p -Laplacian, that is, they satisfy $\tau_p(u) = 0$, where $\tau_p(u) := \text{tr}_g(\nabla^{g \boxtimes h} du) = \tau_p(u)^\gamma \frac{\partial}{\partial y^\gamma} \Big|_u \in C^\infty(u^*T\mathcal{N})$, which in local coordinates reads as

$$\begin{aligned} \tau_p(u)^\gamma &= \text{div}_g(|du|^{p-2} d(u^\gamma)) + |du|^{p-2} g^{ij} {}^h \Gamma_{\alpha\beta}^\gamma(u) \partial_i u^\alpha \partial_j u^\beta \\ &= \frac{1}{\sqrt{\det g}} \partial_i \left(|du|^{p-2} \sqrt{\det g} g^{ij} \partial_j u^\gamma \right) + |du|^{p-2} g^{ij} {}^h \Gamma_{\alpha\beta}^\gamma(u) \partial_i u^\alpha \partial_j u^\beta \end{aligned}$$

for $\gamma = 1, \dots, n$, where ${}^h \Gamma_{\alpha\beta}^\gamma(u)$ denote the Christoffel symbols for the Levi-Civita connection of the metric h . In particular, we will deal with the cases $p = 1, 2$, that is,

$$\begin{aligned} \tau(u)^\gamma &= \tau_2(u)^\gamma = \Delta_g(u^\gamma) + g^{ij} {}^h \Gamma_{\alpha\beta}^\gamma(u) \partial_i u^\alpha \partial_j u^\beta \\ &= g^{ij} \left(\partial_{ij}^2 u^\gamma - g \Gamma_{ij}^k \partial_k u^\gamma + {}^h \Gamma_{\alpha\beta}^\gamma(u) \partial_i u^\alpha \partial_j u^\beta \right) \end{aligned}$$

and, with a shorter notation,

$$(2.13) \quad \tau_1(u) = \text{div}_g \left(\frac{du}{|du|} \right) + \frac{1}{|du|} \Gamma_u(du, du).$$

Alternatively, from an extrinsic perspective, Nash theorem [137] guarantees that there exists an isometric embedding $\iota : \mathcal{N} \hookrightarrow \mathbb{R}^N$ and, by completeness of \mathcal{N} , we even know by [135] that $\iota(\mathcal{N})$ is a closed subset of \mathbb{R}^N (in other words, that the embedding is proper). Then we can work with the extended map

$$\check{u} := \iota \circ u : \mathcal{M} \longrightarrow \mathbb{R}^N.$$

In this setting, with $\nabla \check{u} = (\nabla \check{u}^1, \dots, \nabla \check{u}^N) \in \mathbb{R}^{mN}$, it holds

$$|\nabla \check{u}|^2 = \sum_{\alpha=1}^N g^{ij} \partial_i \check{u}^\alpha \partial_j \check{u}^\alpha \quad \text{and} \quad \tau_p(\check{u}) = \left(\text{div}_g(|\nabla \check{u}|^{p-2} \nabla \check{u}) \right)^\top,$$

where $\mathbb{R}^N \ni X \mapsto X^\top \in T_u \mathcal{N}$ denotes the orthogonal projection onto the tangent space. As the normal component can be represented by means of the second fundamental form \mathcal{A}_u of \mathcal{N} , we get the following expression:

$$\tau_p(\check{u})^\gamma = \text{div}_g(|\nabla \check{u}|^{p-2} \nabla \check{u}^\gamma) + |\nabla \check{u}|^{p-2} g^{ij} \mathcal{A}_u^\gamma(\partial_i \check{u}, \partial_j \check{u}), \quad 1 \leq \gamma \leq N.$$

being $\mathcal{A}_u(X, Y) \in (T_u \mathcal{N})^\perp$ the second fundamental form of \mathcal{N} given by $\mathcal{A}(X, Y) = -(D_X Y)^\perp$, where D is the standard directional derivative in \mathbb{R}^N and X, Y are extended arbitrarily as tangent vector fields to \mathcal{N} in a neighborhood of $u \in \mathcal{N}$. Notice that, with a shorter notation for $p = 1$, we have

$$\tau_1(\check{u}) = \text{div}_g \left(\frac{\nabla \check{u}}{|\nabla \check{u}|} \right) + \frac{1}{|\nabla \check{u}|} \mathcal{A}_u(\nabla \check{u}, \nabla \check{u}),$$

which formally resembles its intrinsic counterpart (2.13). Hereafter, for simplicity, we will always write u instead of \check{u} , unless the meaning is unclear from the context.

2.4.2. p -energy on surfaces. In the case that $\mathcal{M} = \Sigma$, we can further use isothermal coordinates on the surface. Indeed, let $\{(U_\ell, \phi_\ell)\}$ be an atlas of Σ such that the metric g within U_ℓ can be written as $g_{ij} = \varrho_\ell^2 \delta_{ij}$ with $\varrho_\ell \in C^1(U_\ell, \mathbb{R}^+)$. In these coordinates, set $\Omega_\ell := \phi_\ell(U_\ell)$, and thus the p -energy functional (2.11) can be written as

$$(2.14) \quad \mathcal{E}_p(u) = \frac{1}{p} \sum_\ell \int_{\Omega_\ell} (\chi_\ell \cdot \varrho_\ell^{2-p} |\nabla u|^p) \circ \phi_\ell^{-1}(x) \, dx =: \frac{1}{p} \int_\Omega \varrho^{2-p} |\nabla u|^p \, dx,$$

with $\{\chi_\ell\}_\ell$ a partition of unity subordinate to U_ℓ . Notice that, for simplicity, we just write ϱ_ℓ instead of $\chi_\ell \cdot \varrho_\ell$ and, as usual, we omit composition with ϕ_ℓ^{-1} , unless the meaning is unclear from the context.

Thus, observe that the existence of isothermal coordinates up to the boundary (e.g. [164, Lemma 4]) permits to take a finite covering $\{U_\ell\}_{\ell \in I}$ and to suppose that there exists a constant $C_p > 0$ such that

$$(2.15) \quad C_p \leq \varrho_i^{2-p}|_{\Omega_\ell} \leq \frac{1}{C_p} \quad \text{for all } \ell \in I \text{ and } p \geq 1.$$

2.4.3. Sobolev functions. Concerning \mathcal{N} -valued functions, there are two different notions of Sobolev spaces: an extrinsic version and an intrinsic one. For the first, identifying again a function u with its extension $\iota \circ u$,

$$(2.16) \quad W^{1,p}(\mathcal{M}, \mathcal{N}) := \{u \in W^{1,p}(\mathcal{M}, \mathbb{R}^N) : u(x) \in \mathcal{N} \text{ for a.e. } x \in \mathcal{M}\}.$$

It turns out ([60, Proposition 2.7]) that this definition does not depend on the choice of the embedding ι , contrary to the case of higher order Sobolev spaces (see [61]).

On the other hand, in the event that the N -dimensional manifold \mathcal{N} is covered by a normal coordinate chart centered at some point $p \in \mathcal{N}$, \mathcal{N} can be identified with the Euclidean space \mathbb{R}^N endowed with a metric that can be expressed globally in normal Cartesian coordinates. Identifying thus $u : \mathcal{M} \rightarrow \mathcal{N}$ with its vector valued representation, one can define the intrinsic Sobolev space of first order as

$$(2.17) \quad W_{\text{in}}^{1,p}(\mathcal{M}, \mathcal{N}) := W^{1,p}(\mathcal{M}, \mathbb{R}^N).$$

In this case, we recall that $WP^{1,p}(\mathcal{M}, \mathbb{R}^M)$ is the closure of $C_0^\infty(\mathcal{M}, \mathbb{R}^N)$ functions in the standard $W^{1,p}$ -norm.

2.4.4. Barycenter and mollifiers. Finally, we recall that $\mathfrak{b} \in \overline{B_h(p, R)}$ is a barycenter of a Radon measure \mathfrak{m} on $\overline{B_h(p, R)}$ if \mathfrak{b} is a minimizer of the function

$$\overline{B_h(p, R)} \ni q \longmapsto \frac{1}{2} \int_{\mathcal{N}} d_h^2(\cdot, q) \, d\mathfrak{m}.$$

In [1, Theorem 2.1], it is proved that if $R < R_\kappa$ where R_κ is the next bound of the convexity radius:

$$R_\kappa := \begin{cases} \frac{1}{2} \min \left\{ \text{inj}_p \mathcal{N}, \frac{\pi}{\sqrt{\kappa}} \right\} & \text{if } \kappa > 0, \\ \frac{\text{inj}_p \mathcal{N}}{2} & \text{if } \kappa \leq 0, \end{cases}$$

then a unique barycenter exists. Notice that this is true for non-positively simply connected curved manifolds without any restriction on R .

Using the notion of barycenter, we now recall a mollification technique given in [106] for Lipschitz continuous functions, $C^{0,1}(\mathcal{M}, \mathcal{N})$: Firstly, for a given $0 < \delta < R_\kappa$, we set $\mathfrak{r}_h = d_h(p, \cdot)$ and consider the mollifier $\varphi_\delta : \mathcal{N} \times \mathcal{N} \rightarrow \mathbb{R}$ defined as

$$\varphi_\delta(p, q) = \psi\left(\frac{\mathfrak{r}_h(q)}{\delta}\right) \left[\int_{B(0, \delta)} \psi\left(\frac{\mathfrak{r}_h \circ \exp_p(\cdot)}{\delta}\right) d\mathbf{m}_{\exp_p^* h} \right]^{-1},$$

where $B(0, \delta) = \exp_p(B_h(p, \delta))$ is regarded as a ball in \mathbb{R}^n while $\psi \in C^\infty(\mathbb{R})$ is such that $\psi((-\infty, 1/4]) = 1$, $\psi([3/4, \infty)) = 0$ and $\psi' \leq 0$. Then, if we further assume that $f(\mathcal{M}) \subseteq B_h(p, R)$ for some $p \in \mathcal{N}$ and $R < R_\kappa$, we can define $f_\delta : \mathcal{M} \rightarrow \mathcal{N}$ as

$$(2.18) \quad f_\delta(x) := \mathfrak{b}(f(x), \delta) \quad \text{for all } x \in \mathcal{M},$$

where $\mathfrak{b}(f(x), \delta)$ denotes the barycenter of the Radon measure

$$\mathbf{m} = \varphi_\delta(f(x), \cdot) d\mathbf{m}_{\exp_{f(x)}^* h}.$$

By [106, Theorem 4.3], we have that $\{f_\delta\}_\delta \subseteq C^\infty(\mathcal{M}, \mathcal{N})$ for δ small enough. In addition, as $f \in C^{0,1}(\mathcal{M}, \mathcal{N})$ with Lipschitz constant L_f , [106, Theorem 4.4, 4.6] implies that $f_\delta(p) \rightarrow f(p)$ as $\delta \rightarrow 0$ and f_δ are Lipschitz functions with Lipschitz constant CL_f .

Generalized (λ, μ) –linkage

In this chapter, we present a general framework in which the (λ, μ) –linkage holds. More precisely, from the definition of the isotropic (λ, μ) –linkage in [34], we identify the key mathematical tools required to prove this relationship: a suitable selection of the total variation functional and the perimeter functional, which are linked by an expression equivalent to the coarea formula (2.3). In particular, we define these concepts on measure spaces $(X, \mathcal{B}, \mathbf{m})$, introducing the notion of *perimeter measure spaces*. Notably, this approach allows the extension of the (λ, μ) –linkage to the non-Euclidean setting.

In Section 3.1, we formally introduce the framework of *perimeter measure spaces*, denoted by PMS. It is important to emphasize that this setting is broad, encompassing a wide range of scenarios where our extension of the (λ, μ) –linkage is applicable. To illustrate the significance of this framework, we provide a list of examples in Section 3.2, demonstrating the impact of this generalization within the context of PMS.

In Section 3.3, we establish a series of preliminary results leading to Theorem 3.8, which proves the (λ, μ) –linkage for perimeter measure spaces. It is worth noting that the CV model has been less extensively studied in non-Euclidean settings compared to the ROF model, as discussed in Section 3.2 and Chapter 1. Consequently, Theorem 3.8 can be used to define partial minimizers of the CV problem. More specifically, in Section 3.4, we apply the (λ, μ) –linkage, adapting Algorithm 1.1 to approximate a minimizer of the CV problem in the case of locally finite weighted discrete graphs. We provide applications in image segmentation and labeling tasks, highlighting the improvements that the (λ, μ) –linkage framework offers in each case.

3.1. Perimeter Measure Spaces

In this section, we introduce the definition of *perimeter measure spaces*, the framework in which the generalization of the (λ, μ) –linkage is established. First, we define the notions of *perimeter* and *total variation* in measure spaces:

DEFINITION 3.1. *Let $(X, \mathcal{B}, \mathbf{m})$ be a measure space. We say that $P : \mathcal{B} \rightarrow [0, \infty]$ is a perimeter functional if the following conditions are satisfied:*

- (i) $P(\emptyset) = 0$;
- (ii) $P(A) = P(B)$ for every $A, B \in \mathcal{B}$ such that $A \triangle B := (A \setminus B) \cup (B \setminus A)$ satisfies that $\mathbf{m}(A \triangle B) = 0$;
- (iii) $P(A) = P(X \setminus A)$ for every $A \in \mathcal{B}$;
- (iv) P is sub-modular, i.e.,

$$P(A \cup B) + P(A \cap B) \leq P(A) + P(B)$$

for every $A, B \in \mathcal{B}$.

DEFINITION 3.2. *Let $(X, \mathcal{B}, \mathbf{m})$ be a measure space and let $P : \mathcal{B} \rightarrow [0, \infty]$ be a perimeter functional, which assigns a non-negative value to each measurable set in \mathcal{B} . For a function $u \in L^1(X, \mathbf{m})$, the total variation of u with respect to P is defined as*

This chapter contains the results of [130].

$$\mathrm{TV}_P(u) := \begin{cases} \int_{-\infty}^{\infty} P(E_t(u)) dt & \text{if } t \mapsto E_t(u) \text{ is } \mathfrak{m}\text{-measurable,} \\ +\infty & \text{otherwise,} \end{cases}$$

where $E_t(u) := \{x \in X : u(x) > t\}$. In particular, for the characteristic function χ_E of a set $E \in \mathcal{B}$, we have that

$$\mathrm{TV}_P(\chi_E) = P(E).$$

We observe that the definition of TV_P naturally extends the *coarea formula* for $BV(\Omega)$ functions (2.3). Additionally, we highlight a notable property of TV_P : its convexity, which we formalize in the following proposition.

PROPOSITION 3.3. *Let $(X, \mathcal{B}, \mathfrak{m})$ be a measure space. Then TV_P is convex in $L^2(X, \mathfrak{m})$.*

PROOF. The proof of this result essentially follows the lines of [49, Proposition 3.4]. We give the details for the sake of completeness. Before we begin, we observe that $L^2(X, \mathfrak{m}) \subseteq L^1(X, \mathfrak{m})$ since $\mathfrak{m}(X) < +\infty$, so the total variation is well defined for $L^2(X, \mathfrak{m})$ functions.

Firstly, we show that TV_P is a positively one-homogeneous functional. Let $u \in L^1(X, \nu)$ and $\lambda > 0$. Then, we note that

$$\mathrm{TV}_P(\lambda u) = \int_{-\infty}^{\infty} P(E_t(\lambda u)) dt = \int_{-\infty}^{\infty} P(E_{t/\lambda}(u)) dt = \lambda \int_{-\infty}^{\infty} P(E_t(u)) dt = \lambda \mathrm{TV}_P(u).$$

Then, to show the convexity, it suffices to prove that the following inequality holds

$$(3.1) \quad \mathrm{TV}_P(u_1 + u_2) \leq \mathrm{TV}_P(u_1) + \mathrm{TV}_P(u_2) \quad \text{for every } u_1, u_2 \in L^2(X, \nu).$$

We first prove the following representation formula for nonnegative, bounded and integer valued measurable functions u in X :

$$(3.2) \quad \mathrm{TV}_P(u) = \min \left\{ \sum_{i=1}^m P(A_i) : u = \sum_{i=1}^m \chi_{A_i}, A_i \in \mathcal{B}, m \in \mathbb{N} \right\}.$$

Note that, if u satisfies these assumptions, then $u = \sum_{i=1}^M \chi_{E_{i-1}(u)}$ for some $M \in \mathbb{N}$.

Let us suppose that $u = \sum_{i=1}^m \chi_{A_i}$ with $A_i \in \mathcal{B}$. Using the sub-modularity of the perimeter, we find that

$$P(A_j \cap A_k) + P(A_j \cup A_k) + \sum_{i \notin \{j,k\}} P(A_i) \leq \sum_{i=1}^m P(A_i) \quad \text{for any } j \neq k \in \{1, \dots, m\}.$$

By induction on m , we easily deduce that

$$\sum_{i=1}^m P(A_i) \geq \sum_{i=1}^M P(E_{i-1}(u)).$$

Indeed, if $m = 1$ then $A_1 = E_0(u)$ up to a \mathfrak{m} -null set thus $P(A_1) = P(E_0(u))$. Now, suppose that the inequality holds for m . If $u := \sum_{i=1}^{m+1} \chi_{A_i}$ and $v := \sum_{i=1}^m \chi_{A_i}$, by induction hypothesis we have that

$$\sum_{i=1}^{m+1} P(A_i) \geq \sum_{i=1}^m P(E_{i-1}(v)) + P(A_{m+1}).$$

Using the definition of u , we note that

$$E_j(u) = E_j(v) \cup (A_{m+1} \cap E_{j-1}(v)),$$

and by submodularity we deduce that

$$P(E_j(v)) \geq P(E_j(u)) + P(A_{m+1} \cap E_j(v)) - P(A_{m+1} \cap E_{j-1}(v)).$$

Therefore,

$$\begin{aligned} \sum_{i=1}^m P(E_{i-1}(v)) &\geq \sum_{i=1}^m P(E_{i-1}(u)) + P(A_{m+1} \cap E_{i-1}(v)) - P(A_{m+1} \cap E_{i-2}(v)) \\ &= \sum_{i=1}^m P(E_{i-1}(u)) + P(A_{m+1} \cap E_{m-1}(v)) - P(A_{m+1} \cap E_{-1}(v)) \\ &= \sum_{i=1}^m P(E_{i-1}(u)) + P(A_{m+1} \cap E_{m-1}(v)) - P(A_{m+1}) \\ &= \sum_{i=1}^m P(E_{i-1}(u)) + P(E_m(v) \cup (A_{m+1} \cap E_{m-1}(v))) - P(A_{m+1}) \\ &= \sum_{i=1}^m P(E_{i-1}(u)) + P(E_m(u)) - P(A_{m+1}) = \sum_{i=1}^{m+1} P(E_{i-1}(u)) - P(A_{m+1}). \end{aligned}$$

Thus, by taking the infimum, we conclude the proof of (3.2):

$$\inf \left\{ \sum_{i=1}^m P(A_i) : u = \sum_{i=1}^m \chi_{A_i}, A_i \in \mathcal{B} \right\} = \sum_{i=1}^M P(E_{i-1}(u)) = \int_0^{M-1} P(E_t(u)) dt = \text{TV}(u).$$

We now prove (3.1). Assume without loss of generality that $\text{TV}_P(u_i)$ is finite for $i \in \{1, 2\}$.

As a first step, suppose that $0 \leq u_i \leq 1$, $i \in \{1, 2\}$, and consider the following approximations:

$$u_{i,n} := \frac{1}{n} \sum_{k=-1}^n \chi_{E_{\frac{k+t}{n}}(u_i)} \quad \text{for } n \in \mathbb{N}, t \in (0, 1) \text{ and } i \in \{1, 2\}.$$

By construction, $0 \leq u_{i,n} \leq 3$ and $u_{i,n} \rightarrow u_i$ in $L^1(X, \mathfrak{m})$, $i \in \{1, 2\}$. Then, we can extract a subsequence (not relabelled) converging a.e. to u_i . Since $\mathfrak{m}(X) < +\infty$, this also implies $u_{i,n} \rightarrow u_i$ in $L^2(X, \mathfrak{m})$. Therefore, from the lower semicontinuity and the one-homogeneity of the total variation we get

$$\begin{aligned} \text{TV}_P(u_1 + u_2) &\leq \liminf_{n \rightarrow \infty} \text{TV}_P(u_{1,n} + u_{2,n}) \\ &= \liminf_{n \rightarrow \infty} \frac{1}{n} \text{TV}_P \left(\sum_{k=-1}^n \chi_{E_{\frac{k+t}{n}}(u_1)} + \chi_{E_{\frac{k+t}{n}}(u_2)} \right) \\ &\stackrel{(3.2)}{\leq} \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=-1}^n P \left(E_{\frac{k+t}{n}}(u_1) \right) + P \left(E_{\frac{k+t}{n}}(u_2) \right). \end{aligned}$$

Finally, applying [49, Lemma 3.3], we can choose $t \in [0, 1)$ such that, up to a subsequence,

$$\frac{1}{n} \sum_{k=-1}^n \mathbb{P} \left(E_{\frac{k+t}{n}}(u_i) \right) \rightarrow \int_0^1 \mathbb{P}(E_t(u_i)) dt = \text{TV}_{\mathbb{P}}(u_i) \quad \text{as } n \rightarrow \infty, i \in \{1, 2\},$$

which finishes the proof in the case $0 \leq u_i \leq 1$. The case where u_i are bounded, $i = 1, 2$, easily follows by considering $\tilde{u}_i := \frac{u_i - m}{M - m}$ with $m \leq u_i \leq M$ and applying the previous step.

Finally, we consider $u_i \in L^2(X, \mathfrak{m})$, $i \in \{1, 2\}$. In this case, we choose as approximations truncations at level $\pm K$

$$u_i^K := \max\{\min\{u_i, K\}, -K\}.$$

We note that $u_i^K \rightarrow u_i$ in $L^2(X, \mathfrak{m})$ as $K \rightarrow +\infty$. Using the lower semicontinuity of $\text{TV}_{\mathbb{P}}$, the proof is completed as follows:

$$\begin{aligned} \text{TV}_{\mathbb{P}}(u_1 + u_2) &\leq \liminf_{K \rightarrow \infty} \text{TV}_{\mathbb{P}}(u_1^K + u_2^K) \\ &\leq \liminf_{K \rightarrow \infty} \text{TV}_{\mathbb{P}}(u_1^K) + \text{TV}_{\mathbb{P}}(u_2^K) \leq \limsup_{K \rightarrow \infty} \text{TV}_{\mathbb{P}}(u_1^K) + \text{TV}_{\mathbb{P}}(u_2^K) \\ &= \limsup_{K \rightarrow \infty} \int_{-K}^K (\mathbb{P}(E_t(u_1)) + \mathbb{P}(E_t(u_2))) dt = \text{TV}_{\mathbb{P}}(u_1) + \text{TV}_{\mathbb{P}}(u_2). \end{aligned}$$

□

Now we define our framework space using these the following two definitions:

DEFINITION 3.4. *A perimeter measure space $(X, \mathcal{B}, \mathfrak{m}, \mathbb{P})$ is a measure space $(X, \mathcal{B}, \mathfrak{m})$ with $\mathfrak{m}(X) < +\infty$, equipped with a perimeter functional \mathbb{P} such that the total variation functional $\text{TV}_{\mathbb{P}}$ is lower semi-continuous with respect to weak convergence in $L^2(X, \mathfrak{m})$. For brevity, we denote it as $(X, \mathfrak{m}, \mathbb{P})$ and we name it as PMS.*

With this framework, we define the ROF and CV models in PMS:

DEFINITION 3.5. *Let $(X, \mathfrak{m}, \mathbb{P})$ be a PMS, and let f be a measurable function that is not \mathfrak{m} -a.e. constant. The ROF model in PMS is defined as the following minimization problem:*

$$\min_{u \in L^2(X, \mathfrak{m})} \left\{ \mathcal{E}_{\mathbb{P}}^{\lambda}(u) := \text{TV}_{\mathbb{P}}(u) + \frac{\lambda}{2} \int_X (u - f)^2 d\mathfrak{m} \right\}.$$

Similarly, the CV model in PMS is formulated as

$$\min_{\substack{\Sigma \in \mathcal{B} \\ (c_1, c_2) \in [0, 1]^2}} \left\{ \mathcal{F}_{\mathbb{P}}^{\mu}(\Sigma, c_1, c_2) := \mathbb{P}(\Sigma) + \mu \int_{\Sigma} (c_1 - f)^2 d\mathfrak{m} + \mu \int_{X \setminus \Sigma} (c_2 - f)^2 d\mathfrak{m} \right\}$$

REMARK 3.6. *Our definition of the perimeter functional closely aligns with the notion of generalized perimeter introduced in [52]. However, their concept of perimeter is restricted to Lebesgue measurable sets. Furthermore, a similar definition of a perimeter measure space is considered in [78], where the authors analyze a perimeter functional under various assumptions and derive multiple results by exploring different combinations of these assumptions.*

3.2. Examples of PMS

To show the generality of this framework and its broader applicability, we present a comprehensive list of spaces that qualify as *perimeter measure spaces*:

3.2.1. Isotropic and anisotropic Euclidean case. Let $\Omega \subset \mathbb{R}^N$ and $\mathfrak{m} = \mathcal{L}^N$. Recall that for $u \in L^1(\Omega)$, the isotropic total variation and the perimeter are defined in (2.1) and (2.2), respectively; i.e.,

$$\mathrm{TV}_2(u) := |Du|(\Omega), \quad \mathrm{P}_2(E) := \mathrm{Per}(E; \Omega) = \mathrm{TV}_2(\chi_E).$$

By the coarea formula (2.3), it follows that TV_2 , as defined above, coincides with the definition in Definition 3.2. The submodularity of the perimeter is established in [6, Proposition 3.38]. Moreover, the lower semicontinuity with respect to weak convergence in $L^2(X, \mathfrak{m})$ follows directly from the expression in (2.1):

Suppose that $u_n \rightharpoonup u$ weakly in $L^2(X, \mathfrak{m})$. Then, for any $z \in C_0^\infty(\Omega; \mathbb{R}^N)$ such that $\|z(x)\| \leq 1$,

$$\int_X u(x) \mathrm{div}(z)(x) \, dx = \liminf_{n \rightarrow \infty} \int_X u_n(x) \mathrm{div}(z)(x) \, dx \leq \liminf_{n \rightarrow \infty} \mathrm{TV}_2(u_n).$$

Taking the supremum over z , we obtain

$$\mathrm{TV}_2(u) \leq \liminf_{n \rightarrow \infty} \mathrm{TV}_2(u_n).$$

Therefore, $(\Omega, \mathcal{L}^N|_\Omega, \mathrm{P}_2)$ forms a PMS. Now, let us consider an anisotropy $\Phi(\xi)$ as defined in (2.2). Recall that for $u \in L^1(\Omega)$, the anisotropic total variation and perimeter are defined in (2.6) and (2.7), respectively; i.e.,

$$\mathrm{TV}_\Phi(u) := |Du|_\Phi(\Omega), \quad \mathrm{P}_\Phi(E) := \mathrm{Per}_\Phi(E; \Omega) = \mathrm{TV}_\Phi(\chi_E).$$

The coarea formula and the submodularity of the perimeter are discussed in [4, Remark 4.4] and [66, Remark 2.4], respectively. Following the same reasoning as in the isotropic case, we can similarly show that $(\Omega, \mathcal{L}^N|_\Omega, \mathrm{P}_\Phi)$ is a PMS. In this context, we refer to Chapter 4, where we study the ROF and CV models from the anisotropic perspective.

3.2.2. Fractional perimeter. Let $\Omega \subseteq \mathbb{R}^N$ be an open bounded set. Given $s \in (0, 1)$, the fractional Sobolev space $W^{s,1}(\Omega)$ is defined as the set of functions

$$W^{s,1}(\Omega) := \left\{ u \in L^1(\Omega) : \mathrm{TV}_s(u) := \int_\Omega \int_\Omega \frac{|u(x) - u(y)|}{|x - y|^{N+s}} \, dx dy < +\infty \right\}.$$

Moreover, defining the nonlocal perimeter as

$$\mathrm{P}_s(E) := \int_{E \cap \Omega} \int_{(\mathbb{R}^N \setminus E) \cap \Omega} \frac{1}{|x - y|^{N+s}} \, dx dy,$$

for every Lebesgue measurable set $E \subseteq \mathbb{R}^N$, one can show as above that $(\mathbb{R}^N, \mathcal{L}^N|_\Omega, \mathrm{P}_s)$ is a PMS. In fact, the coarea formula for TV_s can be found in [117, Proposition 3.1] or [5, Lemma 10], the submodularity of the perimeter can be proved as in [45, Section 2.1.2] and one also has [8, Theorem 3.4] for a representation formula of TV_s similar to that in (2.1). Let us mention that this nonlocal concept of perimeter was first introduced in [33].

The ROF model for the fractional perimeter in the case $\Omega = \mathbb{R}^N$ has been recently studied in [140] (and in [20, 21] with L^1 fidelity term).

3.2.3. Random walk spaces. Let (X, \mathcal{B}) be a measurable space such that the σ -field \mathcal{B} is countably generated. A random walk on $(X, \mathcal{B}, \mathbf{m})$ is a family of probability measures $(m_x)_{x \in X}$ on \mathcal{B} such that $x \mapsto m_x(B)$ is a measurable function on X for each fixed $B \in \mathcal{B}$. Moreover, a σ -finite measure \mathbf{m} on \mathcal{B} is reversible with respect to the random walk m if the following balance condition holds:

$$dm_x(y)d\mathbf{m}(x) = dm_y(x)d\mathbf{m}(y) \quad \text{for every } x, y \in X.$$

(X, \mathcal{B}) together with a random walk m and a σ -finite measure \mathbf{m} which is reversible with respect to m is called a reversible random walk space and it is denoted by $[X, \mathcal{B}, m, \mathbf{m}]$.

Given $E \in \mathcal{B}$, the m -perimeter of E is defined as

$$P_m(E) := \int_E \int_{X \setminus E} dm_x(y)d\mathbf{m}(x),$$

and the m -total variation of a measurable function $u : X \rightarrow \mathbb{R}$ is defined by

$$\text{TV}_m(u) := \frac{1}{2} \int_X \int_X |u(y) - u(x)| dm_x(y)d\mathbf{m}(x).$$

Then, if $\mathbf{m}(X) < +\infty$, (X, \mathbf{m}, P_m) is a PMS (see [121, 123] for the coarea formula, the submodularity of the perimeter and the lower semicontinuity result). The ROF model in reversible random walk spaces has been studied in [122] and the subdifferential of the m -total variation functional was completely characterized in [121].

We point out that many different examples can be included in this category of reversible random walk spaces. We give a non-exhaustive list of them, including only the more relevant to us and without entering into details, for which we refer to [121, 123]:

- (1) *Nonlocal perimeter with an integrable kernel:* Let $J : \mathbb{R}^N \rightarrow [0, +\infty[$ be a Lebesgue measurable, nonnegative and radially symmetric function such that

$$\int_{\mathbb{R}^N} J(x)dx = 1.$$

Let $\Omega \subseteq \mathbb{R}^N$ be a closed set of finite Lebesgue measure and

$$m_x(A) := \int_A J(x-y)dy + \left(\int_{\mathbb{R}^n \setminus \Omega} J(x-z)dz \right) \delta_x(A)$$

for any $x \in \Omega$ and any Lebesgue measurable set $A \subseteq \Omega$. Then, $m = (m_x)$ is a random walk on Ω with respect to which the Lebesgue measure on Ω is reversible. This is the usual random walk arising from a nonsingular kernel J , but modified so that it does not jump outside of Ω (immediately reflected back to the starting position).

- (2) *Markov chains on a countable space X with a reversible probability measure π :* Given a Markov Kernel $K : X \times X \rightarrow \mathbb{R}$, one defines

$$m_x(A) := \sum_{y \in A} K(x, y) \quad \text{for any } x \in X \text{ and any } A \subseteq X.$$

Then, if \mathbf{m} is a reversible probability measure with respect to K , $[X, \mathcal{B}, m, \mathbf{m}]$ is a reversible random walk space (here \mathcal{B} is the σ -algebra of all subsets of X).

- (3) *Locally finite undirected weighted discrete graphs:* Let $G = (V_G, E_G)$ be a locally finite weighted discrete graph with vertex set V_G and suppose that each edge $(x, y) \in E_G$ has a positive weight $w_{xy} = w_{yx}$ assigned. For each $x \in V_G$, the random walk m is defined as follows:

$$m_x := \frac{\sum_{y \sim x} w_{xy} \delta_y}{\sum_{y \sim x} w_{xy}},$$

with $y \sim x$ denoting that $(x, y) \in E_G$ (alternatively, we can suppose that $w_{xy} = 0$ if $(x, y) \notin E_G$ and consider the sum over all the vertices $y \in V_G$). Then, for $A \subset V_G$, one considers

$$m(A) := \sum_{x \in A} \sum_{y \sim x} w_{xy}.$$

It follows that m is a reversible measure with respect to the random walk m .

3.2.4. Carnot-Carathéodory spaces. Given an open set $\Omega \subseteq \mathbb{R}^N$, $m < N$ and m locally Lipschitz vector fields $X = (X_1, \dots, X_m)$, a distance in Ω is defined as follows:

$$d(x, y) = \inf \left\{ \int_0^1 |\dot{\gamma}(t)| dt : \gamma(0) = x, \gamma(1) = y, \dot{\gamma}(t) = \sum_{i=1}^m a_i(t) X_i, 0 < t < 1 \right\}.$$

Assuming that this distance is everywhere finite, the space of bounded variation $BV_X(\Omega)$ is defined as the space of functions $u \in L^1(\Omega)$ such that

$$TV_X(u) := \sup \left\{ \int_{\Omega} u \operatorname{div}_{X^*}(z) : z \in C_0^1(\Omega, \mathbb{R}^N), \|z\|_{\infty} \leq 1 \right\},$$

where $\operatorname{div}_{X^*}(z) = \sum_{i=1}^m X_i^* z^i$, with X^* the adjoint vector field of X . For any Lebesgue measurable $E \subset \Omega$, its perimeter is defined as

$$P_X(E) := TV_X(\chi_E).$$

The coarea formula and the submodularity of the perimeter can be found in [126, Propositions 4.2 and 4.7] and the lower semicontinuity can be proved as in the Euclidean case. Then, $(\Omega, \mathcal{L}^N, P_X)$ is a PMS. The subdifferential of the total variation functional coupled with different Neumann or Dirichlet conditions in general metric measure spaces (which includes the case of Carnot-Carathéodory spaces) has been recently studied in [120].

3.2.5. Metric graphs. We follow [133] for the definition of a metric graph, which we now briefly recall. Let E be a finite set. Given $\{\ell_e\}_{e \in E} \subset]0, \infty[$, consider the family $[0, \ell_e]_{e \in E}$ of metric measure subspaces of \mathbb{R} (with the Euclidean metric d_e and Lebesgue measure λ_e) and their disjoint union

$$\mathcal{E} := \bigcup_{e \in E} [0, \ell_e].$$

We adopt the notation (x, e) for the element of \mathcal{E} with $x \in [0, \ell_e]$ and $e \in E$. We endow \mathcal{E} with the disjoint union topology. For this, let $\varphi_e : [0, \ell_e] \ni x \mapsto (x, e) \in \mathcal{E}$ be the canonical injection. Then, a set $U \subset \mathcal{E}$ is open if and only if each $\varphi_e^{-1}(U)$ is a union of sets of the form $[0, \varepsilon_1[$, $]\varepsilon_2, \ell_e]$ or $]\varepsilon_3, \varepsilon_4[$, for $\varepsilon_i \in]0, \ell_e[$.

Consider now the set of endpoints of \mathcal{E}

$$\mathcal{V} := \bigcup_{e \in E} \{0, \ell_e\}.$$

Given any equivalence relation \sim on \mathcal{V} , we extend it to an equivalence relation on \mathcal{E} by equality: i.e., two elements $(x_1, e_1), (x_2, e_2) \in \mathcal{E}$ belong to the same equivalence class if and only if $(x_1, e_1) = (x_2, e_2)$ or else $(x_1, e_1), (x_2, e_2) \in \mathcal{V}$ and $(x_1, e_1) \sim (x_2, e_2)$. We continue to denote this equivalence relation on \mathcal{E} by \sim .

We call $\Gamma := \mathcal{E}/\sim$ a *metric graph* and $V := \mathcal{V}/\sim$ its set of vertices. Therefore, a metric graph is uniquely determined by a family $\{\ell_e\}_{e \in E}$ and an equivalence relation on \mathcal{V} . Its vertices are the cells of the partition of \mathcal{V} induced by \sim . Two vertices $\mathbf{v}, \mathbf{w} \in V$ are said to be adjacent if there exists some $e \in E$ such that $\{x, y\} = \{0, \ell_e\}$ for some representatives (x, e) of \mathbf{v} and (y, e) of \mathbf{w} ; in this case we write $\mathbf{v} \sim \mathbf{w}$ and, with an abuse of notation, also $\mathbf{v} \sim e$. The cardinality $\deg(\mathbf{v})$ of the set $\{\mathbf{w} \in V : \mathbf{w} \text{ is adjacent to } \mathbf{v}\}$ is called degree of $\mathbf{v} \in V$. We denote $E_{\mathbf{v}} := \{e \in E : \mathbf{v} \sim e\}$ and $\text{int}(V) := \{\mathbf{v} \in V : \#E_{\mathbf{v}} > 1\}$ (here $\#E_{\mathbf{v}}$ denotes the cardinality of $E_{\mathbf{v}}$). Moreover, Γ is a measure space with respect to the direct sum measure $\mathfrak{m} = \bigoplus_{e \in E} \lambda_e$ (see [133] and the references therein).

A function u on a metric graph Γ is a collection of functions $\{[u]_e\}_{e \in E}$ with $[u]_e :]0, \ell_e[\rightarrow \mathbb{R}$. If $[u]_e$ is integrable for all $e \in E$ then

$$\int_{\Gamma} u(x) dx := \sum_{e \in E} \int_0^{\ell_e} [u]_e(x) dx.$$

The Sobolev space $H^1(\Gamma)$ is defined as the space of functions u such that $[u]_e \in H^1(0, \ell_e)$ for all $e \in E$ and

$$\sum_{e \in E} \|[u]_e\|_{H^1(0, \ell_e)} < +\infty.$$

If $\mathbf{v} \sim e$ and $(0, e)$ (alternatively, (ℓ_e, e)) is a representative of \mathbf{v} , the trace of $[u]_e$ at 0 (alternatively, at ℓ_e) is denoted by $[u]_e(\mathbf{v})$.

The total variation of a measurable function u is defined as

$$\text{TV}_{\Gamma}(u) := \sup \left\{ \int_{\Gamma} u(x) z'(x) dx : z \in X_k(\Gamma), \|z\|_{L^\infty(\Gamma)} \leq 1 \right\},$$

where $X_k(\Gamma)$ is the set of vector fields in $H^1(\Gamma)$ satisfying a Kirchhoff condition on all the vertices of the graph; i.e.,

$$X_K(\Gamma) = \left\{ z \in H^1(\Gamma) : \sum_{e \in E_{\mathbf{v}}} [z]_e(\mathbf{v}) \nu^e(\mathbf{v}) = 0, \forall \mathbf{v} \in \text{int}(V) \right\}.$$

Here, ν^e is the unit outer exterior normal to e ; i.e., $\nu^e(\mathbf{v}) = 1$ if (ℓ_e, e) is a representative of \mathbf{v} and $\nu^e(\mathbf{v}) = -1$ if $(0, e)$ is a representative of \mathbf{v} .

Note that, as in the previous subsections, weak lower semicontinuity holds by the definition of TV_{Γ} . In order to prove the coarea formula and the submodularity of the perimeter, we give a representation formula for TV_{Γ} . We first introduce some notation. Given $\mathbf{v} \in \text{int}(V)$, we consider the maximum and the minimum of the traces of u at \mathbf{v} :

$$u_{\mathbf{v}}^{\max} := \max\{[u]_e(\mathbf{v}) : e \in E_{\mathbf{v}}\}, \quad u_{\mathbf{v}}^{\min} := \min\{[u]_e(\mathbf{v}) : e \in E_{\mathbf{v}}\}.$$

For a function u on Γ such that each $[u]_e$ is a function of bounded variation, we denote $|Du|(\Gamma) := \sum_{e \in E} |D[u]_e|(0, \ell_e)$. With this notation, we obtain the following representation formula:

PROPOSITION 3.7.

$$(3.3) \quad \text{TV}_\Gamma(u) = |Du|(\Gamma) + \sum_{\mathbf{v} \in \text{int}V(\Gamma)} (u_{\mathbf{v}}^{\max} - u_{\mathbf{v}}^{\min}).$$

PROOF. To prove (3.3) we use the following integration by parts formula [119]:

$$(3.4) \quad \int_\Gamma uz' dx = -(z, Du)(\Gamma) + \sum_{\mathbf{v} \in \text{int}(V)} \sum_{e \in E_{\mathbf{v}}} [z]_e(\mathbf{v}) \nu^e(\mathbf{v}) [u]_e(\mathbf{v}),$$

with

$$(z, Du)(\Gamma) := \sum_{e \in E} ([z]_e, D[u]_e)(0, \ell_e);$$

$([z]_e, D[u]_e)$ being the one dimensional Anzellotti Radon measure product (see (2.4)). We recall that $|(z, Du)| \leq \|z\|_\infty |Du|$ as measures.

From (3.4) we directly have

$$\text{TV}_\Gamma(u) \leq |Du|(\Gamma) + \sum_{\mathbf{v} \in \text{int}(V)} \max \left\{ \sum_{e \in E_{\mathbf{v}}} w_e [u]_e(\mathbf{v}) : w_e \in [-1, 1], \sum_{e \in E_{\mathbf{v}}} w_e = 0 \right\}.$$

The last maximum is easily computed by the simplex method and we obtain

$$\text{TV}_\Gamma(u) \leq |Du|(\Gamma) + \sum_{\mathbf{v} \in \text{int}(V)} (u_{\mathbf{v}}^{\max} - u_{\mathbf{v}}^{\min}).$$

Let us see the opposite inequality. Since

$$|D[u]_e|(0, \ell_e) = \sup \left\{ (z, Du)(0, \ell_e) = - \int_0^{\ell_e} [u]_e z' : z \in C_c^1(0, \ell_e) \right\},$$

we can find $z^n := \{[z^n]_e\}_{e \in E}$ with $[z^n]_e \in C_c^1(0, \ell_e)$ such that $\text{dist}(\text{supp}([z^n]_e), \{0, \ell_e\}) > \frac{1}{n}$ for all $e \in E_{\mathbf{v}}$, and

$$|Du|(\Gamma) \leq (z^n, Du)(\Gamma) + \frac{1}{n}.$$

We now fix $\mathbf{v} \in \text{int}(V)$ and take

$$e_{\max} \in \text{argmax}\{[u]_e(\mathbf{v}) : e \in E_{\mathbf{v}}\}, \quad e_{\min} \in \text{argmin}\{[u]_e(\mathbf{v}) : e \in E_{\mathbf{v}}\},$$

with $e_{\max} \neq e_{\min}$. Suppose for simplicity that $(0, e_{\min})$ and $(\ell_{e_{\max}}, e_{\max})$ are representatives of \mathbf{v} (the other cases follow similarly) and define

$$[\tilde{z}^n]_{e_{\max}} := \begin{cases} \frac{nx-1}{n} & \text{if } 0 \leq x \leq \frac{1}{n} \\ [z^n]_{e_{\max}} & \text{otherwise} \end{cases}, \quad [\tilde{z}^n]_{e_{\min}} := \begin{cases} \frac{1-nx}{n} & \text{if } 0 \leq x \leq \frac{1}{n} \\ [z^n]_{e_{\min}} & \text{otherwise} \end{cases}.$$

Then, repeating this with each $\mathbf{v} \in \text{int}(V)$ (changing $(0, \frac{1}{n})$ with $(\ell_{e_{\min}} - \frac{1}{n}, \ell_{e_{\min}})$ or with $(\ell_{e_{\max}} - \frac{1}{n}, \ell_{e_{\max}})$ when necessary) we end up with \tilde{z}^n such that

$$\begin{aligned} & (\tilde{z}^n, Du)(\Gamma) - \sum_{\mathbf{v} \in \text{int}(V)} \sum_{e \in E_{\mathbf{v}}} [z]_e(\mathbf{v}) \nu^e(\mathbf{v}) [u]_e(\mathbf{v}) \\ &= (z^n, Du)(\Gamma) + \sum_{\mathbf{v} \in \text{int}(V)} (u_{\mathbf{v}}^{\max} - u_{\mathbf{v}}^{\min}) \\ & \quad + \sum_{\mathbf{v} \in \text{int}(V)} \left(([\tilde{z}^n]_{e_{\max}}, D[u]_{e_{\max}}) \left(0, \frac{1}{n}\right) + ([\tilde{z}^n]_{e_{\min}}, D[u]_{e_{\min}}) \left(0, \frac{1}{n}\right) \right) \\ & \geq |Du|(\Gamma) + \sum_{\mathbf{v} \in \text{int}(V)} (u_{\mathbf{v}}^{\max} - u_{\mathbf{v}}^{\min}) - \sum_{\mathbf{v} \in \text{int}(V)} (|D[u]_{e_{\max}}| + |D[u]_{e_{\min}}|) \left(0, \frac{1}{n}\right) - \frac{1}{n}. \end{aligned}$$

Letting $n \rightarrow +\infty$, with the use of (3.4), we conclude that

$$\text{TV}_{\Gamma}(u) \geq |Du|(\Gamma) + \sum_{\mathbf{v} \in \text{int}(V)} (u_{\mathbf{v}}^{\max} - u_{\mathbf{v}}^{\min}).$$

Therefore,

$$\begin{aligned} \text{TV}_{\Gamma}(u) &= |Du|(\Gamma) + \sum_{\mathbf{v} \in \text{int}(V)} (u_{\mathbf{v}}^{\max} - u_{\mathbf{v}}^{\min}) \\ &= |Du|(\Gamma) + \sum_{\mathbf{v} \in \text{int}(V)} \max \left\{ \sum_{e \in E_{\mathbf{v}}} w_e [u]_e(\mathbf{v}) : w_e \in [-1, 1], \sum_{e \in E_{\mathbf{v}}} w_e = 0 \right\}. \end{aligned}$$

□

We define $\text{P}_{\Gamma}(A) := \text{TV}_{\Gamma}(\chi_A)$ for any \mathbf{m} -measurable $A \subset \Gamma$. Here, given a \mathbf{m} -measurable set $A \subseteq \Gamma$ the function $\chi_A = \{[\chi_A]_e\}_{e \in E}$ on Γ is defined by

$$[\chi_A]_e(x) := \begin{cases} 1 & \text{if } (x, e) \text{ is a representative of an element in } A, \\ 0 & \text{if } (x, e) \text{ is not a representative of any element in } A. \end{cases}$$

Then, $(\Gamma, \mathbf{m}, \text{P}_{\Gamma})$ is a PMS. Indeed, observe that, since the classical total variation satisfies the coarea formula and the usual perimeter is submodular, thanks to the last equality, we only need to show that:

- i) the trace at the endpoints of an interval of a one dimensional BV function satisfies the layer cake formula, in particular,

$$[u]_e(\mathbf{v}) = \int_0^{+\infty} [\chi_{E_t(u)}]_e(\mathbf{v}) dt - \int_0^{+\infty} [\chi_{\Gamma \setminus E_t(u)}]_e(\mathbf{v}) dt$$

for every $\mathbf{v} \in \text{int}(V)$ and $e \in E_{\mathbf{v}}$ (here $\chi_{E_t(u)} = \{\chi_{E_t([u]_e)}\}_{e \in E}$).

- ii) the trace of the characteristic function of a set is submodular.

These two properties follow directly from the continuity of a precise representative of a one dimensional BV function outside of the jump set since, working edge by edge, we have that the precise representative of the extension of $[u]_e$ with constant value $[u]_e(\mathbf{v})$ to the right of ℓ_e (or left of 0) is continuous at the point ℓ_e (or 0) and the same thing happens with the precise representative of the extension of χ_A of any finite perimeter set A .

3.3. (λ, μ) -linkage in PMS

Hereinafter, we consider that we are working on a generic PMS $(X, \mathbf{m}, \mathbf{P})$. In this section, we prove a (λ, μ) -linkage which holds for $(X, \mathbf{m}, \mathbf{P})$:

THEOREM 3.8. *Let f be a \mathbf{m} -measurable function and let u be the minimizer of \mathcal{E}_P^λ . If a tuple (Σ^*, c_1^*, c_2^*) satisfies the conditions*

$$(3.5) \quad c_1^* = \int_{\Sigma} f d\mathbf{m}, \quad c_2^* = \int_{X \setminus \Sigma} f d\mathbf{m}, \quad \Sigma^* = E_{\frac{c_1^* + c_2^*}{2}}(u), \quad 0 < \mathbf{m}(\Sigma^*) < \mathbf{m}(X);$$

then (Σ^*, c_1^*, c_2^*) is a partial minimizer of \mathcal{F}_P^μ with $\mu = \lambda/(2(c_1^* - c_2^*))$.

The first step in obtaining the above result is to prove the existence and uniqueness of minimizers of \mathcal{E}_P^λ for a given perimeter measure space $(X, \mathcal{B}, \mathbf{m})$. This question is addressed in the following result:

THEOREM 3.9. *There exists an unique minimizer u of \mathcal{E}_P^λ such that it is the only function in $L^1(X, \mathbf{m})$ satisfying $0 \in \partial(\mathcal{E}_P^\lambda)(u)$.*

REMARK 3.10. *Recall that*

$$\partial(\mathcal{E}_P^\lambda)(u) := \left\{ w \in L^2(X, \mathbf{m}) : \mathcal{E}_P^\lambda(v) - \mathcal{E}_P^\lambda(u) \geq \int_X w(v - u) d\mathbf{m} \text{ for all } v \in L^2(X, \mathbf{m}) \right\}.$$

PROOF. Since the TV functional is lower semicontinuous with respect to the weak convergence in $L^2(X, \nu)$, existence of a minimizer follows from the direct method of the calculus of variations. Indeed, let $\{u_k\}_{k \in \mathbb{N}} \subseteq L^2(X, \nu)$ be a minimizing sequence. Then,

$$\begin{aligned} \int_X |u_k|^2 d\mathbf{m} &\leq 2 \int_X (u_k - f)^2 d\mathbf{m} + 2 \int_X |f|^2 d\mathbf{m} \\ &\leq \frac{4}{\lambda} \sup_{k \in \mathbb{N}} \left\{ \mathcal{E}_P^\lambda(u_k) \right\} + 2 \int_X |f|^2 d\mathbf{m}. \end{aligned}$$

Hence, up to a subsequence, $\{u_k\}_k$ converges weakly in $L^2(X, \mathbf{m})$ to a function $u \in L^2(X, \mathbf{m})$. Finally, by applying the lower semicontinuity of the total variation, we conclude that u minimizes the functional \mathcal{E}_P^λ . Moreover, by Proposition 3.3, \mathcal{E}_P^λ is a strictly convex functional and consequently, \mathcal{E}_P^λ has a unique minimizer. Observe that $0 \in \partial(\mathcal{E}_P^\lambda)(u)$ is equivalent to u being a minimizer. \square

COROLLARY 3.11. *Let u be the minimizer of \mathcal{E}_P^λ and f be a \mathbf{m} -measurable function such that $0 \leq f \leq 1$ \mathbf{m} -a.e. Then $0 \leq u \leq 1$ \mathbf{m} -a.e.*

PROOF. This range bound is a direct consequence of the fact that the truncation of any function $v \in L^2(X, \mathbf{m})$ by $T_0^1(s) := \max\{\min\{1, s\}, 0\}$ satisfies

$$\mathcal{E}_P^\lambda(T_0^1(v)) = \int_0^1 \mathbf{P}(E_t(v)) dt + \frac{\lambda}{2} \int_X |T_0^1(v(x)) - f(x)|^2 d\mathbf{m}(x) \leq \mathcal{E}_P^\lambda(v).$$

\square

The next step is to adapt a series of results from the isotropic Euclidean case [48] (Proposition 2.1, Lemma 2.4, and Lemma 2.5), which allow us to establish a relation between the functional \mathcal{E}_P^λ and the functional $\tilde{\mathcal{E}}_P^\lambda : \mathcal{B} \rightarrow \mathbb{R}$, defined as

$$(3.6) \quad \tilde{\mathcal{E}}_P^\lambda(\Sigma) := \mathbf{P}(\Sigma) - \lambda \int_{\Sigma} f d\mathbf{m}.$$

The first result establishes a relation between $\tilde{\mathcal{E}}_P^\lambda(\Sigma)$ and an auxiliary problem:

PROPOSITION 3.12. *Given a minimizer $u \in L^2(X, \mathbf{m})$ of*

$$(3.7) \quad \min_{\substack{u \in L^2(X, \mathbf{m}) \\ 0 \leq u \leq 1}} \left\{ \text{TV}_P(u) - \lambda \int_X u f \, d\mathbf{m} \right\},$$

we have that $E_t(u) := \{x \in X : u(x) > t\}$ is a minimizer of $\tilde{\mathcal{E}}_P^\lambda$ for any $t \in [0, 1)$.

PROOF. First of all, by the lower semi-continuity of TV_P with respect to the weak convergence in $L^2(X, \mathbf{m})$, the direct method of the calculus of variations yields the existence of a minimizer solving (3.7). Now observe that for any $u \in L^2(X, \mathbf{m})$ with $0 \leq u \leq 1$ \mathbf{m} -a.e.,

$$\begin{aligned} \int_X u f \, d\mathbf{m} &= \int_X \left(\int_0^{u(x)} ds \right) f(x) \, d\mathbf{m}(x) \\ &= \int_X \int_0^1 \chi_{E_s(u)}(x) f(x) \, ds \, d\mathbf{m}(x) = \int_0^1 \int_{E_s(u)} f(x) \, d\mathbf{m}(x) \, ds. \end{aligned}$$

Therefore, it follows that

$$(3.8) \quad \text{TV}_P(u) - \lambda \int_X u f \, d\mathbf{m} = \int_0^1 \tilde{\mathcal{E}}_P^\lambda(E_s(u)) \, ds \geq \min_{\Sigma \in \mathcal{B}} \left\{ \tilde{\mathcal{E}}_P^\lambda(\Sigma) \right\}.$$

Consequently, the minimum in (3.7) is greater than or equal to the minimum Σ^* of (3.6). However, Σ^* is greater than (3.7) since minimizing (3.6) is just (3.7) over characteristic functions of Borel sets. Therefore, they coincide.

Moreover, from (3.8) it follows that, if u is a minimizer of (3.7) then $E_t(u)$ is a minimizer of (3.6) for a.e. $t \in (0, 1)$.

Let us see that, in fact, this is true for every $t \in [0, 1)$. Let $t \in [0, 1)$ and let $(t_n)_{n \geq 1}$ be a decreasing sequence such that $E_{t_n}(u)$ is a minimizer of (3.6) for every $n \geq 1$ and $t_n \downarrow t$ as $n \rightarrow \infty$. Since we have

$$\chi_{E_{t_n}(u)} \xrightarrow{n} \chi_{E_t(u)} \quad \text{in } L^2(X, \mathbf{m}),$$

from the lower semi-continuity of TV_P with respect to the weak convergence in $L^2(X, \mathbf{m})$, we conclude that

$$\tilde{\mathcal{E}}_P^\lambda(E_t(u)) \leq \liminf_{n \rightarrow \infty} \tilde{\mathcal{E}}_P^\lambda(E_{t_n}(u)) = \min_{\Sigma \in \mathcal{B}} \left\{ \tilde{\mathcal{E}}_P^\lambda(\Sigma) \right\}.$$

□

As a consequence of Proposition 3.12, for a fixed $\tau \in \mathbb{R}$ and a minimizer u of the problem

$$\min_{\substack{u \in L^2(X, \mathbf{m}) \\ 0 \leq u \leq 1}} \left\{ \text{TV}_P(u) - \lambda \int_X u(\tau - f) \, d\mathbf{m} \right\},$$

the sets $E_t(u)$ for $t \in [0, 1)$ are minimizers of

$$\mathcal{E}_{P, \tau}^\lambda(\Sigma) := P(\Sigma) + \lambda \int_\Sigma (\tau - f) \, d\mathbf{m}, \quad \Sigma \in \mathcal{B}.$$

LEMMA 3.13. *Let $\tau_0 < \tau_1$. If Σ_{τ_0} and $\Sigma_{\tau_1} \in \mathcal{B}$ are minimizers of $\mathcal{E}_{\mathbb{P},\tau_0}^\lambda$ and $\mathcal{E}_{\mathbb{P},\tau_1}^\lambda$, respectively; then $\Sigma_{\tau_1} \subseteq \Sigma_{\tau_0}$ up to a \mathbf{m} -null set.*

PROOF. Since Σ_{τ_0} and Σ_{τ_1} are minimizers, we have that

$$\begin{aligned}\mathcal{E}_{\mathbb{P},\tau_0}^\lambda(\Sigma_{\tau_0}) &\leq \mathbf{P}(\Sigma_{\tau_0} \cup \Sigma_{\tau_1}) + \lambda \int_{\Sigma_{\tau_0} \cup \Sigma_{\tau_1}} (\tau_0 - f) \mathbf{d}\mathbf{m}, \\ \mathcal{E}_{\mathbb{P},\tau_1}^\lambda(\Sigma_{\tau_1}) &\leq \mathbf{P}(\Sigma_{\tau_0} \cap \Sigma_{\tau_1}) + \lambda \int_{\Sigma_{\tau_0} \cap \Sigma_{\tau_1}} (\tau_1 - f) \mathbf{d}\mathbf{m}.\end{aligned}$$

Adding these equations we get

$$\begin{aligned}&\mathbf{P}(\Sigma_{\tau_0}) + \mathbf{P}(\Sigma_{\tau_1}) + \lambda \int_{\Sigma_{\tau_0}} (\tau_0 - f) \mathbf{d}\mathbf{m} + \lambda \int_{\Sigma_{\tau_1}} (\tau_1 - f) \mathbf{d}\mathbf{m} \\ &\leq \mathbf{P}(\Sigma_{\tau_0} \cup \Sigma_{\tau_1}) + \mathbf{P}(\Sigma_{\tau_0} \cap \Sigma_{\tau_1}) + \lambda \int_{\Sigma_{\tau_0} \cup \Sigma_{\tau_1}} (\tau_0 - f) \mathbf{d}\mathbf{m} + \lambda \int_{\Sigma_{\tau_0} \cap \Sigma_{\tau_1}} (\tau_1 - f) \mathbf{d}\mathbf{m}\end{aligned}$$

which, by the sub-modularity of \mathbf{P} , yields

$$\int_{\Sigma_{\tau_0}} (\tau_0 - f) \mathbf{d}\mathbf{m} + \int_{\Sigma_{\tau_1}} (\tau_1 - f) \mathbf{d}\mathbf{m} \leq \int_{\Sigma_{\tau_0} \cup \Sigma_{\tau_1}} (\tau_0 - f) \mathbf{d}\mathbf{m} + \int_{\Sigma_{\tau_0} \cap \Sigma_{\tau_1}} (\tau_1 - f) \mathbf{d}\mathbf{m}$$

and consequently, we have

$$\int_{\Sigma_{\tau_1} \setminus \Sigma_{\tau_0}} (\tau_1 - f) \mathbf{d}\mathbf{m} \leq \int_{\Sigma_{\tau_1} \setminus \Sigma_{\tau_0}} (\tau_0 - f) \mathbf{d}\mathbf{m}.$$

Therefore, we conclude the proof since $(\tau_1 - \tau_0)\mathbf{m}(\Sigma_{\tau_1} \setminus \Sigma_{\tau_0}) \leq 0$. \square

REMARK 3.14. *It follows that the minimizer Σ_τ of $\mathcal{E}_{\mathbb{P},\tau}^\lambda$ is unique up to a \mathbf{m} -null set, except for at most countably many values of τ . Indeed, for each $\tau \in \mathbb{R}$, fix a minimizer Σ_τ of $\mathcal{E}_{\mathbb{P},\tau}^\lambda$. Then, the function $\tau \mapsto \mathbf{m}(\Sigma_\tau)$ is monotone and, consequently, continuous except for at most countably many values of τ .*

Now, let us show that if τ is a continuity point, then Σ_τ is unique up to a \mathbf{m} -null set. Suppose, on the contrary, that there exist two distinct minimizers Σ_τ^1 and Σ_τ^2 such that $\mathbf{m}(\Sigma_\tau^1 \triangle \Sigma_\tau^2) > 0$. Then, for any $\tau' < \tau$, we have $\Sigma_\tau^1 \cup \Sigma_\tau^2 \subseteq \Sigma_{\tau'}$ up to a \mathbf{m} -null set, and for any $\tau^* > \tau$, we have $\Sigma_\tau^1 \cap \Sigma_\tau^2 \supseteq \Sigma_{\tau^*}$ up to a \mathbf{m} -null set. However, this implies that

$$\mathbf{m}(\Sigma_{\tau'}) \geq \mathbf{m}(\Sigma_{\tau^*}) + \mathbf{m}(\Sigma_\tau^1 \triangle \Sigma_\tau^2)$$

so τ cannot be a continuity point.

According to the above remark, we can define \mathfrak{Q} as a countable dense subset of \mathbb{R} such that the minimizer Σ_τ of $\mathcal{E}_{\mathbb{P},\tau}^\lambda$ is unique for every $\tau \in \mathfrak{Q}$. Thus, the next result is deduced:

LEMMA 3.15. *Let Σ_τ be a minimizer of $\mathcal{E}_{\mathbb{P},\tau}^\lambda$. Then, the function*

$$u(x) := \sup \{ \tau \in \mathfrak{Q} : x \in \Sigma_\tau \}$$

is the unique minimizer of $\mathcal{E}_{\mathbb{P}}^\lambda$ in $L^2(X, \mathbf{m})$.

PROOF. Let us first see that $u \in L^2(X, \mathbf{m})$. Since $\mathcal{E}_{\mathbb{P},\tau}^\lambda(\Sigma_\tau) \leq 0 = \mathcal{E}_{\mathbb{P},\tau}^\lambda(\emptyset)$, we have that

$$(3.9) \quad \int_0^T \tau \mathbf{m}(\Sigma_\tau) \mathbf{d}\tau \leq \int_0^T \int_{\Sigma_\tau} f \mathbf{d}\mathbf{m} \mathbf{d}\tau.$$

for some $T \in \mathbb{R}$. Now, by Fubini's theorem,

$$\begin{aligned} \int_0^T \tau \mathbf{m}(\Sigma_\tau) d\tau &= \int_0^T \int_{\Sigma_0} \tau \chi_{\Sigma_\tau}(x) d\mathbf{m}(x) d\tau = \int_{\Sigma_{\lambda,0}} \int_0^T \tau \chi_{[0, u_\lambda(x)]}(\tau) d\tau d\mathbf{m}(x) \\ &= \int_{\Sigma_0} \int_0^{\min\{u(x), T\}} \tau d\tau d\mathbf{m}(x) = \int_{\Sigma_0} \frac{1}{2} (\min\{u(x), T\})^2 d\mathbf{m}(x). \end{aligned}$$

Similarly, we find that

$$\int_0^T \int_{\Sigma_\tau} f(x) d\mathbf{m}(x) d\tau = \int_{\Sigma_0} \min\{u(x), T\} f(x) d\mathbf{m}(x).$$

Consequently, by (3.9) we have

$$\begin{aligned} \frac{1}{2} \int_{\Sigma_0} (\min\{u, T\})^2 d\mathbf{m} &\leq \int_{\Sigma_0} \min\{u, T\} f d\mathbf{m} \\ &\leq \left(\int_{\Sigma_0} (\min\{u, T\})^2 d\mathbf{m} \int_{\Sigma_0} f^2 d\mathbf{m} \right)^{\frac{1}{2}}. \end{aligned}$$

From this, letting $T \rightarrow \infty$, it follows that

$$(3.10) \quad \int_{\{u>0\}} u^2 d\mathbf{m} \leq 4 \int_{\{u \geq 0\}} f^2 d\mathbf{m}.$$

Similarly, we get that

$$(3.11) \quad \int_{\{u<0\}} u^2 d\mathbf{m} \leq 4 \int_{\{u \leq 0\}} f^2 d\mathbf{m}.$$

Indeed, since $\mathbf{P}(E) = \mathbf{P}(X \setminus E)$ for every $E \in \mathcal{B}$, we have that $\{-u > -\tau\} = \{u < \tau\}$ is a minimizer of

$$\min_{\Sigma \in \mathcal{B}} \left\{ \mathbf{P}(\Sigma) + \lambda \int_{\Sigma} (f - \tau) d\mathbf{m} \right\} = \min_{\Sigma \in \mathcal{B}} \left\{ \mathbf{P}(\Sigma) + \lambda \int_{\Sigma} (-\tau - (-f)) d\mathbf{m} \right\}.$$

Therefore, it follows that if we replace f with $-f$ in $\mathcal{E}_{\mathbf{P}, \tau}^\lambda$ then we obtain $-u$ instead of u . It follows from (3.10) and (3.11) that $u \in L^2(X, \mathbf{m})$.

We define the set Σ_τ^- for $\tau \in \mathbb{R}$ as follows

$$\Sigma_\tau^- := \bigcup_{\substack{\tau' > \tau, \\ \tau' \in \mathcal{Q}}} \Sigma_{\tau'}.$$

Note that Σ_τ^- coincides with $E_\tau(u)$ up to a \mathbf{m} -null set. Next, we show that Σ_τ^- is a minimizer of $\mathcal{E}_{\mathbf{P}, \tau}^\lambda$. To do this, consider a decreasing sequence $\{\tau_n\}_{n \in \mathbb{N}} \subset \mathcal{Q}$ such that $\tau_n \downarrow \tau$ as $n \rightarrow \infty$. Since for any $\Sigma \in \mathcal{B}$ we have

$$\tilde{\mathcal{E}}_{\mathbf{P}, \tau_n}^\lambda(\Sigma) \geq \tilde{\mathcal{E}}_{\mathbf{P}, \tau_n}^\lambda(\Sigma_{\tau_n})$$

and $\chi_{\Sigma_{\tau_n}}$ is nonincreasing, i.e.

$$\chi_{\Sigma_{\tau_n}} \xrightarrow{n} \chi_{\Sigma_\tau^-} \quad \text{in } L^2(X, \mathbf{m}),$$

the lower semi-continuity of P with respect to the weak convergence in $L^2(X, \mathbf{m})$ yields

$$\mathcal{E}_{P,\tau}^\lambda(\Sigma) \geq \mathcal{E}_{P,\tau}^\lambda(\Sigma_\tau^-),$$

for any $\Sigma \in \mathcal{B}$. Therefore, Σ_τ^- is a minimizer of $\mathcal{E}_{P,\tau}^\lambda$.

Now let $v \in L^2(X, \mathbf{m})$. Since Σ_τ^- is a minimizer of $\tilde{\mathcal{E}}_{P,\tau}^\lambda$ we have that, for every $T > 0$,

$$(3.12) \quad \begin{aligned} & \int_{-T}^T \left(P(\Sigma_\tau^-) + \lambda \int_{\Sigma_\tau^-} (\tau - f) d\mathbf{m} \right) d\tau \\ & \leq \int_{-T}^T \left(P(E_\tau(v)) + \lambda \int_{E_\tau(v)} (\tau - f) d\mathbf{m} \right) d\tau. \end{aligned}$$

Now, for any $w \in L^2(X, \mathbf{m})$, by Fubini's theorem we have

$$\begin{aligned} & \int_{-T}^T \int_{E_\tau(w)} (\tau - f(x)) d\mathbf{m}(x) d\tau = \int_X \int_{-T}^T \chi_{E_\tau(w)}(x) (\tau - f(x)) d\tau d\mathbf{m}(x) \\ & = \int_X \int_{-T}^{\min\{w(x), T\}} (\tau - f(x)) d\tau d\mathbf{m}(x) = \frac{1}{2} \int_X ((\min\{w, T\} - f)^2 - (-T - f)^2) d\mathbf{m}. \end{aligned}$$

Thus, we obtain the equality

$$(3.13) \quad \int_{-T}^T \int_{E_\tau(w)} (\tau - f(x)) d\mathbf{m}(x) d\tau + \frac{1}{2} \int_X (T + f)^2 d\mathbf{m} = \frac{1}{2} \int_X (w - f)^2 d\mathbf{m} + \varphi(w, T)$$

where $\varphi(w, T)$ is defined as

$$\varphi(w, T) := \frac{1}{2} \int_X ((\min\{w, T\} - f)^2 - (w - f)^2) d\mathbf{m}.$$

Observe that

$$\lim_{T \rightarrow \infty} \varphi(w, T) = 0$$

for any $w \in L^2(X, \mathbf{m})$. Therefore, from (3.12) and (3.13), it follows that

$$\int_{-T}^T P(E_\tau(u)) d\tau + \frac{\lambda}{2} \int_X (u - f)^2 d\mathbf{m} + \varphi(u, T) \leq \int_{-T}^T P(E_\tau(v)) d\tau + \frac{\lambda}{2} \int_X (v - f)^2 d\mathbf{m} + \varphi(v, T).$$

Then, letting $T \rightarrow \infty$, we get

$$\text{TV}_P(u) + \frac{\lambda}{2} \int_X (u - f)^2 d\mathbf{m} \leq \text{TV}_P(v) + \frac{\lambda}{2} \int_X (v - f)^2 d\mathbf{m}.$$

Therefore, since $v \in L^2(X, \mathbf{m})$ is arbitrary, u is a minimizer of \mathcal{E}_P^λ . □

REMARK 3.16. Note that, in the case that $0 \leq f \leq 1$:

- if $\tau > 1$, then \emptyset is the unique (up to \mathbf{m} -null sets) minimizer of $\mathcal{E}_{P,\tau}^\lambda$.
- if $\tau < 0$, then X is the unique (up to \mathbf{m} -null sets) minimizer of $\mathcal{E}_{P,\tau}^\lambda$.

Therefore, $0 \leq u \leq 1$ and the proof of Lemma 3.15 becomes much simpler.

Consequently, we have defined a linkage between the minimizers of \mathcal{E}_P^λ and $\mathcal{E}_{P,\tau}^\lambda$:

PROPOSITION 3.17. *The function u is minimizer of \mathcal{E}_P^λ if and only if $E_\tau(u)$ is a minimizer of $\mathcal{E}_{P,\tau}^\lambda$ for every $\tau \in \mathbb{R}$.*

As in [48, Proposition 2.6], we observe that the change of τ from almost any value to all value in \mathbb{R} is achieved by approximation. Next, we finally show the (λ, μ) -linkage in PMS.

PROOF OF THEOREM 3.8. Firstly, we characterize the minimizer of $\mathcal{F}_P^\mu(\cdot, c_1, c_2)$ for $0 \leq c_2 < c_1 \leq 1$ and $\mu > 0$. Let u be the minimizer of \mathcal{E}_P^λ with $\lambda = 2\mu(c_1 - c_2)$. Thanks to Proposition 3.17 we observe that $\Sigma^* := E_\tau(u)$ with $\tau = (c_1 + c_2)/2$ is the minimizer of $\mathcal{E}_{P,\tau}^\lambda$. Let us suppose that $0 < \mathbf{m}(\Sigma^*) < \mathbf{m}(X)$. Then, it is direct to show that Σ^* is also a minimizer of $\mathcal{E}_{P,\tau}^\lambda + C$ for any $C \in \mathbb{R}$; in particular for

$$C := \mu \int_X (c_2 - f)^2 \mathrm{d}\mathbf{m}.$$

On the other hand, for any $\Sigma \in \mathcal{B}$,

$$\begin{aligned} \mathcal{E}_{P,\tau}^\lambda(\Sigma) + C &= \mathbf{P}(\Sigma) + \lambda \int_\Sigma (\tau - f) \mathrm{d}\mathbf{m} + C \\ &= \mathbf{P}(\Sigma) + \mu \int_\Sigma ((c_1 - f)^2 - (c_2 - f)^2) \mathrm{d}\mathbf{m} + C \\ &= \mathbf{P}(\Sigma) + \mu \left(\int_\Sigma (c_1 - f)^2 \mathrm{d}\mathbf{m} + \int_{X \setminus \Sigma} (c_2 - f)^2 \mathrm{d}\mathbf{m} \right) = \mathcal{F}_P^\mu(\Sigma, c_1, c_2). \end{aligned}$$

Therefore $\mathcal{F}_P^\mu(\Sigma^*, c_1, c_2) \leq \mathcal{F}_P^\mu(\Sigma, c_1, c_2)$ for any $\Sigma \in \mathcal{B}$. In this sense, we note that the minimizers of $\mathcal{F}_P^\mu(\Sigma^*, \cdot, \cdot)$ can be computed direct by

$$c_1^* := \int_{\Sigma^*} f \mathrm{d}\mathbf{m}, \quad c_2^* := \int_{X \setminus \Sigma^*} f \mathrm{d}\mathbf{m}.$$

Therefore, if a tuple (Σ^*, c_1^*, c_2^*) satisfies (3.5), it is a partial minimizer provided that $c_2^* < c_1^*$. We note that, if $\tau^* = (c_1^* - c_2^*)/2$, we have that

$$\mathcal{E}_{P,\tau^*}^\lambda(\Sigma^*) \leq \mathcal{E}_{P,\tau^*}^\lambda(\emptyset) = 0,$$

which implies that $\tau^* \leq c_1^*$. If $c_1^* = c_2^*$, then \emptyset is a minimizer of $\mathcal{E}_{P,\tau^*}^\lambda$ thus, by uniqueness (up to a \mathbf{m} -null set) of the minimizer, we get $\mathbf{m}(\Sigma^*) = 0$, a contradiction. Therefore, $c_2^* < c_1^*$. \square

3.4. Applications

In this section, we show how our theoretical results extend the applicability of the CV model for image segmentation to settings beyond those for which it was originally designed (see Section 1.3). Specifically, we consider a locally finite weighted discrete graph $G = (V_G, E_G)$, where the edge weights $\{w_{xy}\}_{x,y \in V_G}$ are defined using the Gaussian kernel

$$\eta_\sigma(x, y) := \exp\left(-\frac{d(x, y)}{\sigma^2}\right),$$

for some distance function d on V_G . By Example 3.2.3, we know that $(G, \mathbf{m}, \mathbf{P})$ is a PMS, with

$$\mathbf{m}(A) := \sum_{x \in A} \sum_{y \sim x} w_{xy}, \quad \mathbf{P}(A) = \sum_{x \in A} \sum_{y \in V_G \setminus A} w_{xy}, \quad \text{for all } A \subseteq V_G.$$

In this setting, for $f : V_G \rightarrow [0, 1]$, $\Sigma \subseteq V_G$, $u \in L^2(V_G, \mathbf{m})$, and $c_1, c_2 \in [0, 1]$, the functionals \mathcal{E}_P^λ and \mathcal{F}_P^μ can be as follows:

$$\begin{aligned}\mathcal{F}_P^\mu(\Sigma, c_1, c_2) &= P(\Sigma) + \mu \sum_{x \in \Sigma} (c_1 - f(x))^2 \mathbf{m}(\{x\}) + \mu \sum_{x \in V_G \setminus \Sigma} (c_2 - f(x))^2 \mathbf{m}(\{x\}), \\ \mathcal{E}_P^\lambda(u) &= \frac{1}{2} \sum_{x, y \in V_G} |u(x) - u(y)| w_{xy} + \frac{\lambda}{2} \sum_{x \in V_G} (u(x) - f(x))^2 \mathbf{m}(\{x\}).\end{aligned}$$

Based on the results in Section 3.3, we establish that minimizing the ROF functional suffices to obtain a partial minimizer of \mathcal{F}_P^μ . To compute an approximate minimizer of \mathcal{E}_P^λ , we utilize the implementation provided in [40], which is based on the Iteratively Reweighted Least Squares (IRLS) method [80].

In our experiments, we specify the parameters ε , σ , and K used in this method, which we denote as $\text{IRLS}_{(\varepsilon, \sigma, K)}$. Here, σ corresponds to the parameter of the kernel η_σ , while K represents the number of active edges per vertex (i.e., edges with $w_{xy} \neq 0$), determined by selecting the K nearest neighbours for each node. Based on these considerations, we now present a strategy, formulated using Theorem 3.8, to estimate a minimizer of \mathcal{F}_P^μ :

Algorithm 3.1 \mathcal{F}_P^μ approximate minimizer

Require: $k = 1$, $\mu > 0$, $(c_1, c_2) \in [0, 1]^2$ s.t. $c_1 > c_2$.

$w \leftarrow \mathcal{E}_P^\lambda$ minimizer using $\text{IRLS}_{(\varepsilon, \sigma, K)}$ with $\lambda = 2\mu(c_1 - c_2)$.

$\Sigma_0 \leftarrow \emptyset$.

$\Sigma_1 \leftarrow \{x \in V : w(x) > \frac{1}{2}(m_1 - m_0)\}$.

while $|\Sigma_k \Delta \Sigma_{k-1}|^2 > \varepsilon_{tol} \wedge k < n_{max}$ **do**

$m_0 \leftarrow |\Sigma_k|^{-1} \sum_{x \in \Sigma_k} f(x)$, $m_1 \leftarrow |V \setminus \Sigma_k|^{-1} \sum_{x \in V \setminus \Sigma_k} f(x)$.

$w \leftarrow \mathcal{E}_P^\lambda$ minimizer using $\text{IRLS}_{(\varepsilon, \sigma, K)}$ with $\lambda = 2\mu(c_1 - c_2)$.

$\Sigma_{k+1} \leftarrow \{x \in V : w(x) > \frac{1}{2}(m_1 - m_0)\}$, $k \leftarrow k + 1$.

end while

return Σ_k, w

where n_{max} and ε_{tol} denote the maximum number of iterations and the tolerance of the algorithm, respectively. In our experiments, we set $n_{max} = \varepsilon_{tol} = 10$ and we take d as the Euclidean distance. We note that the above scheme is equivalent to those presented in [34, 129], proposed for the classical CV model in its isotropic and anisotropic forms, respectively.

3.4.1. Nonlocal image segmentation. As a first example, we define $f : \Omega \rightarrow \{0, 1\}$ such that the image represented by f consists of diagonal lines with added noise. The noise fulfills the condition that the closer it is to a line, the denser it is (cf. figures 3.1a and 3.1b). Suppose that $\Omega := \{1, \dots, n\} \times \{1, \dots, m\}$, where (n, m) is the image size. Taking into account the particular features of this image, we divide Ω into disjoint subsets $\{V_i\}_{i=1}^{10}$ as follows:

$$V_i := \begin{cases} \{x \in \Omega : \bar{f}(x) \leq D_i(\bar{f}(\Omega))\} & \text{if } i = 1, \\ \{x \in \Omega \setminus V_{i-1} : \bar{f}(x) \leq D_i(\bar{f}(\Omega))\} & \text{if } i > 1, \end{cases} \quad \text{s.t.} \quad \bar{f}(x) := \sum_{y \in B_1(x)} \frac{f(y)}{|B_1(x)|},$$

$B_1(x)$ is the ℓ^∞ unit ball in Ω centered at x , and $D_i(\bar{f}(\Omega))$ is the i -th decile of the finite set $\bar{f}(\Omega) \subseteq \mathbb{R}$. Then, we minimize the CV functional on each $(V_i, \nu_{G_i}, \text{Per}_{G_i})$ (here $G_i = (V_i(G_i), E(G_i))$). With this procedure, the segmentation process is carried out independently

on each element of the partition, which is constructed taking into account the density of 1's in the neighborhood of each node of Ω .

We apply the Algorithm 3.1 with $\text{IRLS}_{(3,0,3,10)}$ and compare our approach with the classical Euclidean CV segmentation. We computed the latter by making use of the algorithm proposed in [34], conceptually equivalent to ours. In Figures 3.1c and 3.1d, we present the segmentations that differ the least, in quadratic error terms, from the original denoised image shown in Figure 3.1b. In these examples, we can see that these approaches produce noticeably different segmentations, especially where the density of the noise is highest. In fact, if we compute the Frobenius norm of the differences between the denoising results and Figure 3.1b, we see that our approach, in this case, is 13.27% more accurate than the classical one. This is due to the way in which the segmentation process is carried out. The classical approach handles the image as a whole, solving both problems from a global viewpoint. Our approach, however, solves both problems in specific regions of the image, which have been previously identified through a particular criterion (the noise density around each pixel). Therefore, our proposal provides higher flexibility in the segmentation or denoising process.

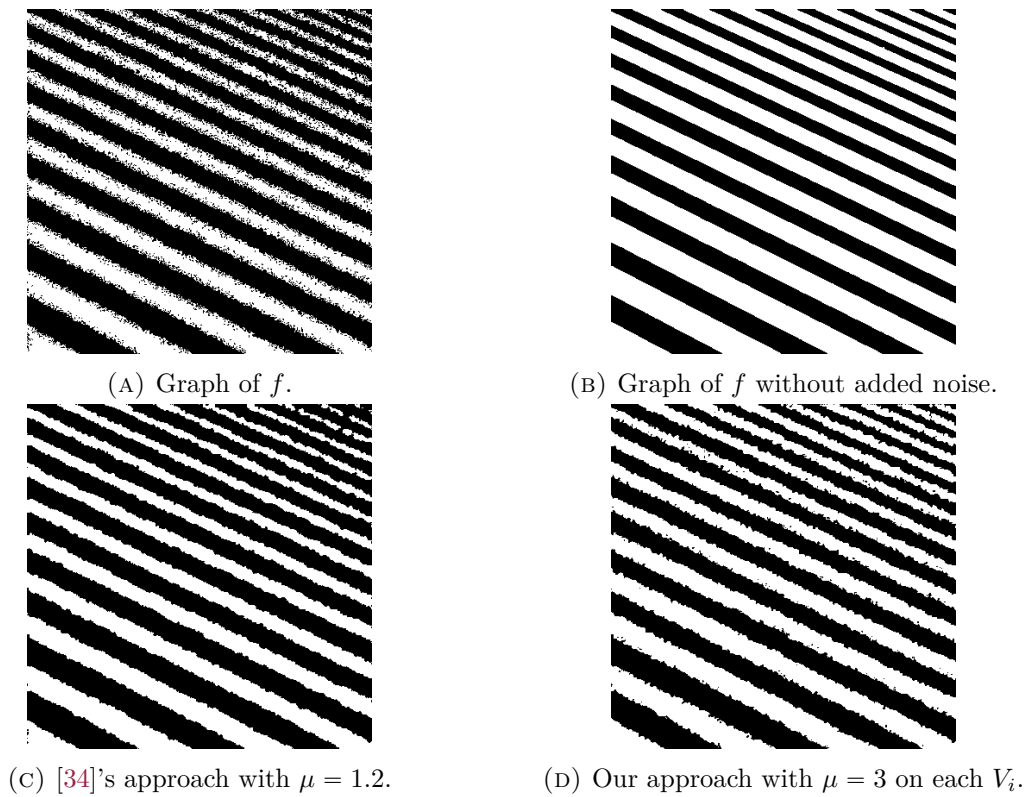


FIGURE 3.1. Comparison of approximate solutions to the CV model

3.4.2. Labeling. Additionally, extending the ROF and CV models to perimeter measure spaces allows us to use these models in other fields beyond image processing. In the next example, we show an application of these models in a multiclass classification problem in higher-dimensional spaces: the *labeling problem*.

We consider again the PMS $(G, \mathbf{m}, \mathbf{P})$ associated to a weighted graph. The labeling problem consists in assigning a label from the label set $\{y_i\}_{i=1}^k$ to each vertex in V_G . As the initial condition, we assume that labels have already been assigned to each of the vertices in $\{x_i\}_{i=1}^m$ for a given $m \leq k$, i.e., $y_{j_i} \in \{y_j\}_{j=1}^k$ is already assigned to x_i for $1 \leq i \leq m$. One of the methods most commonly used to solve this problem is the so-called Laplacian learning

algorithm. It finds an approximate solution as follows: First, an approximate solution $u = (u_i)_{i=1}^k : V_G \rightarrow [0, 1]^k$ of the following PDE on the graph is found:

$$(3.14) \quad \begin{cases} Lu(x_i) = 0 & \text{if } i > m, \\ u(x_i) = e_{j_i} & \text{otherwise,} \end{cases}$$

where $\{e_i\}_{i=1}^k$ is the canonical basis of \mathbb{R}^k and L is the graph Laplacian functional defined as

$$Lu(x) := \sum_{y \in V} w_{xy}(u(x) - u(y)) \quad \text{for } x \in V.$$

Then, a label is assigned to each $x \in V_G \setminus \{x_i\}_{i=1}^m$ corresponding to the largest component of $u(x)$. Unfortunately, this labeling method can be inconsistent and is largely influenced by the number of vertices per label initially known [36, 136].

Alternatively, we use the CV model to improve the results. First, we obtain an approximate solution u of (3.14) by using the well-known algorithm proposed in [167]. However, we then follow a different approach to assign the labels, first using the CV model on a modified version of each u_i , $1 \leq i \leq k$. To begin with, let

$$\tilde{u} = (\tilde{u}_i)_{i=1}^k : \tilde{u}_i(x) := u_i(x) - w_i, \quad x \in V_G,$$

where w_i is the mean of $\{u_i(x)\}_{x \in V_G}$; and then normalize between 0 and 1 (without renaming) the values of $\{\tilde{u}_i(x)\}_{1 \leq i \leq k}$ for each $x \in V_G$. This first step balances out the prevalence of the different labels. Then, for each $1 \leq i \leq k$, we use Algorithm 3.1 with $f = \tilde{u}_i$ and denote the output by (Σ_i, \tilde{w}_i) . Finally, we define $v := (\tilde{w}_i \chi_{\Sigma_i})_{i=1}^k : V \rightarrow \mathbb{R}^k$ and assign a label to $x \in V_G \setminus \{x_i\}_{i=1}^m$ according to the largest component of $v(x)$.

In order to evaluate the effectiveness of this post-processing application, we compare it with the PoissonMBO learning [35], which, to the best of our knowledge, is one of the best ways to solve the labeling problem (see, for example, [35, Section 4]). To compare the accuracy, we use three databases composed by $k = 10$ categories: MNIST [111], FashionMNIST [163] (in both cases, $n = 60000$ and $V \subseteq \{0, 1\}^{784}$) and CIFAR-10 [108] ($n = 70000$ and $V \subseteq \{0, \dots, 255\}^{3072}$). We perform experiments with varying number n_k of initially labeled elements per label. To compute the approximate solutions of the Laplacian (u according to the previous notation) and PoissonMBO learnings, we use the source code provided in [35]; while $v = (\tilde{w}_i \chi_{\Sigma_i})_{i=1}^{10}$ is computed by minimizing \mathcal{F}_P^5 (via Algorithm 3.1 with IRLS $_{(3,1,4)}$). Note that, to reduce the computational cost of obtaining v , we have split the set of vertices V_G into 30 disjoint subsets $\{V_i\}_{i=1}^{30}$ of equal size and used the CV model on each of the V_i with $f = \tilde{u}|_{V_i}$.

The results are shown in Tables 3.1, 3.2 and 3.3; where the same scheme as the one in [35] is used. To be more precise, we show the average accuracy for each of the approaches over 100 trials together with the standard deviation inside the brackets.

| n_k | 1 | 2 | 3 | 4 | 5 |
|---------------------|------------|------------|-------------|-------------|-------------|
| PoissonMBO learning | 96.1 (2.5) | 97.0 (0.2) | 97.2 (0.1) | 97.2 (0.1) | 97.2 (0.1) |
| Laplacian learning | 16.8 (6.1) | 27.5 (7.8) | 41.1 (12.4) | 57.9 (11.5) | 69.1 (12.3) |
| CV post-processing | 83.3 (4.7) | 94.2 (2.6) | 95.5 (0.3) | 96.1 (0.3) | 96.3 (0.3) |

TABLE 3.1. Average accuracy for the MNIST dataset

| n_k | 1 | 2 | 4 | 8 | 16 |
|---------------------|------------|------------|------------|------------|------------|
| PoissonMBO learning | 61.9 (5.8) | 67.0(4.9) | 72.2 (2.5) | 74.2 (2.1) | 76.9 (1.8) |
| Laplacian learning | 17.4 (7.2) | 32.0 (8.1) | 51.6 (5.9) | 69.7 (3.3) | 74.6 (1.3) |
| CV post-processing | 64.3 (6.7) | 65.4 (5.8) | 69.5 (3.3) | 72.1 (1.7) | 76.3 (1.3) |

TABLE 3.2. Average accuracy for the FashionMNIST dataset

| n_k | 10 | 20 | 40 | 80 | 160 |
|---------------------|------------|------------|------------|------------|------------|
| PoissonMBO learning | 61.6 (2.3) | 64.6 (1.7) | 66.7 (0.7) | 68.5 (0.6) | 70.4 (0.5) |
| Laplacian learning | 21.5 (7.3) | 38.2 (7.9) | 55.0 (4.7) | 62.8 (1.8) | 66.5 (1.2) |
| CV post-processing | 57.8 (3.3) | 62.0 (2.1) | 66.2 (1.5) | 68.3 (0.9) | 70.5 (0.5) |

TABLE 3.3. Average accuracy for the CIFAR-10 dataset

According to the tables, the application of the CV post-processing to the Laplacian approximation provides a considerable increase in accuracy in the labeling process. This increase is significant in all three datasets, specially in the cases where the number n_k is small. Moreover, the accuracy obtained is comparable to that of the PoissonMBO learning. Consequently, we see that our CV model proposal can improve existing processes in settings different from image segmentation, such as the classical Laplacian learning algorithm for labeling.

Anisotropic (λ, μ) –linkage

In this chapter, we address the (λ, μ) –connection defined in Theorem 1.5 within anisotropic formulations of the ROF and CV problems. Furthermore, we study the relationship between J_f and some minimizers of the CV problem in a specific case. Throughout this chapter, we assume that Φ is an anisotropy as defined in Section 2.3.2. Accordingly, the anisotropic ROF problem is defined as

$$\min_{u \in L^2(\Omega)} \left\{ \mathcal{E}_\Phi^\lambda(u) := \text{TV}_\Phi(u) + \frac{\lambda}{2} \int_\Omega (u - f)^2 dx \right\},$$

and the anisotropic Chan–Vese problem is formulated as

$$\min_{\substack{\Sigma \subseteq \Omega \\ c_1, c_2 \in [0,1]}} \left\{ \mathcal{F}_\Phi^\mu(\Sigma, c_1, c_2) := \text{Per}_\Phi(\Sigma; \Omega) + \mu \int_\Sigma (c_1 - f)^2 dx + \mu \int_{\Omega \setminus \Sigma} (c_2 - f)^2 dx \right\},$$

which are the anisotropic counterparts of problems (1.3) and (1.7), respectively. In particular, we focus on the case where $\Phi(\xi) = |\xi|_1$. For the sake of clarity, we denote the above functionals by \mathcal{E}_1^λ and \mathcal{F}_1^μ , the anisotropic total variation by $|Du|_1$ or TV_1 , and the anisotropic perimeter by Per_1 .

Firstly, we establish the (λ, μ) –connection between \mathcal{E}_1^λ and \mathcal{F}_1^μ when $f \in \text{PCR}(\Omega)$, based on the characterization of the minimizer of \mathcal{E}_1^λ (see Section 2.3.4). Specifically, in Section 4.1 we prove the following result:

THEOREM 4.1. *Let $f \in \text{PCR}(\Omega)$, and assume that Ω is not the first component of any minimizer of \mathcal{F}_1^μ . Then, there exist $0 \leq \tilde{c}_2 < \tilde{c}_1 \leq 1$ and $\Sigma \subseteq \Omega$ such that, if u is the minimizer of \mathcal{E}_1^λ with $\lambda = 2\mu(\tilde{c}_1 - \tilde{c}_2)$ satisfying*

$$c_1 = \int_\Sigma f dx, \quad c_2 = \int_{\Omega \setminus \Sigma} f dx, \quad \Sigma = \left\{ x \in \Omega : u(x) \geq \frac{\tilde{c}_1 + \tilde{c}_2}{2} \right\},$$

then $(\Sigma, \tilde{c}_1, \tilde{c}_2)$ is a minimizer of \mathcal{F}_1^μ , and $(\chi_\Sigma, \tilde{c}_1, \tilde{c}_2)$ is a minimizer of $\tilde{\mathcal{F}}_1^\mu$ (see Equation (4.2)).

In this case, we would like to emphasize that, unlike the isotropic case (see Theorem 1.5), this result defines a minimizer through the (λ, μ) –linkage, whereas in the PMS setting (see Chapter 3) only a *partial minimizer* is obtained, which encompasses the isotropic (λ, μ) –linkage.

Additionally, we study the proposed (λ, μ) –connection for the n -phase case, which we have introduced in Theorem 1.6. To do so, we extend the functional $\mathcal{F}^{\mu,n}$ to the anisotropic setting, i.e., the minimization

$$(4.1) \quad \min_{\substack{\mathbf{\Lambda} \in \mathcal{P}_n(\Omega) \\ \mathbf{c} \in [0,1]^n}} \left\{ \mathcal{F}_\Phi^{\mu,n}(\mathbf{\Lambda}, \mathbf{c}) := \sum_{i=1}^n \text{Per}_\Phi(\Lambda_i; \Omega) + \mu_i \int_{\Lambda_i} (c_i - f)^2 dx \right\}$$

This chapter contains the results of [128, 129].

as well as the functional $\mathcal{E}^{\lambda, n}$, which we generalize as

$$\min_{\substack{\Sigma \in \mathcal{P}_n^*(\Omega) \\ \tau \in \Gamma^{n-1}([0,1])}} \left\{ \mathcal{E}_{\Phi}^{\lambda, n}(\Sigma, \tau) := \sum_{i=1}^{n-1} \left(\text{Per}_{\Phi}(\Sigma_i; \Omega) + \lambda \int_{\Sigma_i} (\tau_i - f) dx \right) \right\}$$

where $\mathcal{P}_n(\Omega)$, $\mathcal{P}_n^*(\Omega)$, and $\Gamma^n([0,1])$ are defined as in Chapter 1. In Section 4.2, we prove that this relationship, formulated for $\mathcal{E}_{\Phi}^{\lambda, n}$ and a variant of $\mathcal{F}_{\Phi}^{\mu, n}$, does not hold for any anisotropy $\Phi(\xi)$. However, we observe that the functionals $\mathcal{E}_{\Phi}^{\lambda}$ and $\mathcal{E}_{\Phi}^{\lambda, n}$ do define a valid relationship for any anisotropy. Utilizing these relations, in Section 4.4, we compare the relationships between $\mathcal{E}_{\Phi}^{\lambda}$, \mathcal{F}_{Φ}^{μ} , $\mathcal{E}_{\Phi}^{\lambda, n}$, and $\mathcal{F}_{\Phi}^{\mu, n}$ for both the isotropic case and the anisotropic case $\Phi(\xi) = |\xi|_1$, highlighting the advantages of the anisotropic approach.

Finally, we investigate whether the preservation of the jump set of the minimizer u of \mathcal{E}^{λ} , introduced in Theorem 1.1 (i.e., $J_u \subseteq J_f$), enables a characterization of the structure of the first component of \mathcal{F}^{μ} , namely Λ . Specifically, in Section 4.3 we discuss whether, if (Λ, c_1, c_2) minimizes \mathcal{F}_{Φ}^{μ} , it follows that $\partial\Lambda \subseteq J_f$. Additionally, we provide a weaker version of this inclusion, which holds whenever $f \in \text{PCR}(\Omega)$ and $\Phi(\xi) = |\xi|_1$.

4.1. Anisotropic (λ, μ) -linkage for $\Phi(\xi) = |\xi|_1$

In this section we prove the existence of a minimizer for \mathcal{F}_1^{μ} and provide the (λ, μ) -linkage for our anisotropic case. First, we need to consider a relaxation of \mathcal{F}_1^{μ} :

DEFINITION 4.2. *Let $\Omega \subseteq \mathbb{R}^2$ be an open set, and let f be a function such that $f(\Omega) \subseteq [0, 1]$. The relaxed functional $\tilde{\mathcal{F}}_1^{\mu}$ of \mathcal{F}_1^{μ} is defined as follows:*

$$\begin{aligned} \tilde{\mathcal{F}}_1^{\mu} &: L^2(\Omega) \times [0, 1]^2 \rightarrow [0, +\infty], \\ \tilde{\mathcal{F}}_1^{\mu}(u, c_1, c_2) &= \text{TV}_1(u) + \mu \int_{\Omega} (u(c_1 - f)^2 + (1 - u)(c_2 - f)^2) dx + \int_{\Omega} \mathbb{I}_{[0,1]}(u) dx, \end{aligned}$$

where $\mu > 0$. In this functional, the partition $\{\Lambda, \Omega \setminus \Lambda\}$ is relaxed by the functions u and $1 - u$, providing a convex behavior with respect to u . To ensure that the range of the first component lies within $[0, 1]$, we include the indicator function of $[0, 1]$, defined as:

$$\mathbb{I}_{[0,1]}(x) := \begin{cases} 0 & \text{if } x \in [0, 1], \\ +\infty & \text{otherwise.} \end{cases}$$

PROPOSITION 4.3. *There exists a triplet $(u, c_1, c_2) \in BV(\Omega) \times [0, 1]^2$ such that it is a minimizer of $\tilde{\mathcal{F}}_1^{\mu}$ and satisfies $u(\Omega) \subseteq [0, 1]$.*

PROOF. The proof follows directly from the direct method in the calculus of variations. For the sake of completeness, we provide a sketch of the argument. Since $\tilde{\mathcal{F}}_1^{\mu} \geq 0$ on $L^2(\Omega) \times [0, 1]^2$, the infimum

$$m = \inf \left\{ \tilde{\mathcal{F}}_1^{\mu}(w, a, b) : (w, a, b) \in L^2(\Omega) \times [0, 1]^2 \right\}$$

is well-defined. Let $\{(u_n, c_{1,n}, c_{2,n})\}_{n \in \mathbb{N}}$ be a minimizing sequence for $\tilde{\mathcal{F}}_1^{\mu}$. Since $u_n(\Omega) \subseteq [0, 1]$, there exists a subsequence (not relabeled) and $u \in L^2(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $L^2(\Omega)$ and $u(x) \in [0, 1]$ a.e. in Ω .

Moreover, by the lower semicontinuity of the anisotropic total variation and the coercivity of the $|\cdot|_1$ norm, we may assume that $u \in BV(\Omega)$ and that $u_n \rightarrow u$ in $L^1(\Omega)$. On the other hand, since $c_{i,n} \in [0, 1]$ for $i = 1, 2$, there exists a subsequence (not relabeled) and $c_i \in [0, 1]$ such that $c_{i,n} \rightarrow c_i$ for $i = 1, 2$.

Finally, from the lower semicontinuity of the anisotropic total variation, we obtain:

$$\tilde{\mathcal{F}}_1^\mu(u, c_1, c_2) \leq \liminf_{n \rightarrow \infty} \tilde{\mathcal{F}}_1^\mu(u_n, c_{1,n}, c_{2,n}) = m,$$

which shows that (u, c_1, c_2) is indeed a minimizer of $\tilde{\mathcal{F}}_1^\mu$. \square

Now let us fix $c_1, c_2 \in [0, 1]$ such that $c_2 \leq c_1$; and consider $\tilde{\mathcal{F}}_{1,c}^\mu$ as depending solely on $u \in L^2(\Omega)$ by fixing the values of c_1 and c_2 , i.e. $\tilde{\mathcal{F}}_{1,c}^\mu(\cdot) := \tilde{\mathcal{F}}_1^\mu(\cdot, c_1, c_2)$. We observe that if $c_1 = c_2$, the only minimizers of $\tilde{\mathcal{F}}_{1,c}^\mu(\cdot)$ are constant functions. Taking this into account, we proceed to prove the linkage between \mathcal{E}_1^λ and $\tilde{\mathcal{F}}_{1,c}^\mu$:

THEOREM 4.4. *Let $f \in \text{PCR}(\Omega)$. Given $0 \leq c_2 < c_1 \leq 1$, let w be the minimizer of \mathcal{E}_1^λ with parameter $\lambda = 2\mu(c_1 - c_2)$. Then, if*

$$\Sigma := \left\{ x \in \Omega : w(x) > \frac{c_1 + c_2}{2} \right\},$$

$\tilde{u} = \chi_\Sigma$ is a minimizer of $\tilde{\mathcal{F}}_{1,c}^\mu$.

PROOF. First, note that the functional $\tilde{\mathcal{F}}_{1,c}^\mu$ is convex and lower semicontinuous in $L^2(\Omega)$. Therefore, the existence of a minimizer $\tilde{u} \in BV(\Omega)$ to $\tilde{\mathcal{F}}_{1,c}^\mu$ is guaranteed. Moreover, applying [128, Theorem 2.1], the associated Euler-Lagrange equation is given by

$$(4.2) \quad \text{div}(z_{\tilde{u}}) = \mu(c_1 - c_2)(c_1 + c_2 - 2f) + h,$$

where $z_{\tilde{u}} \in \partial\text{TV}_1(\tilde{u})$ and $h \in \partial\mathbb{I}_{[0,1]}(\tilde{u})$. On the other hand, since u is a minimizer of \mathcal{E}_1^λ , the Euler-Lagrange equation for this functional holds, i.e., there exist $z_u \in \partial\text{TV}_1(u) \subseteq \partial\text{TV}_1(\tilde{u})$ by (2.10) such that

$$(4.3) \quad \text{div}(z_u) = 2\mu(c_1 - c_2)(u - f).$$

We set $z_{\tilde{u}} := z_u \in \partial\text{TV}_1(\tilde{u})$ and we prove that (4.2) is satisfied by z_u . By showing this, we establish that \tilde{u} is a minimizer of $\tilde{\mathcal{F}}_{1,c}^\mu$.

From (4.3), equation (4.2) is satisfied if and only if

$$2\mu(c_1 - c_2) \left(u - \frac{c_1 + c_2}{2} \right) = h.$$

It remains to verify that $h \in \partial\mathbb{I}_{[0,1]}(\tilde{u})$. We distinguish two cases:

- If $u < \frac{c_1 + c_2}{2}$, $\tilde{u} = 0$ and $h \in] - \infty, 0] = \partial\mathbb{I}_{[0,1]}(0)$.
- If $u \geq \frac{c_1 + c_2}{2}$, $\tilde{u} = 1$ and $h \in [0, +\infty[= \partial\mathbb{I}_{[0,1]}(1)$.

Thus, \tilde{u} satisfies (4.2), and the proof is complete. \square

REMARK 4.5. *The Euler-Lagrange system corresponding to functional $\tilde{\mathcal{F}}_1^\mu$ consists in (4.2) for u , and*

$$\int_{\Omega} u(c_1 - f) \, dx = \int_{\Omega} (1 - u)(c_2 - f) \, dx = 0.$$

Therefore, if (χ_Σ, c_1, c_2) is a minimizer of $\tilde{\mathcal{F}}_1^\mu$, then

$$(4.4) \quad c_1 = \int_{\Sigma} f \, dx, \quad c_2 = \int_{\Omega \setminus \Sigma} f \, dx.$$

We conclude with the proof of the (λ, μ) -linkage in our anisotropic setting:

PROOF OF THEOREM 4.1. First, observe that for any set $\Lambda \subseteq \Omega$ with finite perimeter, it follows that

$$\tilde{\mathcal{F}}_1^\mu(\chi_\Lambda, c_1, c_2) = \mathcal{F}_1^\mu(\Lambda, c_1, c_2),$$

which leads to the next inequality:

$$(4.5) \quad \min_{\substack{v \in L^2(\Omega) \\ c_1, c_2 \in [0,1]}} \tilde{\mathcal{F}}_1^\mu(v, c_1, c_2) \leq \min_{\substack{\Lambda \subseteq \Omega \\ c_1, c_2 \in [0,1]}} \mathcal{F}_1^\mu(\Lambda, c_1, c_2).$$

From Proposition 4.3, we know that minimizers of $\tilde{\mathcal{F}}_1^\mu$ exist, and each component of a minimizer satisfies the associated Euler-Lagrange equation from Remark 4.5. Let $(\tilde{u}, \tilde{c}_1, \tilde{c}_2)$ be one such minimizer. If $\tilde{c}_1 = \tilde{c}_2$, then \tilde{u} must be a constant function and then, (4.5) implies that $(\Omega, \tilde{c}_1, \tilde{c}_1)$ is a minimizer of \mathcal{F}_1^μ . Thus, we can assume $\tilde{c}_1 < \tilde{c}_2$ by hypothesis.

By Theorem 4.4, the set Σ satisfies that

$$(4.6) \quad \tilde{\mathcal{F}}_1^\mu(\tilde{u}, \tilde{c}_1, \tilde{c}_2) = \tilde{\mathcal{F}}_1^\mu(\chi_\Sigma, \tilde{c}_1, \tilde{c}_2) \geq \min_{c_1, c_2 \in [0,1]} \mathcal{F}_1^\mu(\Sigma, c_1, c_2).$$

Thus, $(\tilde{c}_1, \tilde{c}_2)$ satisfies (4.4). Combining (4.5) and (4.6), we conclude:

$$\min_{\substack{\Lambda \subseteq \Omega \\ c_1, c_2 \in [0,1]}} \mathcal{F}_1^\mu(\Lambda, c_1, c_2) \leq \tilde{\mathcal{F}}_1^\mu(\chi_\Sigma, \tilde{c}_1, \tilde{c}_2) \leq \min_{\substack{\Lambda \subseteq \Omega \\ c_1, c_2 \in [0,1]}} \mathcal{F}_1^\mu(\Lambda, c_1, c_2),$$

which proves that $(\Sigma, \tilde{c}_1, \tilde{c}_2)$ is a minimizer of \mathcal{F}_1^μ . \square

4.2. Anisotropic multiphase (λ, μ) -linkage

In this section, we prove that the linkage formulated in Theorem 1.6 is no valid by providing a general counterexample for any anisotropy Φ . Furthermore, we propose an alternative relationship between \mathcal{E}_Φ^λ and $\mathcal{E}_\Phi^{\lambda, n}$ that holds for any anisotropy. First, we define the variant of (1.10) for a given anisotropy Φ as follows

$$\hat{\mathcal{F}}_\Phi^{\mu, n}(\Omega, \mathbf{c}) := \sum_{i=1}^n \left(\text{Per}_\Phi(\cup_{j=1}^i \Omega_j; \Omega) + \mu_i \int_{\Omega_i} (c_i - f)^2 \, dx \right),$$

where $\Omega \in \mathcal{P}_n(\Omega)$, $\mathbf{c} \in [0, 1]^n$ and $\boldsymbol{\mu} \in [0, +\infty)^n$. Next, we hypothesize the anisotropic extension of the relationship proposed by [34] in the isotropic case:

CONJECTURE 4.6. *Let $(\Sigma, \boldsymbol{\tau}) \in \mathcal{P}_n^*(\Omega) \times [0, 1]^n$ be a pair such that*

$$\mathcal{E}_\Phi^{\lambda, n}(\Sigma, \boldsymbol{\tau}) \leq \mathcal{E}_\Phi^{\lambda, n}(\Lambda, \boldsymbol{\tau})$$

for any $\Lambda \in \mathcal{P}_n^*(\Omega)$. Suppose that $\Omega := \{\Omega_i\}_{i=1}^n = \{\Sigma_i \setminus \Sigma_{i-1}\}_{i=1}^n$ and $\boldsymbol{\tau} := \{\tau_i\}_{i=0}^{n-1}$ fulfills

$$\tau_i = \frac{c_{i+1} + c_i}{2}, \quad \text{with } c_i := \int_{\Omega_i} f \, dx, \quad \text{for } i \in \{1, \dots, n-1\}.$$

If $c_i > c_{i+1}$ for each i , then Ω is a minimizer of $\hat{\mathcal{F}}_\Phi^{\mu, n}(\cdot, \mathbf{c})$ with the values of $\boldsymbol{\mu} := \{\mu_i\}_{i=1}^n$:

$$(4.7) \quad \mu_1 = \frac{\lambda}{2(c_1 - c_2)}, \quad \mu_n = \frac{\lambda}{2(c_{n-1} - c_n)}, \quad \mu_i = \frac{\lambda(c_{i-1} - c_{i+1})}{2(c_{i-1} - c_i)(c_i - c_{i+1})} \quad 1 < i < n.$$

Before refuting this conjecture, we would like to remark a relationship that holds true between \mathcal{E}_Φ^λ , $\mathcal{E}_\Phi^{\lambda, n}$ and $\mathcal{F}_\Phi^{\mu, n}$. If $n = 2$ and u is the solution to \mathcal{E}_Φ^λ , we deduce that the upper level set

$$\Sigma_\tau := \{x \in \Omega : u(x) > \tau\}$$

is the minimizer of $\mathcal{E}_\Phi^{\lambda, 2}(\{\emptyset, \cdot, \Omega\}, \tau)$. The proof is provided for a general framework in Proposition 3.17, where this case is covered when $X = \Omega$, $\mathbf{m} = \mathcal{L}^2$ and $\mathbf{P}(\Sigma) = \text{Per}_\Phi(\Sigma; \Omega)$. This observation is significant because, for a fixed $\tau \in \mathbb{T}^{n-1}([0, 1])$, any $\Sigma \in \mathcal{P}_n^*(\Omega)$ satisfies the following expression:

$$\mathcal{E}_\Phi^{\lambda, n}(\Sigma, \tau) = \sum_{i=1}^{n-1} \mathcal{E}_\Phi^{\lambda, 2}(\{\emptyset, \Sigma_{\tau_i}, \Omega\}, \tau_i),$$

which implies that minimizing $\mathcal{E}_\Phi^{\lambda, n}$ is equivalent to minimizing $\mathcal{E}_\Phi^{\lambda, 2}$ ($n - 1$) times. Moreover, the minimizers of $\mathcal{E}_\Phi^{\lambda, 2}$ are determined by the minimizer of \mathcal{E}_Φ^λ . We summarize this result in the following corollary:

COROLLARY 4.7. *Let w be the \mathcal{E}_Φ^λ minimizer. Then, for a $\tau \in \mathbb{T}^{n-1}([0, 1])$ such that $\tau_i > \tau_{i+1}$, the minimizer $\Sigma \in \mathcal{P}_n^*(\Omega)$ of $\mathcal{E}_\Phi^{\lambda, n}(\cdot, \tau)$ is given by*

$$\Sigma_i = \{x \in \Omega : w(x) > \tau_i\}, \quad \text{for } i = 1, \dots, n - 1.$$

It is worth noting that this alternative relationship between \mathcal{E}_Φ^λ and $\mathcal{E}_\Phi^{\lambda, n}$ is inherently interesting due to its validity for any anisotropy. To explore the connection between these functionals and $\mathcal{F}_\Phi^{\mu, n}$, we examine a specific case. By Theorem 4.1, when $\Phi = |\cdot|_1$, $n = 2$, and μ is a constant vector, the following corollary holds:

COROLLARY 4.8. *Let w be the minimizer of \mathcal{E}_1^λ and Σ_τ the level set of w that solves $\mathcal{E}_1^{\lambda, 2}(\{\emptyset, \cdot, \Omega\}, \tau)$, where τ is defined as*

$$\tau = \frac{c_1 + c_2}{2}, \quad \text{with } c_1 := \int_{\Sigma_\tau} f \, dx, \quad c_2 := \int_{\Omega \setminus \Sigma_\tau} f \, dx.$$

Then, (Σ_τ, c_1, c_2) is a minimizer of \mathcal{F}_1^μ if $\mu = \lambda/(c_1 - c_2)$.

Apart from this case, the proposed relationship in Conjecture 4.6 is not valid for any anisotropy $\Phi(\xi)$, including the isotropic case itself. To present a general counterexample, we require some preliminary results. These results are an adaptation of the results in [16, Section 9]. Since our aim is to show some very particular examples, we only consider some specific settings.

DEFINITION 4.9. *Let $A \subseteq \mathbb{R}^2$ be a bounded open set with Lipschitz boundary. We say that A is calibrable with respect to the anisotropy Φ if there exists $z \in X_A$ such that $\Phi^0(z) \leq 1$ a.e. in A , $\text{div}(z)$ is constant in A and $[z, \nu^A] = -\Phi^0(\nu^A)$ at ∂A .*

DEFINITION 4.10. *Let $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^+ \cup \{+\infty\}$ be an anisotropy, and $A, B \subseteq \mathbb{R}^N$. We define $d_\Phi(A, B)$ as*

$$d_\Phi(A, B) = \inf\{\Phi(x - y) : x \in A, y \in B\}.$$

Likewise, we define the ball $B_R^{\Phi^0}(x)$ as

$$B_R^{\Phi^0}(x) := \{y \in \mathbb{R}^2 : \Phi^0(x - y) < R\}.$$

LEMMA 4.11. *Let $A \subseteq \mathbb{R}^2$ be a bounded open set with Lipschitz boundary, and let $\Omega := B_R^{\Phi^0}(0)$ be such that $\bar{A} \subseteq \Omega$. Then, there exists $z \in X_{\Omega \setminus \bar{A}}$ such that $\Phi^0(z) \leq 1$ a.e. in $\Omega \setminus \bar{A}$, $\operatorname{div}(z)$ is constant in $\Omega \setminus \bar{A}$, $[z, \nu^\Omega] = 0$ at $\partial\Omega$ and $[z, \nu^A] = -\Phi^0(\nu^A)$ at ∂A if and only if*

$$(4.8) \quad \Omega \setminus \bar{A} \in \arg \min_{E \in \mathcal{E}} \left\{ \frac{\operatorname{Per}_\Phi(E; \Omega \setminus \bar{A}) - \mathcal{H}^1(\partial^* E \cap \partial A)}{\mathcal{L}^2(E)} \right\},$$

with $\mathcal{E} := \{E \subseteq \Omega \setminus \bar{A} : \operatorname{Per}_\Phi(E) < +\infty, \mathcal{L}^2(E) > 0\}$.

PROOF. The proof follows the same ideas as the proof of [16, Theorem 5]. Suppose first that there is a vector field z satisfying the hypothesis of the Lemma. Then, by Green–Gauss formula (2.5),

$$\int_{\Omega \setminus \bar{A}} \operatorname{div}(z) \, dx = - \int_{\partial(\Omega \setminus \bar{A})} [z, \nu^A] \, d\mathcal{H}^1 = \operatorname{Per}_\Phi(A; \Omega).$$

and thus,

$$\operatorname{div}(z) = \frac{\operatorname{Per}_\Phi(A; \Omega)}{\mathcal{L}^2(\Omega \setminus \bar{A})}.$$

Applying once again Green–Gauss formula, for any $E \in \mathcal{E}$, we obtain

$$\mathcal{L}^2(E) \frac{\operatorname{Per}_\Phi(A; \Omega)}{\mathcal{L}^2(\Omega \setminus \bar{A})} = \int_E \operatorname{div}(z) \, dx = \int_{\partial E} [z, \nu] \, d\mathcal{H}^1 \geq \mathcal{H}^1(\partial^* E \cap \partial A) - \operatorname{Per}_\Phi(E; \Omega \setminus \bar{A}),$$

which shows the first implication.

Let us suppose now that $\Omega \setminus \bar{A}$ is a minimizer of the functional

$$E \mapsto \frac{\operatorname{Per}_\Phi(E; \Omega \setminus \bar{A}) - \mathcal{H}^1(\partial^* E \cap \partial A)}{\mathcal{L}^2(E)},$$

among all sets in \mathcal{E} . We now define the functional

$$F(\xi) := \int_{\Omega \setminus \bar{A}} (\operatorname{div}(\xi))^2 \, dx, \quad \text{with } \xi \in X_{\Omega \setminus \bar{A}}.$$

Therefore, arguing as in [17, Proposition 6.1 and Theorem 6.7] and [18, Proposition 3.5 and Theorem 5.3], one can prove that the following variational problem has a solution with unique divergence:

$$\min \left\{ F(\xi) : \begin{array}{l} \xi \in X_{\Omega \setminus \bar{A}}, \quad \Phi^0(\xi) \leq 1 \text{ a.e. in } \Omega \setminus \bar{A} \\ [\xi, \nu^\Omega] = 0 \text{ at } \partial\Omega \quad \text{and} \quad [\xi, \nu^A] = -\Phi^0(\nu^A) \text{ at } \partial A \end{array} \right\}.$$

Furthermore, for any minimizer ξ_{\min} , it holds that

$$\operatorname{div}(\xi_{\min}) \in L^\infty(\Omega \setminus \bar{A}) \cap BV(\Omega \setminus \bar{A}).$$

For any $\mu \in \mathbb{R}$, let Q_μ denote the μ -upper level set of $\operatorname{div}(\xi_{\min})$ within $\Omega \setminus \bar{A}$. The set Q_μ has finite perimeter, and the following equality is satisfied:

$$(4.9) \quad \int_{Q_\mu} \operatorname{div}(\xi_{\min}) \, dx = \mathcal{H}^1(\partial^* Q_\mu \cap \partial A) - \operatorname{Per}_\Phi(Q_\mu; \Omega \setminus \bar{A}).$$

If $\operatorname{div}(\xi_{\min})$ is not constant in $\Omega \setminus \bar{A}$ and equal to $\operatorname{Per}_\Phi(A)/\mathcal{L}^2(\Omega \setminus \bar{A})$, there would exist

$$\mu_0 > \frac{\operatorname{Per}_\Phi(A)}{\mathcal{L}^2(\Omega \setminus \bar{A})}$$

such that Q_{μ_0} is nonempty. Substituting into (4.9), it follows that $\Omega \setminus \bar{A}$ cannot be a minimizer of the given functional. \square

THEOREM 4.12. *Let $A \subseteq \mathbb{R}^2$ be a bounded open set with Lipschitz boundary such that $A = \cup_{i=1}^m C_i$ with C_i convex and calibrable and such that $\operatorname{dist}_\Phi(C_i, C_j) > 0$ for $i \neq j$, and let $\Omega := B_R^{\Phi^0}(0)$ be such that $A \subseteq \Omega$. Let $1 \leq k \leq m$, $\{i_1, \dots, i_k\} \subseteq \{1, \dots, m\}$ and consider the following variational problem*

$$(P)_{i_1, \dots, i_k} := \min_{E \in \mathcal{E}_{i_1, \dots, i_k}} \left\{ \operatorname{Per}_\Phi(E; \Omega) + \frac{\operatorname{Per}_\Phi(A; \Omega)}{\mathcal{L}^2(\Omega \setminus \bar{A})} \mathcal{L}^2(E \setminus \bar{A}) \right\},$$

with $\mathcal{E}_{i_1, \dots, i_k} := \left\{ \cup_{j=1}^k C_{i_j} \subseteq E \subseteq \Omega \setminus (\cup_{j=k+1}^m C_{i_j}) : \operatorname{Per}_\Phi(E) < +\infty \right\}$.

Then there exists $z \in X_{\Omega \setminus \bar{A}}$ such that $\Phi^0(z) \leq 1$ a.e. in $\Omega \setminus \bar{A}$, $\operatorname{div}(z)$ is constant in $\Omega \setminus \bar{A}$, $[z, \nu^\Omega] = 0$ at $\partial\Omega$ and $[z, \nu^A] = -\Phi^0(\nu^A)$ at ∂A if and only if $\cup_{j=1}^k C_{i_j}$ is a solution to $(P)_{i_1, \dots, i_k}$ for any $\{i_1, \dots, i_k\} \subseteq \{1, \dots, m\}$.

PROOF. Suppose first that there exist a vector field z as in the hypothesis and let us consider $E \in \mathcal{E}_{i_1, \dots, i_k}$ and $D := E \setminus \bar{A}$. Then, by Green–Gauss formula,

$$\frac{\operatorname{Per}_\Phi(A; \Omega)}{\mathcal{L}^2(\Omega \setminus \bar{A})} \mathcal{L}^2(D) = \int_D \operatorname{div}(z) \, dx \geq -\operatorname{Per}_\Phi(E; \Omega \setminus \bar{A}) + H^1\left(\partial^* D \cap \left(\cup_{j=1}^k \partial C_{i_j}\right)\right).$$

Therefore,

$$\begin{aligned} \sum_{j=1}^k \operatorname{Per}_\Phi(C_{i_j}; \Omega) &\leq \sum_{j=1}^k \operatorname{Per}_\Phi(C_{i_j}; \Omega) + \frac{\operatorname{Per}_\Phi(A; \Omega)}{\mathcal{L}^2(\Omega \setminus \bar{A})} \mathcal{L}^2(D) + \operatorname{Per}_\Phi(E; \Omega \setminus \bar{A}) \\ &\quad - \mathcal{H}^1\left(\partial^* D \cap \left(\cup_{j=1}^k \partial C_{i_j}\right)\right) \leq \frac{\operatorname{Per}_\Phi(A; \Omega)}{\mathcal{L}^2(\Omega \setminus \bar{A})} \mathcal{L}^2(D) + \operatorname{Per}_\Phi(E; \Omega), \end{aligned}$$

thus showing that $\cup_{j=1}^k C_{i_j}$ is a solution to $(P)_{i_1, \dots, i_k}$.

Now, let us suppose that $\cup_{j=1}^k C_{i_j}$ is a solution to $(P)_{i_1, \dots, i_k}$ for any subset $\{i_1, \dots, i_k\}$ of $\{1, \dots, m\}$. We only need to show that, in this specific case of A , (4.8) holds. Let E be a set of finite perimeter such that $E \subseteq \Omega \setminus \bar{A}$ and let C_{i_j} for $j = 1, \dots, k$ be the ones such that $\partial E \cap C_{i_j} \neq \emptyset$. Then, by minimality,

$$\begin{aligned} \sum_{j=1}^k \text{Per}_{\Phi}(C_{i_j}; \Omega) &= \text{Per}_{\Phi}\left(\bigcup_{j=1}^k C_{i_j}; \Omega\right) \leq \text{Per}_{\Phi}\left(E \cup \left(\bigcup_{j=1}^k C_{i_j}\right); \Omega\right) + \frac{\text{Per}_{\Phi}(A; \Omega)}{\mathcal{L}^2(\Omega \setminus \bar{A})} \mathcal{L}^2(E) \\ &\leq \text{Per}_{\Phi}(E; \Omega) + \sum_{j=1}^k \text{Per}_{\Phi}(C_{i_j}; \Omega) - \mathcal{H}^1(\partial^* E \cap \partial A) + \frac{\text{Per}_{\Phi}(A; \Omega)}{\mathcal{L}^2(\Omega \setminus \bar{A})} \mathcal{L}^2(E). \end{aligned}$$

Therefore, we conclude that

$$\text{Per}_{\Phi}(E; \Omega) - \mathcal{H}^1(\partial^* E \cap \partial A) + \frac{\text{Per}_{\Phi}(A; \Omega)}{\mathcal{L}^2(\Omega \setminus \bar{A})} \mathcal{L}^2(E) \geq 0,$$

which implies (4.8). \square

COROLLARY 4.13. *Let $A \subseteq \mathbb{R}^2$ be a bounded open set with Lipschitz boundary such that $A = \bigcup_{i=1}^m C_i$ with C_i disjoint, convex and calibrable, and let $\Omega := B_R^{\Phi^0}(0)$ be such that $\bar{A} \subseteq \Omega$. Suppose that*

$$(4.10) \quad d_{\Phi}(C_i, (\bigcup_{j \neq i} C_j) \cup (\mathbb{R}^2 \setminus \bar{\Omega})) > \text{Per}_{\Phi}(C_i; \Omega) \quad \text{for any } i = 1, \dots, m.$$

Let $f := \sum_{i=1}^m \alpha_i \chi_{C_i}$ with $\alpha_i > 0$. Then, for

$$(4.11) \quad \lambda \geq \max_{i=1, \dots, m} \left\{ \frac{1}{\alpha_i} \left(\frac{\text{Per}_{\Phi}(A; \Omega)}{\mathcal{L}^2(\Omega \setminus \bar{A})} + \frac{\text{Per}_{\Phi}(C_i; \Omega)}{\mathcal{L}^2(C_i)} \right) \right\},$$

the solution to $\mathcal{E}_{\Phi}^{\lambda}$ model is given by

$$u = \sum_{i=1}^m \left(\alpha_i - \frac{\text{Per}_{\Phi}(C_i; \Omega)}{\lambda \mathcal{L}^2(C_i)} \right) \chi_{C_i} + \frac{\text{Per}_{\Phi}(A; \Omega)}{\lambda \mathcal{L}^2(\Omega \setminus \bar{A})} \chi_{\Omega \setminus A}.$$

PROOF. Since each C_i is calibrable, we can construct a vector field $z_i \in X_{C_i}$ such that

$$\text{div}(z_i) = \frac{\text{Per}_{\Phi}(C_i; \Omega)}{\mathcal{L}^2(C_i)} \quad \text{and} \quad [z_i, \nu^{C_i}] = -\Phi^0(\nu^{C_i}).$$

Moreover, it is easy to show that the hypothesis in Theorem 4.12 are satisfied. Therefore, we can construct a vector field $z_{\text{out}} \in X_{\Omega \setminus \bar{A}}$ such that

$$\text{div}(z_{\text{out}}) = \frac{\text{Per}_{\Phi}(A; \Omega)}{\mathcal{L}^2(\Omega \setminus \bar{A})}, \quad [z_{\text{out}}, \nu^{C_i}] = -\Phi^0(\nu^{C_i}) \quad \text{and} \quad [z_{\text{out}}, \nu^{\Omega}] = 0.$$

Furthermore, observe that, in this case $0|_u C_i \geq 0|_u \Omega \setminus \bar{A}$. Therefore, it is easy to show that, considering z as

$$z := \sum_{i=1}^m z_i \chi_{C_i} + z_{\text{out}} \chi_{\Omega \setminus \bar{A}},$$

then $-\text{div}(z) \in \partial \text{TV}_{\Phi}(u)$ and therefore, u is the minimizer of $\mathcal{E}_{\Phi}^{\lambda}$. \square

EXAMPLE 4.14. *In this example, using a case of 3-phases segmentation, we show that the relationship from Conjecture 4.6 does not hold. Let Ω and f be as follows*

$$\Omega := B_R^{\Phi^0}(0), \quad f := \chi_{C_1} + \chi_{C_2} + \frac{1}{2} \chi_{C_3}$$

with $C_1 := B_1^{\Phi^0}((0, -L))$, $C_2 := B_{1/2}^{\Phi^0}((0, 0))$ and $C_3 := B_2^{\Phi^0}((0, L))$.

We take R and L large enough so that $A := \cup_{i=1}^3 C_i$ satisfies that $\bar{A} \subseteq \Omega$, C_i fulfils (4.10) for each i and $\lambda = 10$ is feasible with respect to (4.11). Then, by Corollary 4.13, the minimizer of $\mathcal{E}_{\Phi}^{\lambda}$ is exactly given by

$$(4.12) \quad u = \frac{8}{10}\chi_{C_1} + \frac{6}{10}\chi_{C_2} + \frac{4}{10}\chi_{C_3} + \frac{\text{Per}_{\varphi}(A; \Omega)}{10\mathcal{L}^2(\Omega \setminus \bar{A})}\chi_{\Omega \setminus \bar{A}}$$

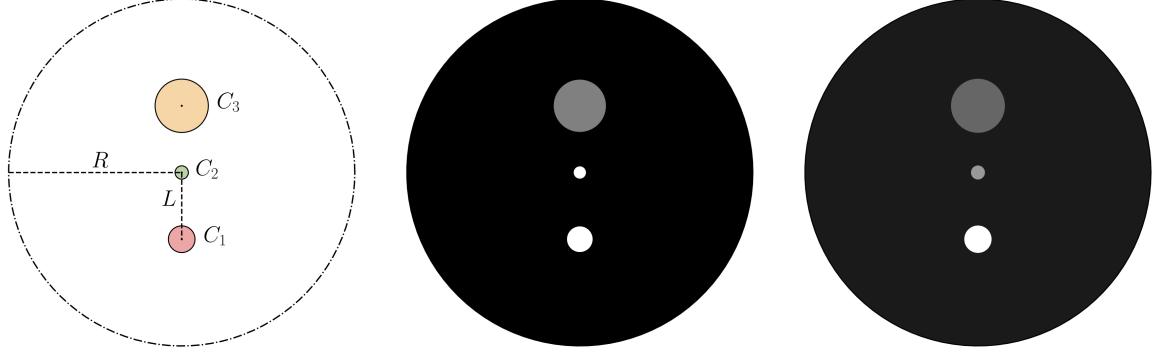


FIGURE 4.1. Illustration of the example in the isotropic case. *Left*: Parts of f and u . *Center*: Image modeled by f . *Right*: Image modeled by u .

Now, we define c_1 , c_2 and c_3 as follows:

$$c_1 = \int_{C_1} f \, dx, \quad c_2 = \int_{C_2 \cup C_3} f \, dx, \quad c_3 = \int_{\Omega \setminus \bar{A}} f \, dx.$$

Then, $c_1 = 1$, $c_2 = 9/17$ and $c_3 = 0$. By enlarging R if necessary, we assume that the latter term of (4.12) is bounded on top by $c_2/2$. Thus, we may assure that

$$\Sigma_1 := \left\{ x \in \Omega : u(x) > \frac{c_1 + c_2}{2} =: \tau_1 \right\} = C_1, \quad \Sigma_2 := \left\{ x \in \Omega : u(x) > \frac{c_2 + c_3}{2} =: \tau_2 \right\} = A.$$

By Corollary (4.7), we know that the set $\{\emptyset, \Sigma_1, \Sigma_2, \Omega\}$ is the minimizer of $\mathcal{E}_{\Phi}^{\lambda, 3}(\cdot, \{\tau_i\}_{i=1}^2)$. Assuming that Conjecture 4.6 holds, the partition

$$\{\Omega_i\}_{i=1}^3 := \{\Sigma_1, \Sigma_2 \setminus \Sigma_1, \Omega \setminus \Sigma_2\} = \{C_1, C_2 \cup C_3, \Omega \setminus \bar{A}\}$$

satisfies the inequality

$$(4.13) \quad \widehat{\mathcal{F}}_{\Phi}^{\mu, n}(\{\Omega_i\}_{i=1}^3, \{c_i\}_{i=1}^3) \leq \widehat{\mathcal{F}}_{\Phi}^{\mu, n}(\{\Lambda_i\}_{i=1}^3, \{c_i\}_{i=1}^3), \quad \forall \{\Lambda_i\}_{i=1}^3 \in \mathcal{P}_3(\Omega),$$

where $\mu = \{85/8, 1445/72, 85/9\}$, as defined in (4.7). However, if we define the following disjoint partition

$$\{\Omega_i^*\}_{i=1}^3 := \{C_1 \cup C_2, C_3, \Omega \setminus \bar{A}\},$$

we find that

$$\widehat{\mathcal{F}}_{\Phi}^{\mu, n}(\{\Omega_i^*\}_{i=1}^3, \{c_i\}_{i=1}^3) = \frac{725}{72}\mathcal{L}^2(C_1) < \widehat{\mathcal{F}}_{\Phi}^{\mu, n}(\{\Omega_i\}_{i=1}^3, \{c_i\}_{i=1}^3) = \frac{733}{72}\mathcal{L}^2(C_1),$$

which contradicts (4.13), for any anisotropy $\Phi(\xi)$.

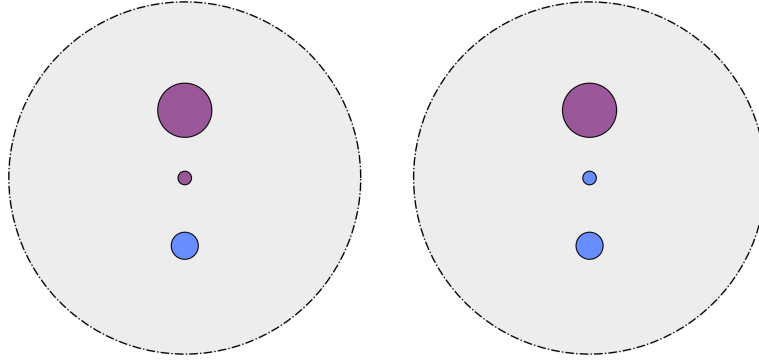


FIGURE 4.2. Illustration of $\{\Omega_i\}_i$ (left) and $\{\Omega_i^*\}_i$ (right), where Ω_1 and Ω_1^* are represented in blue, and Ω_2 and Ω_2^* are represented in purple.

4.3. Jump properties

In this section we study two characterizations of the boundary of the first component of the minimizers of Chan–Vese problem, thus relating it to the jump set of f

4.3.1. Strong jump property. Firstly, we introduce the *strong jump* property:

DEFINITION 4.15. *Let Λ be a minimizer of $\mathcal{F}_\Phi^\mu(\cdot, c_1, c_2)$ for some $c_1, c_2 \in [0, 1]$. We say that Λ satisfies the strong jump property if $\partial\Lambda \subseteq J_f$.*

Since $\partial\Lambda = J_{\chi_\Lambda}$, we note that the above property can be interpreted as the adaptation of the jump characterization provided in Theorem 1.1 where the minimizer u of \mathcal{E}^λ satisfies $J_u \subseteq J_f$. However, we need to remark that some assumptions on f are required, as in Theorem 1.1. The strong jump property has been previously studied in [128] in the one-dimensional setting. There, we required that f satisfies for every $c \in (0, 1)$ that

$$\mathcal{L}^1(\partial\{x \in (0, 1) \setminus J_f : f(x) = c\}) = 0,$$

an assumption that excludes certain pathological behaviors of f , such as the presence of a fat Cantor set as a level set. Under this range condition, in [128] we establish two key properties of the minimizers of \mathcal{F}^μ in the one-dimensional case:

- (a) The boundary of any solution is concentrated on the topological boundary of a single level set of f .
- (b) If f is piecewise constant, then the boundary of any solution is contained within the jump set of f .

These properties are significant as they formalize a strong jump property when f is a piecewise constant function, while also characterizing the behavior of $\partial\Lambda$ in a more general setting. However, we observe that it is not possible to extend either of these two properties to the two-dimensional setting for a generic anisotropy $\Phi(\xi)$. This limitation is shown in the following series of examples:

EXAMPLE 4.16. *A isotropic counterexample in two dimensions is provided by*

$$f = \chi_\Lambda \quad \text{with} \quad \Lambda = [-\tfrac{1}{2}, \tfrac{1}{2}]^2 \quad \text{and} \quad \Omega = [-1, 1]^2.$$

In this case, J_f is precisely the boundary of the square $[-\frac{1}{2}, \frac{1}{2}]^2$, while it can be shown that for any $\lambda > \frac{16}{3}$, the first component of a minimizer of \mathcal{F}^μ cannot be either Ω or Λ . This demonstrates that properties (a) and (b) do not hold.

In fact, the minimal \mathcal{F}^μ energy for these two candidates is:

$$\mathcal{F}^\mu(\Omega, 1/4, 0) = \frac{3}{4}\lambda, \quad \mathcal{F}^\mu(\Lambda, 1, 0) = 4$$

Therefore, for $\lambda > \frac{16}{3}$, Ω is not the first component of a minimizer. On the other hand, it is possible to modify the corners of the square to reduce the perimeter without significantly altering the fidelity term in the energy. To achieve this, we modify Λ , defining a new set Λ_δ by removing the corners using circumferential arches of radius δ , tangent to every two contiguous sides of $\partial\Lambda$. In Figure 4.3 we provide a sketch of the definition of Λ_δ .

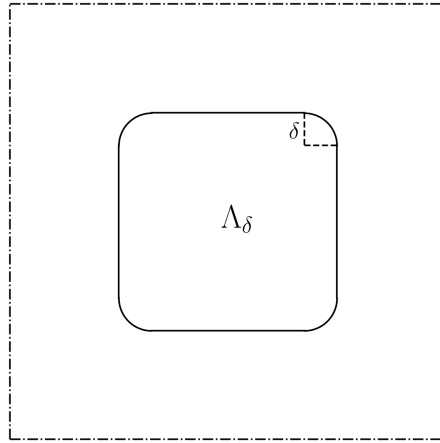


FIGURE 4.3. Sketch of Λ_δ

Then, one can verify that $\mathcal{F}^\mu(\Lambda_\delta, c_1, c_2)$ has less energy, for c_1, c_2 and δ such that

$$c_1 = 1, \quad c_2 = \frac{(4 - \pi)\delta^2}{3 + (4 - \pi)\delta^2}, \quad \text{and} \quad \frac{3\delta}{3 + (4 - \pi)\delta^2} < \frac{2}{\lambda}.$$

This phenomenon is related to the non-calibrability of the set Λ with respect to the isotropic norm in the total variation.

EXAMPLE 4.17. Next, we show that the properties (a) and (b) are not satisfied in the anisotropic two-dimensional case either for $\Phi(\xi) = |\xi|_1$, using the previous result and the characterization of \mathcal{E}_1^λ 's solutions presented in Section 2.3.4.

Let us consider $\Omega = [-1, 1]^2$ and $f = \chi_{\Lambda_1 \cup \Lambda_2}$, where

$$\Lambda_1 = [-\frac{1}{2}, \frac{1}{2}]^2, \quad \Lambda_2 = [\frac{1}{4}, \frac{1}{2}] \times [\frac{1}{2}, \frac{3}{4}].$$

According to [110, Theorem 5], the \mathcal{E}_1^λ minimizer with $\lambda = 16$ is

$$u = \frac{3}{4}\chi_{\Lambda_1} + \frac{1}{2}\chi_{\Lambda_2} + \frac{9}{94}\chi_{\Omega \setminus (\Lambda_1 \cup \Lambda_2)}.$$

By Theorem 4.1, we know that $(\Lambda_1, 1, 1/48)$ is a minimizer of \mathcal{F}_1^μ with $\mu = 768/47$, such that $\partial\Lambda_1$ is not contained in the jump set of f , as it is shown in Figure 4.4. Consequently, $\partial\Lambda_1$ is not in a sole level set of f .

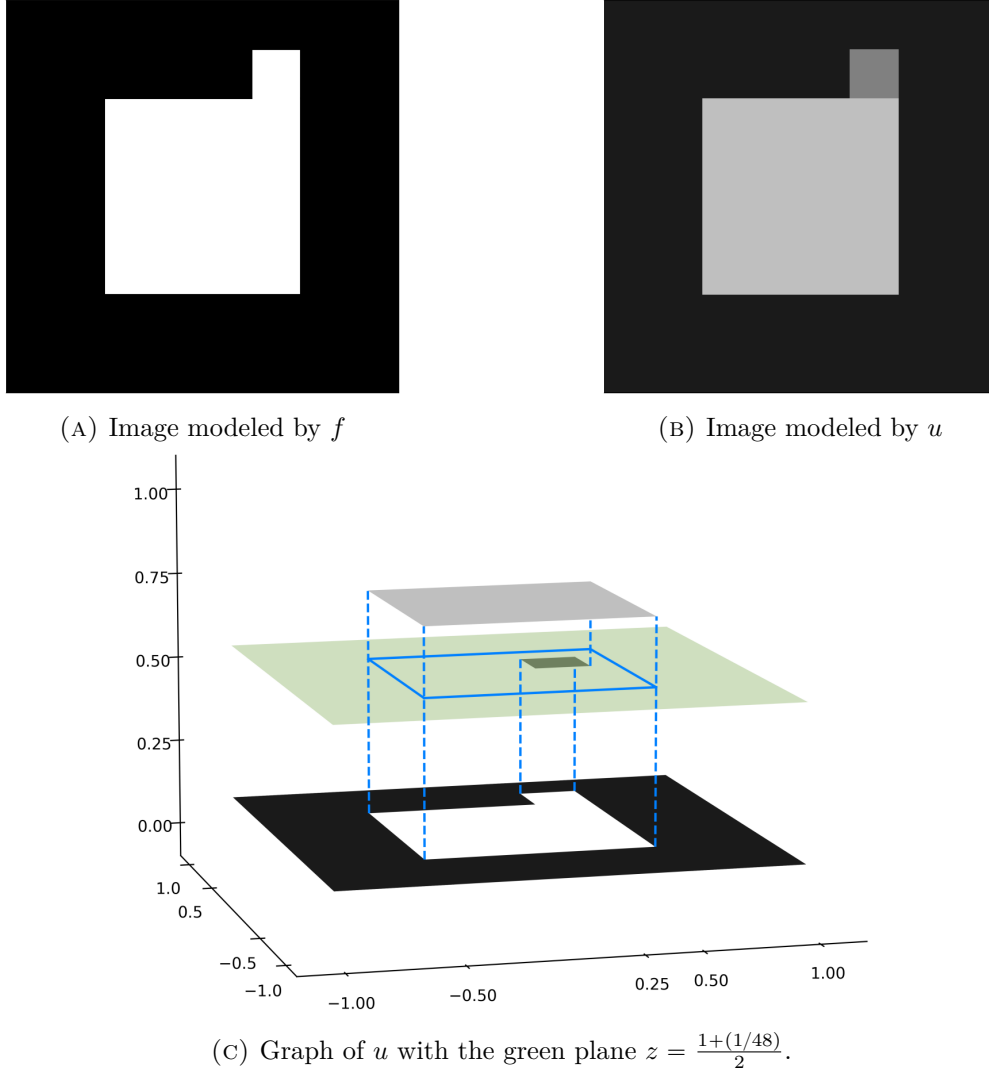


FIGURE 4.4. Sketch of the example

4.3.2. Weak jump property. As we have observed in the previous section, the strong jump property does not hold in general, though it is the natural extension of Theorem (1.1). Then, we need to provide a weaker jump property. Having in mind the fine properties of the minimizer of \mathcal{E}_1^λ when $f \in \text{PCR}(\Omega)$ (see Section 2.3.4), we define the next property:

DEFINITION 4.18. *Let $f \in \text{PCR}(\Omega)$ and Σ be a minimizer of $\mathcal{F}_\Phi^\mu(\cdot, c_1, c_2)$ for some $c_1, c_2 \in [0, 1]$. We say that Σ satisfies the weak jump property if $\partial\Sigma \subseteq G_f$.*

Now, we adapt the *weak jump property* to the n -phases setting:

DEFINITION 4.19. *Let $f \in \text{PCR}(\Omega)$ and $\Sigma = \{\Sigma_i\}_{i=1}^n$ be a minimizer of $\mathcal{F}_\Phi^{\mu, n}(\cdot, \mathbf{c})$ for some $\mathbf{c} \in [0, 1]^n$. We say that Σ satisfies the weak jump property if $\partial\Sigma_i \subseteq G_f$ for every i .*

Let us discuss this property when $\Phi(\xi) = |\xi|_1$. In the case biphasic, by Theorem 4.1 we know that if $f \in \text{PCR}(\Omega)$, then there exists a partial minimizer Σ of \mathcal{F}_1^μ such that $\partial\Sigma \subseteq G_f$ thanks to the (λ, μ) -linkage. Consequently, the weak jump property 4.18 holds.

In the case of n -phases, we cannot use the (λ, μ) -linkage as a tool to prove the property 4.19; we need to do it directly. Firstly, for $\mu_i = \mu_j = \mu$ for all $1 \leq i < j \leq n$, we reformulate $\mathcal{F}_1^{\mu, n}$ as follows

$$\mathcal{F}_1^{\mu,n}(\Sigma, \mathbf{c}) = \sum_{i=1}^n \text{Per}_1(\Sigma_{i-1}; \text{int}(\Sigma_i)) + \mu \int_{\Sigma_i \setminus \Sigma_{i-1}} (x_i - f)^2 dx,$$

with $\mu > 0$ and $\Sigma \in \mathcal{P}_n^*(\Omega)$. Hereinafter, we associate $\mathcal{F}_1^{\mu,n}$ with the above definition instead of with that in (4.1). The first step to prove that the weak jump property holds for a minimizer of $\mathcal{F}_1^{\mu,n}$ is proving the existence of minimizers:

PROPOSITION 4.20. *Let $\Omega \subseteq \mathbb{R}^2$ be an open set and let f be a function such that $f(\Omega) \subseteq [0, 1]$. Then, there exists $\Sigma \in \mathcal{P}_n^*(\Omega)$ and $\mathbf{c} \in [0, 1]^n$ such that*

$$(\Sigma, \mathbf{c}) \in \arg \min \{ \mathcal{F}_1^{\mu,n}(\Lambda, \mathbf{a}) : \Lambda \in \mathcal{P}_n^*(\Omega), \mathbf{a} \in [0, 1]^n \}.$$

PROOF. The proof is very similar to that of Proposition 4.3. In particular, for a minimizing sequence $(\Sigma^k, \mathbf{c}^k) = ((\Sigma_0^k, \dots, \Sigma_n^k); (c_1^k, \dots, c_n^k))$, we can take $u_i^k := \chi_{\Sigma_i^k}$ and work exactly as in Proposition 4.3. Finally, from the convergence a.e. obtained from the lower semicontinuity of the anisotropic total variation, we conclude that the weak limits in $L^2(\Omega)$, u_i are of the form $u_i = \chi_{\Sigma_i}$ and that $\Sigma := (\Sigma_0, \dots, \Sigma_n) \in \mathcal{P}_n^*(\Omega)$. The rest of the proof is analogous and we will omit it. \square

THEOREM 4.21. *Let $\mathbf{c} = \{c_i\}_{i=1}^n$ satisfy $1 \geq c_i \geq c_{i+1} \geq 0$ and let $f \in \text{PCR}(\Omega)$. Then there exists a minimizer Σ of $\mathcal{F}_1^{\mu,n}(\cdot, \mathbf{c})$ which satisfies the weak jump property.*

Before proving this result, we present the following lemma, which can be derived from [110, Lemma 2, Step 3]:

LEMMA 4.22. *Let $f \in \text{PCR}(\Omega)$, and let E and F be two rectangular polygons such that $E \subseteq F$. Define the functional*

$$H(A) := \text{Per}_1(A; F) + \mu \int_A (c - f)^2 dx$$

for all subsets $A \subseteq F$, where $\mu, c > 0$. Then, there exists a set $\hat{E} \subseteq F$ such that $G(\hat{E}) \subseteq (G_f \cup G(F))$ and $H(\hat{E}) \leq H(E)$.

PROOF. Let $G := G_f \cup G(F)$ and $E_0 := E$. We suppose that $G(E) \not\subseteq G$. Let L be a line of $G(E_0)$ such that is not contained in G . For clarity, we assume that it is horizontal, i.e. $L := \mathbb{R} \times \{y\}$. By definition of E , we know that $L \cap E = C \times \{y\}$ where $C = \{C_i\}_i$ is a union of non-contiguous intervals. Now, we consider $y_- < y < y_+$ such that $L_- = \mathbb{R} \times \{y_-\}$ and $L_+ = \mathbb{R} \times \{y_+\}$ are contained in $G \cap G(E_0)$ and there is no line in $G \cap G(E_0)$ between them (see Figure 4.5 for an illustrative example).

Let us fix the index i and define $E'_0 = E_0 \setminus C_i \times [y_-, y_+]$. In this case, we note that f is constant on $Q_j \cap (C_i \times [y_-, y_+]) = C_{i,j} \times [y_-, y_+] \neq \emptyset$ for $Q_j \in \{Q_j\}_{j=1}^{m_i} \subseteq \mathcal{Q}_f$. Now, we define $f_{i,j}$ such that $f(C_{i,j} \times [y_-, y_+]) =: \{f_{i,j}\}$ and $y_i \in \{y_-, y_+\}$ such that $C_i \times \{y_i\}$ is contained in E_0 . In this context, we define $p_i(t), v_{i,j}(t) : [y_-, y_+] \rightarrow [0, +\infty)$ as

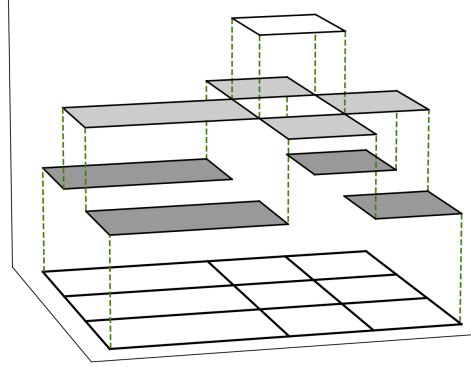
$$\begin{aligned} p_i(t) &:= \text{Per}_1(C_i \times [y_i, t] \cup E'_0; F), \\ v_{i,j}(t) &:= \mu(c - f_{i,j}) \mathcal{L}^2(C_{i,j} \times [y_i, t]). \end{aligned}$$

Observe that both are linear and monotone in (y_-, y_+) and they reach their minima in the set $\{y_-, y_+\}$. Consequently, the function

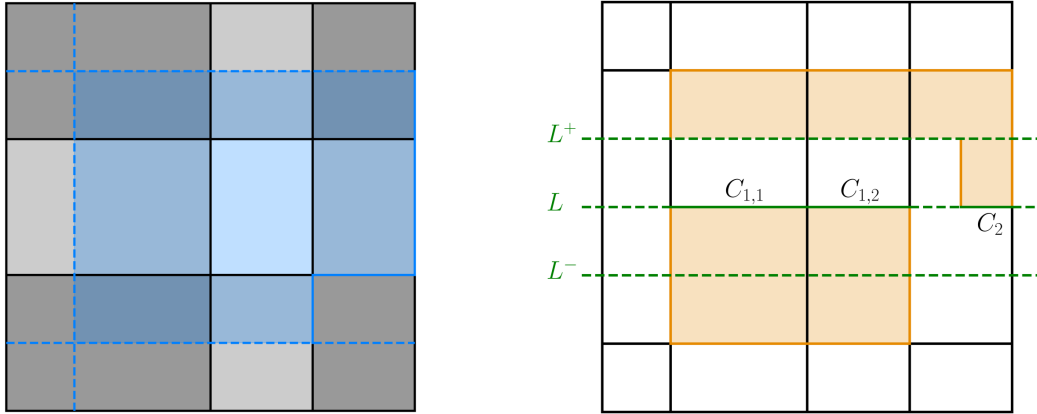
$$h_i(t) := p_i(t) + \sum_{j=1}^{m_i} v_{i,j}(t)$$

has a minimum $\alpha_i \in \{y_-, y_+\}$. Thus, the resulting set $E_1 := E'_0 \cup C_i \times [y_i, \alpha_i]$ satisfies that $H(E_1) \leq H(E_0)$. In addition, E_1 has one less side contained in G with respect to E , at least.

Recursively on each k , we define a set E_{k+1} applying the previous argument on $G(E_k)$. By finiteness, we conclude that there is a finite number k^* such that $G(E_{k^*}) \subseteq G$ and $H(E_{k^*}) \leq H(E)$.



(A) Graph of a function f , with the grid G_f projected onto the $z = 0$ plane.



(B) Grid G_f is the black one. $F \subseteq \Omega$ is the red-shaded region. Dashed lines are those that are in $G(F) \setminus G_f$.

(C) Grid $G := G_f \cap G(F)$ is the black one. Set $E \subseteq F$ is the blue-shaded region. Note that $L \in G(E) \setminus G$ and $C = \partial E \cap L$ is split as in the proof.

FIGURE 4.5. Concept example in the proof of Lemma 4.22.

□

PROOF OF THEOREM 4.21. We define $\mathcal{F}_{1,c}^{\mu,n}(\Sigma) := \mathcal{F}_1^{\mu,n}(\cdot, \mathbf{c})$ and express it as follows:

$$\mathcal{F}_{1,c}^{\mu,n}(\Sigma) = \sum_{i=1}^{n-1} \left(\text{Per}_1(\Sigma_i; \text{int}(\Sigma_{i+1})) + \mu \int_{\Sigma_i} f_i(x) dx \right) + \int_{\Omega} (c_n - f)^2 dx,$$

where $f_i(x) = (c_i - c_{i+1})(c_i + c_{i+1} - 2f(x))$. Following the approach in [110, Lemma 2], and given $\Sigma \in \mathcal{P}_n^*(\Omega)$ and $\varepsilon > 0$, we construct $\Sigma^* \in \mathcal{P}_n^*(\Omega)$ such that $\Sigma_i^* \in R(G_f)$ and

$$\mathcal{F}_{1,c}^{\mu,n}(\Sigma^*) < \mathcal{F}_{1,c}^{\mu,n}(\Sigma) + \varepsilon.$$

We divide the proof into three steps:

1. *Smoothing*: We construct a variant of Σ , denoted by $\tilde{\Sigma}$ such that $\tilde{\Sigma} \in \mathcal{P}_n^*(\Omega)$ and their components have smooth boundaries. For that, we define $\tilde{\Sigma}_{i,\delta}$ as

$$\tilde{\Sigma}_{i,\delta} = \{x \in \Omega : \psi_\delta * \chi_{\Sigma_i} \geq t\},$$

where ψ_δ is a standard smooth approximation of the identity, δ is a positive parameter close to 0 and $t \in (0, \frac{1}{2})$. As a consequence of this definition and following the idea of [110, Lemma 2], we obtain that $\tilde{\Sigma}_{i,\delta}$ is smooth and it is possible to set values t and δ such that

$$\mathcal{F}_{1,c}^{\mu,n}(\tilde{\Sigma}) < \mathcal{F}_{1,c}^{\mu,n}(\Sigma) + \varepsilon,$$

with $\tilde{\Sigma} = \{\tilde{\Sigma}_i\}_{i=1}^{n-1} \in \mathcal{P}_n^*(\Omega)$.

2. *Squaring*: Next, we construct a variant of $\tilde{\Sigma}$, denoted by $\hat{\Sigma}$, such that $\hat{\Sigma} \in \mathcal{P}_n^*(\Omega)$ and their components have rectangular boundaries. For this end, we apply a squaring process on the components of $\tilde{\Sigma}$ in each cell of G_f , avoiding an increase of the energy at $\mathcal{F}_{1,c}^{\mu,n}$. So, we consider a closed cell C of G_f and we suppose that

$$\text{int}(C) \cap (\cup_{i=1}^{n-1} \partial \tilde{\Sigma}_i) = \text{int}(C) \cap (\cup_{i=m_1}^{m_2} \partial \tilde{\Sigma}_i).$$

We split $\{\tilde{\Sigma}_i\}_{i=m_1}^{m_2}$ into those sets whose index satisfies $f_i(\cdot) > 0$ in the cell C and the rest, denoting them by $\{\tilde{\Sigma}_i\}_{i=m_1}^{i_0-1}$ and $\{\tilde{\Sigma}_i\}_{i=i_0}^{m_2}$ respectively, providing a well defined division thanks to the condition $c_i \geq c_{i+1}$, and the fact that f is constant in $\text{int}(C)$. To treat this division, we define $C_i := \Sigma_i \cap C$ and we use the idea of [110, Lemma 2, Step 2].

First, we work with the sets $\tilde{\Sigma}_i \in \{\tilde{\Sigma}_i\}_{i=i_0}^{m_2}$. For each $x \in \partial C_i$, we define an open square $U(x)$ such that $C_i \cap U(x)$ can be defined as the subgraph of a smooth function and that $U(x)$ intersects at most one edge of Ω . As ∂C_i is compact and $\{U(x) : x \in \partial C_i\}$ is an open covering, there exists a finite open covering $\{U_k\}_{k=1}^l$ of ∂C_i .

On this covering, we define a covering of rectangles $\{W_k^i\}_{k=1}^{l_i}$ such that $W_k^i \subseteq C_i$ is the smallest closed rectangle which contains $C_i \cap U_k$. Now, we can suppose without loss of generality that $U_k = (a_k, b_k) \times (c_k, d_k)$ and that $C_i \cap U_k$ is the subgraph of a smooth function $g_k : (a_k, b_k) \rightarrow (c_k, d_k)$. Then, we note that

$$\int_{W_k^i \cap \partial C_i} |\nu^{C_i}|_1 d\mathcal{H}^1 = \int_{a_k}^{b_k} (1 + |g_k'|) dx \geq |\alpha_k - g_k(a_k)| + |b_k - a_k| + |\alpha_k - g_k(b_k)|$$

where α_k is the supremum of g_k . Therefore,

$$\text{Per}_1(C_i; \Omega) = \mathcal{H}^1(\partial C_i \setminus W_k) + \int_{W_k^i \cap \partial C_i} |\nu^{C_i}|_1 d\mathcal{H}^1 \geq \text{Per}_1(C_i \cup W_k^i).$$

Then, since $C_i \subseteq \text{int}(\tilde{\Sigma}_{i+1})$, we get that $C_i \cup (\cup_k W_k^i) = C_i \cup W^i$ does not increase the 1-perimeter of C_i in $\text{int}(\tilde{\Sigma}_{i+1})$ but its volume, i.e.

$$(4.14) \quad \begin{aligned} \text{Per}_1(C_i \cup W^i; \text{int}(\tilde{\Sigma}_{i+1})) &\leq \text{Per}_1(C_i; \text{int}(\tilde{\Sigma}_{i+1})), \\ \mathcal{L}^2(C_i \cup W^i) &\geq \mathcal{L}^2(C_i). \end{aligned}$$

Now, we apply a similar argument on $\{\tilde{\Sigma}_i\}_{i=m_1}^{i_0-1}$. In this case, we repeat the previous approach on $C_{i+1} \setminus \text{int}(C_i)$ instead of on C_i , defining a finite rectangular covering $\{W_k^i\}_{k=1}^{l_i}$ of $\partial(C_{i+1} \setminus \text{int}(C_i))$. Consequently, $C_i \setminus \text{int}(\cap_k W_k^i) = C_i \setminus \text{int}(W^i)$ is a rectangular polygon and

$$(4.15) \quad \begin{aligned} \text{Per}_1(C_i \setminus \text{int}(W^i); \text{int}(\tilde{\Sigma}_{i+1})) &\leq \text{Per}_1(C_i; \text{int}(\tilde{\Sigma}_{i+1})), \\ \mathcal{L}^2(C_i \setminus \text{int}(W^i)) &\leq \mathcal{L}^2(C_i). \end{aligned}$$

Then, we define

$$\hat{\Sigma}^c = \{\hat{\Sigma}_i^c\}_{i=1}^n := \begin{cases} \tilde{\Sigma}_i \setminus \text{int}(W^i) & \text{if } i \in \{m_1, \dots, i_0 - 1\} \\ \tilde{\Sigma}_i \cup W^i & \text{if } i \in \{i_0, \dots, m_2\} \\ \tilde{\Sigma}_i & \text{otherwise} \end{cases}$$

which satisfies that $\mathcal{F}_{1,c}^{\mu,n}(\hat{\Sigma}^c) \leq \mathcal{F}_{1,c}^{\mu,n}(\tilde{\Sigma})$ given the behaviour of f_i at each index i and the above inequalities (4.14) and (4.15). Moreover, their components satisfy that their boundaries are rectangular at cell C . Adjusting the covers W^i , we assure the inclusion condition $\hat{\Sigma}_i^c \subseteq \tilde{\Sigma}_{i+1}^c$. If we repeat this process on each cell of G_f adding to the changes on $\tilde{\Sigma}$, we define a collection $\hat{\Sigma} \in \mathcal{P}_n^*(\Omega)$ such that their components have rectangular boundaries and

$$\mathcal{F}_{1,c}^{\mu,n}(\hat{\Sigma}) \leq \mathcal{F}_{1,c}^{\mu,n}(\tilde{\Sigma}).$$

3. *Aligning:* All in a row, we define a collection, denoted by $\Sigma^* \in \mathcal{P}_n^*(\Omega)$, whose components are rectilinear polygons on $R(G_f)$. For that, we modify the boundaries of the components of $\hat{\Sigma}$ averting an increase of the energy at $\mathcal{F}_{1,c}^{\mu,n}$.

Let be $G^i := G_f \cup G(\hat{\Sigma}_{i+1})$ for $i \in \{1, \dots, n-1\}$. First, we apply Lemma 4.22 on $\hat{\Sigma}_1$, and we obtain a set $\hat{\Sigma}_{(1),1}$ such that $G(\hat{\Sigma}_{(1),1}) \subseteq G^2$. Then, we define the collection $\hat{\Sigma}_{(1)}$, which is equal to $\hat{\Sigma}$ with the exception of the first element that is replaced by $\hat{\Sigma}_{(1),1}$. As consequence of Lemma 4.22, $\mathcal{F}_{1,c}^{\mu,n}(\hat{\Sigma}_{(1)}) \leq \mathcal{F}_{1,c}^{\mu,n}(\hat{\Sigma})$.

Likewise, we apply a similar strategy to modify $\hat{\Sigma}_2$. By Lemma 4.22, we obtain a set $\hat{\Sigma}_{(2),2}$ such that $G(\hat{\Sigma}_{(2),2}) \subseteq G^3$. In this case, we apply a slight tweak in the above strategy. Now, we modify $\hat{\Sigma}_{(1),1}$ with respect to $\hat{\Sigma}_{(2),2}$. In particular, we modify each side $s \in G(\hat{\Sigma}_{(1),1}) \cap G(\hat{\Sigma}_2)$ in the same way that it is moved from $\hat{\Sigma}_2$ to $\hat{\Sigma}_{(2),2}$. Then, we obtain a modified version $\hat{\Sigma}_{(1),1}$, which we denote by $\hat{\Sigma}_{(2),1}$, with this adjustment if required. Consequently, we define the collection $\hat{\Sigma}_{(2)}$, which is equal to $\hat{\Sigma}$ with the exception of the first two elements that are replaced by $\hat{\Sigma}_{(2),1}$ and $\hat{\Sigma}_{(2),2}$, respectively. In addition, the inequality

$$\mathcal{F}_{1,c}^{\mu,n}(\hat{\Sigma}_{(2)}) \leq \mathcal{F}_{1,c}^{\mu,n}(\hat{\Sigma}_{(1)}) \leq \mathcal{F}_{1,c}^{\mu,n}(\hat{\Sigma})$$

holds by the condition $c_i > c_{i+1}$.

Rehashing this idea for each i up to $n-1$, we obtain a collection $\hat{\Sigma}_{(n-1)}$ such that the boundaries of their elements are contained in G^{n-1} and we have $\mathcal{F}_{1,c}^{\mu,n}(\hat{\Sigma}_{(n-1)}) \leq \mathcal{F}_{1,c}^{\mu,n}(\hat{\Sigma})$. Moreover, since $G(\hat{\Sigma}_n) = \partial\Omega \subseteq G_f$, we get $G^{n-1} = G_f$.

As a result, the collection $\Sigma^* = \{\Sigma_i^*\}_{i=1}^n := \hat{\Sigma}_{(n-1)} \in \mathcal{P}_n^*(\Omega)$ satisfies that

$$(4.16) \quad \mathcal{F}_{1,c}^{\mu,n}(\Sigma^*) \leq \mathcal{F}_{1,c}^{\mu,n}(\Sigma) + \varepsilon \quad \text{and} \quad \partial\Sigma_i^* \subseteq G_f \quad \text{for each } i.$$

We complete the proof as in [110, Theorem 3]. Let us consider a minimizing sequence $\{\Sigma_k\}_k \subseteq \mathcal{P}_n^*(\Omega)$. By the previous argument, there exist $\{\Sigma_k^*\}_k = \{\{\Sigma_{k,i}^*\}_{i=1}^n\}_k$ such that $\Sigma_{k,i}^* \in R(G_f)$; and by (4.16), we have that $\mathcal{F}_{1,c}^{\mu,n}(\Sigma_k) \leq \mathcal{F}_{1,c}^{\mu,n}(\Sigma_k^*) + \varepsilon$. Thus, as $\mathcal{F}_{1,c}^{\mu,n}$ is bounded from below by 0 and $R(G_f)$ is a finite set, we conclude that $\{\Sigma_k^*\}_k$ has a constant subsequence $\{\Sigma_{k_n}^*\}_{k_n} \equiv \{\Sigma^*\}$, which minimizes $\mathcal{F}_{1,c}^{\mu,n}$. \square

REMARK 4.23. Observe that given $(\tilde{\Sigma}, \mathbf{c})$ a minimizer to $\mathcal{F}_1^{\mu, n}$, then $\tilde{\Sigma}$ is a minimizer to $\mathcal{F}_1^{\mu, n}(\cdot, \mathbf{c})$ and that \mathbf{c} can be reordered in such a way that $1 \geq c_i \geq c_{i+1} \geq 0$ for all $i = 1, \dots, n-1$. Therefore, by Theorem 4.21, we obtain that there exists $\Sigma \in \mathcal{P}_n^*(\Omega)$ such that its components Σ_i satisfy that $\partial \Sigma_i \subseteq G_f$ and (Σ, \mathbf{c}) is also a minimizer to $\mathcal{F}_1^{\mu, n}$. Therefore, the weak jump property is fulfilled.

4.4. Applications

In this section, we show that the previous analysis not only offers a characterization of the minimizers of the CV and ROF problems, but also efficient segmentation algorithms, leveraged by Theorem 4.4 in the 2-phases case and by Corollary 4.7 in the n -phases case. In the existing literature, the CV and ROF models have been studied from different perspectives, as we have commented previously in Sections 1.2.4.3 and 1.3.3. In our case, we provide algorithms to find partial minimizers of \mathcal{F}_1^μ and $\mathcal{E}_\Phi^{\lambda, n}$ by means of a minimizer of \mathcal{E}_1^λ and \mathcal{E}_Φ^λ , respectively.

4.4.1. Algorithms. Using Theorem 4.4 and Corollary 4.7, we can derive partial minimizers of \mathcal{F}_1^μ and $\mathcal{E}_1^{\lambda, n}$ from the minimizer of \mathcal{E}_1^λ . Building on this idea, we propose the Algorithms 4.1 and 4.2. It is important to note that Algorithm 4.2 is formulated for any anisotropy, as explained in the Corollary 4.7. Nevertheless, for this section, we use its variant with $\Phi(\xi) = |\xi|_1$ to ensure a consistent comparison between isotropic and anisotropic cases.

Algorithm 4.1 \mathcal{F}_1^μ approximate minimizer

initiation: $k = 0$, $\mu > 0$, $(c_1, c_2) \in [0, 1]$ such that $c_1 > c_2$.

$w_0 \leftarrow \mathcal{E}_1^{2\mu(c_1-c_2)}$ minimizer.

$\Lambda \leftarrow \mathcal{F}_1^\mu(\cdot, c_1, c_2)$ minimizer using w_0 via Theorem 4.4.

while $(|\Lambda_k \Delta \Lambda_{k-1}|^2 > \varepsilon_{tol}) \wedge (k < n_{max})$ **do**

$c_1 \leftarrow \int_{\Lambda_k} f \, dx$, $c_2 \leftarrow \int_{\Omega \setminus \Lambda_k} f \, dx$, $k \leftarrow k + 1$

$w_k \leftarrow \mathcal{E}_1^{2\mu(c_1-c_2)}$ minimizer.

$\Lambda \leftarrow \mathcal{F}_1^\mu(\cdot, c_1, c_2)$ minimizer using w_k via Theorem 4.4.

end while

return (Λ_k, c_1, c_2)

Algorithm 4.2 $\mathcal{E}_\Phi^{\lambda, n}$ approximate minimizer

initiation: $k = 0$, $\lambda > 0$, $\{\tau_i\}_{i=1}^{n-1} \in [0, 1]^{n-1}$ such that $\tau_i > \tau_{i+1}$.

$w \leftarrow \mathcal{E}_\Phi^\lambda$ minimizer.

$\Sigma^0 \leftarrow \mathcal{E}_\Phi^{\lambda, n}$ minimizer via Corollary 4.7.

while $(\sum_{i=0}^n |\Sigma_i^k \Delta \Sigma_i^{k-1}|^2 > \varepsilon_{tol}) \wedge (k < n_{max})$ **do**

$\tau_i \leftarrow \int_{\Sigma_{i+1}^k \setminus \Sigma_i^k} f \, dx$, $k \leftarrow k + 1$.

$\Sigma^k \leftarrow \mathcal{E}_\Phi^{\lambda, n}$ minimizer via Corollary 4.7.

end while

return Σ^k

Algorithm 4.3 $\mathcal{F}_1^{\mu,n}$ approximate minimizer.

initiation: $k = 0$, $\mu > 0$, $\mathbf{c} = \{c_i\}_{i=1}^{n-1} \in [0, 1]^{n-1}$ such that $c_i > c_{i+1}$.

$\Sigma^0 \leftarrow \mathcal{F}_1^{\mu,n}(\cdot, \mathbf{c})$ approximate minimizer.

$\Sigma^{*,0} \leftarrow$ variant of Σ^0 via Theorem 4.21.

while $(\sum_{i=0}^n |\Sigma_i^{*,k} \Delta \Sigma_i^{*,k-1}|^2 > \varepsilon_{tol}) \wedge (k < n_{max})$ **do**

$c_i \leftarrow \int_{\Sigma_i^{*,k} \setminus \Sigma_{i-1}^{*,k}} f \, dx$, $k \leftarrow k + 1$;

$\Sigma^k \leftarrow \mathcal{F}_1^{\mu,n}(\cdot, \mathbf{c})$ approximate minimizer.

$\Sigma^{*,k} \leftarrow$ variant of Σ^k via Theorem 4.21.

end while

return $(\Sigma^{*,k}, \mathbf{c})$

REMARK 4.24. In [34], the convergence of the aforementioned algorithms is established within the isotropic framework. This proof can be easily adapted to the anisotropic setting for any anisotropic norm, thereby ensuring the convergence of the algorithms described above.

Similarly, Theorem 4.21 provides a method for constructing minimizers of $\mathcal{F}_1^{\mu,n}$. While this approach is not computationally feasible at present, we define a theoretical algorithm 4.3 due to its potential future relevance in numerical applications. To ensure the consistency of Algorithm 4.3, we need to prove that the inequality $c_i \geq c_{i+1}$ holds in every iteration, a question which is treated in the next result:

PROPOSITION 4.25. Let $\mathbf{c} = \{c_i\}_{i=1}^n$ satisfy $1 \geq c_i \geq c_{i+1} \geq 0$, and let Σ be the minimizer of $\mathcal{F}_1^{\mu,n}(\cdot, \mathbf{c})$ obtained in Theorem 4.21. If \tilde{c}_i is defined as

$$\tilde{c}_i := \int_{\Sigma_i \setminus \Sigma_{i-1}} f(x) \, dx,$$

then it is satisfied that $\tilde{c}_i > \tilde{c}_{i+1}$ for $i = 1, \dots, n-1$.

PROOF. First of all, note that, given Σ_{i-1} and Σ_{i+1} , Σ_i is a minimizer of

$$\text{Per}_1(\Sigma_{i-1}; \text{int}(E)) + \text{Per}_1(E; \text{int}(\Sigma_{i+1})) + \mu(c_i - c_{i+1}) \int_E (c_i + c_{i+1} - 2f) \, dx,$$

for all sets $\Sigma_{i-1} \subseteq E \subseteq \Sigma_{i+1}$.

Since Σ_{i-1} is an admissible set, and Σ_i is a minimizer, it follows that

$$\begin{aligned} & \text{Per}_1(\Sigma_{i-1}; \text{int}(\Sigma_i)) + \text{Per}_1(\Sigma_i; \text{int}(\Sigma_{i+1})) + \int_{\Sigma_i} (c_i + c_{i+1} - 2f) \, dx \\ & \leq \text{Per}_1(\Sigma_{i-1}; \text{int}(\Sigma_i)) + \int_{\Sigma_{i-1}} (c_i + c_{i+1} - 2f) \, dx, \end{aligned}$$

which yields

$$\text{Per}_1(\Sigma_i, \text{int}(\Sigma_{i+1} \setminus \Sigma_{i-1})) + \mu(c_i - c_{i+1}) \int_{\Sigma_i \setminus \Sigma_{i-1}} (c_i + c_{i+1} - 2f) \, dx \leq 0.$$

In case that

$$\int_{\Sigma_i \setminus \Sigma_{i-1}} (c_i + c_{i+1} - 2f) dx = 0,$$

then $\text{Per}_1(\Sigma_i; \text{int}(\Sigma_{i+1} \setminus \Sigma_{i-1})) = 0$. Consequently, Σ_i coincides either with Σ_{i-1} or Σ_{i+1} and this phase is removed, passing from n to $(n - 1)$ phases. Therefore, we can suppose that

$$\tilde{c}_i > \frac{c_i + c_{i+1}}{2}.$$

By contradiction, if $\tilde{c}_i \leq \tilde{c}_{i+1}$, it follows that

$$\int_{\Sigma_{i+1} \setminus \Sigma_i} (c_i + c_{i+1} - 2f) dx < 0.$$

This implies that

$$\begin{aligned} & \text{Per}_1(\Sigma_{i-1}; \text{int}(\Sigma_{i+1})) + \mu(c_i - c_{i+1}) \int_{\Sigma_{i+1}} (c_i + c_{i+1} - 2f) dx, \\ & < \text{Per}_1(\Sigma_{i-1}; \text{int}(\Sigma_i)) + \text{Per}_1(\Sigma_i; \text{int}(\Sigma_{i+1})) + \int_{\Sigma_i} (c_i + c_{i+1} - 2f) dx, \end{aligned}$$

a contradiction with the fact Σ_i is a minimizer. \square

4.4.2. Examples. In the following series of examples, we compare the performance of Algorithms 4.1 and 4.2 with analogous isotropic methods proposed in [34]. Additionally, we demonstrate a potential application of Theorems 4.1 and 4.21. For these experiments, we set $n_{max} = 200$ and $\varepsilon_{tol} = 10^{-3}$. The parameters c_i and τ_i are initialized using fuzzy c-means clustering, and \mathcal{E}_Φ^λ is solved in each case using the appropriate variant of the standard Split-Bregman algorithm. In Figures 4.6, 4.7, 4.8, and 4.9 we present comparisons of the best approximations (in the Frobenius norm) obtained by these algorithms and their isotropic counterparts when applied to various noisy images. To assess segmentation quality, the results of each model are compared to a reference segmentation derived from the original, unaltered image. The isotropic and anisotropic results are denoted by u_2 and u_1 , respectively.

The anisotropic model consistently outperforms the isotropic model across all noisy scenarios. Specifically, the two-phase anisotropic segmentation yields a more accurate partition, closely aligning with the reference segmentation, as highlighted in Figures 4.6 and 4.7. Furthermore, the anisotropic multiphase segmentation exhibits greater robustness to noise, resulting in more stable phase divisions, as illustrated in Figures 4.8 and 4.9. Specifically, in Figures 4.6e and 4.7e, there is a noticeable reduction in misclassified pixels compared to Figures 4.6d and 4.7d, where blue/red regions indicate pixels incorrectly segmented as white/black. Moreover, the anisotropic multiphase model exhibits superior performance in preserving finer image details, such as crosswalk lines (Figure 4.8) and window structures (Figure 4.9), as evidenced by Figures 4.8d-4.8e and 4.9d-4.9e.

Overall, the improvement rates between models, summarized in Table 4.1, support these observations, confirming that the anisotropic model is a reliable and robust option for image segmentation in noisy environments.

| Example | Fig. 4.6 | Fig. 4.7 | Fig. 4.8 | Fig. 4.9 |
|-------------------------------------|----------|----------|----------|----------|
| $\ u_c - u_1\ _F / \ u_c - u_2\ _F$ | 0.8107 | 0.6257 | 0.859 | 0.8945 |

TABLE 4.1. Differences between isotropic and anisotropic approaches

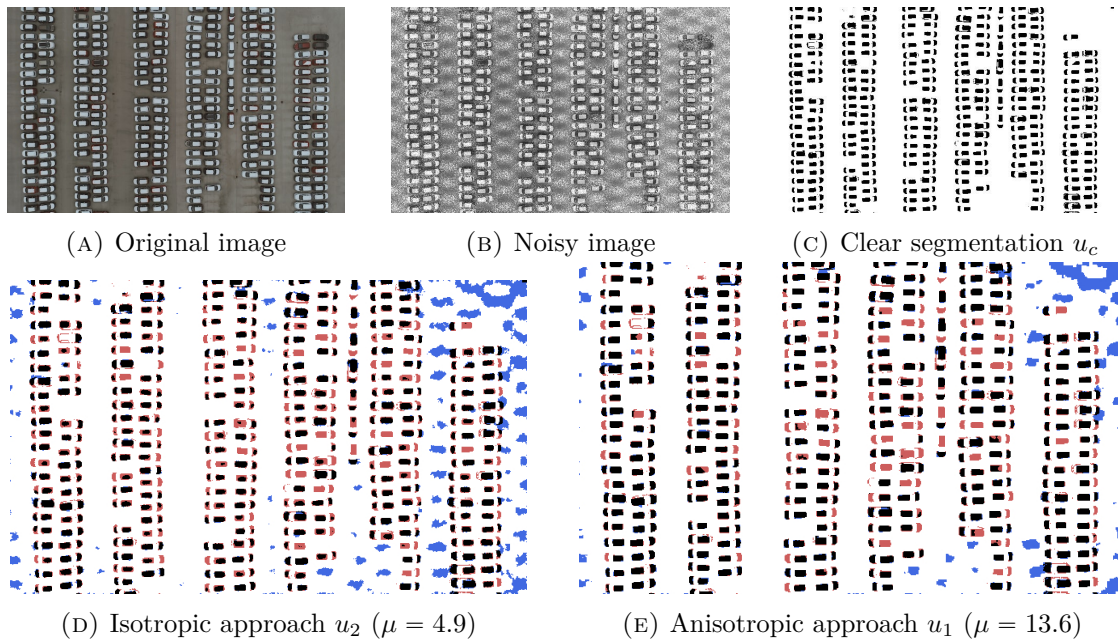


FIGURE 4.6. 2-phase comparison

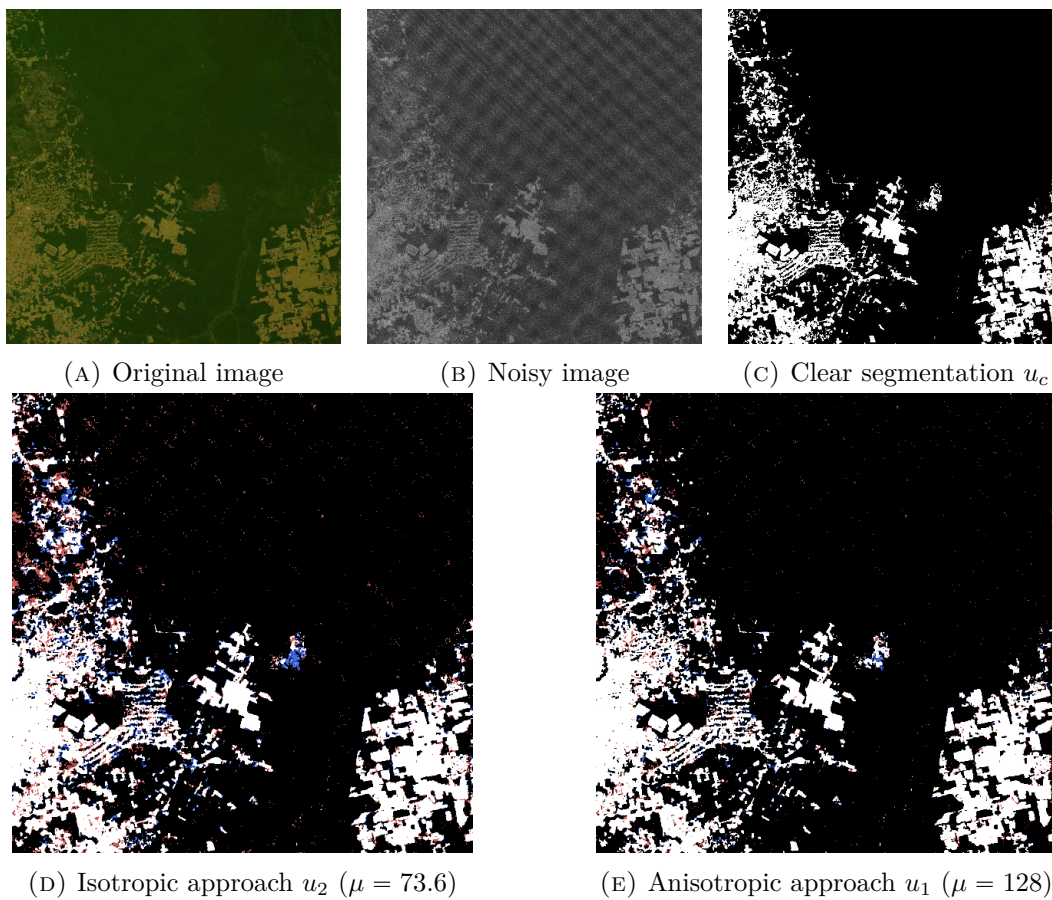


FIGURE 4.7. 2-phase comparison

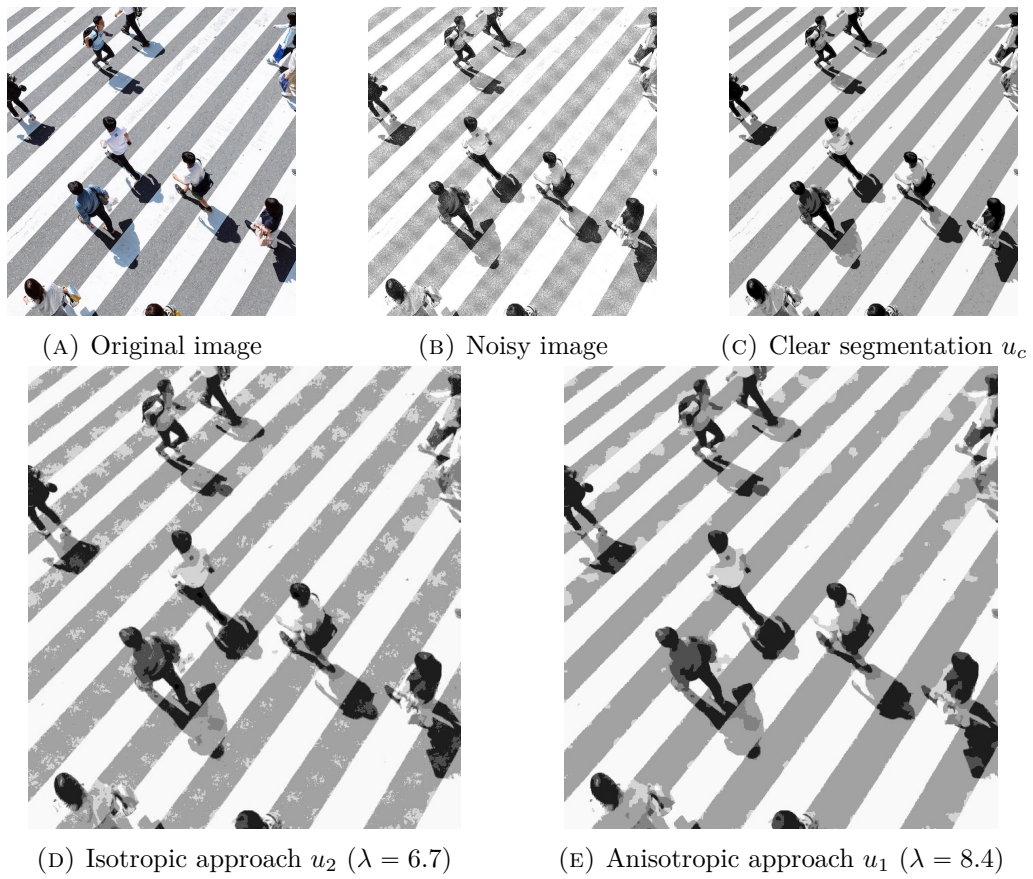


FIGURE 4.8. 6-phases comparison

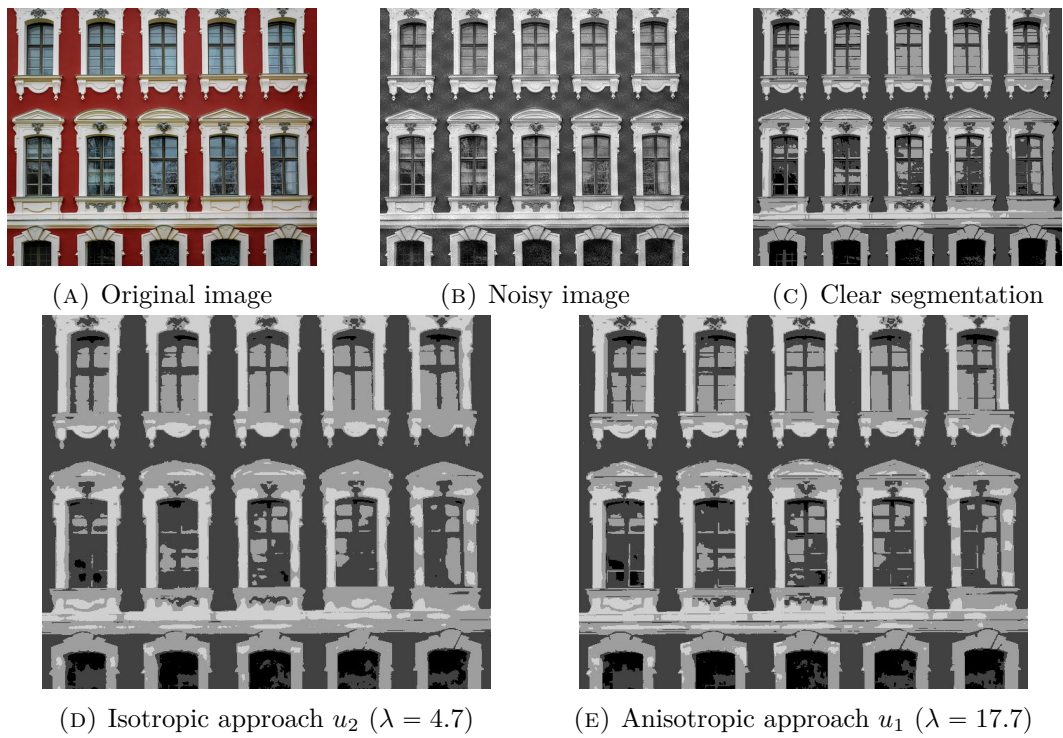


FIGURE 4.9. 5-phases comparison

Finally, we consider the application of Theorems 4.1 and 4.21 as tools for reliability analysis. Based on the preceding results, segmentations obtained using the \mathcal{F}_1^μ or $\mathcal{F}_1^{\mu,n}$ functionals can be considered effective if it closely aligns with the grid of cells, effectively resembling the structure of an exact minimizer. While this alignment does not necessarily guarantee an accurate approximation of the exact minimizer, it provides a clear indication of the performance of segmentations, particularly highlighting cases where the segmentation is inadequate.

To illustrate this application, we analyze Figure 4.10, where the image in 4.10a is segmented using the \mathcal{F}_1^μ model with two different methods: the anisotropic version of Getreuer’s implementation (based on the isotropic version described in [81]) and our proposed Algorithm 4.1. Upon comparing the results, we observe notable differences between the segmentations in 4.10c and 4.10d. Specifically, 4.10c exhibits inaccuracies due to misalignment with the grid structure of the image 4.10a, a discrepancy that could potentially be mitigated by refining the initialization parameters. Notably, this inaccuracy remains evident even without direct comparison to the exact minimizer shown in 4.10b.

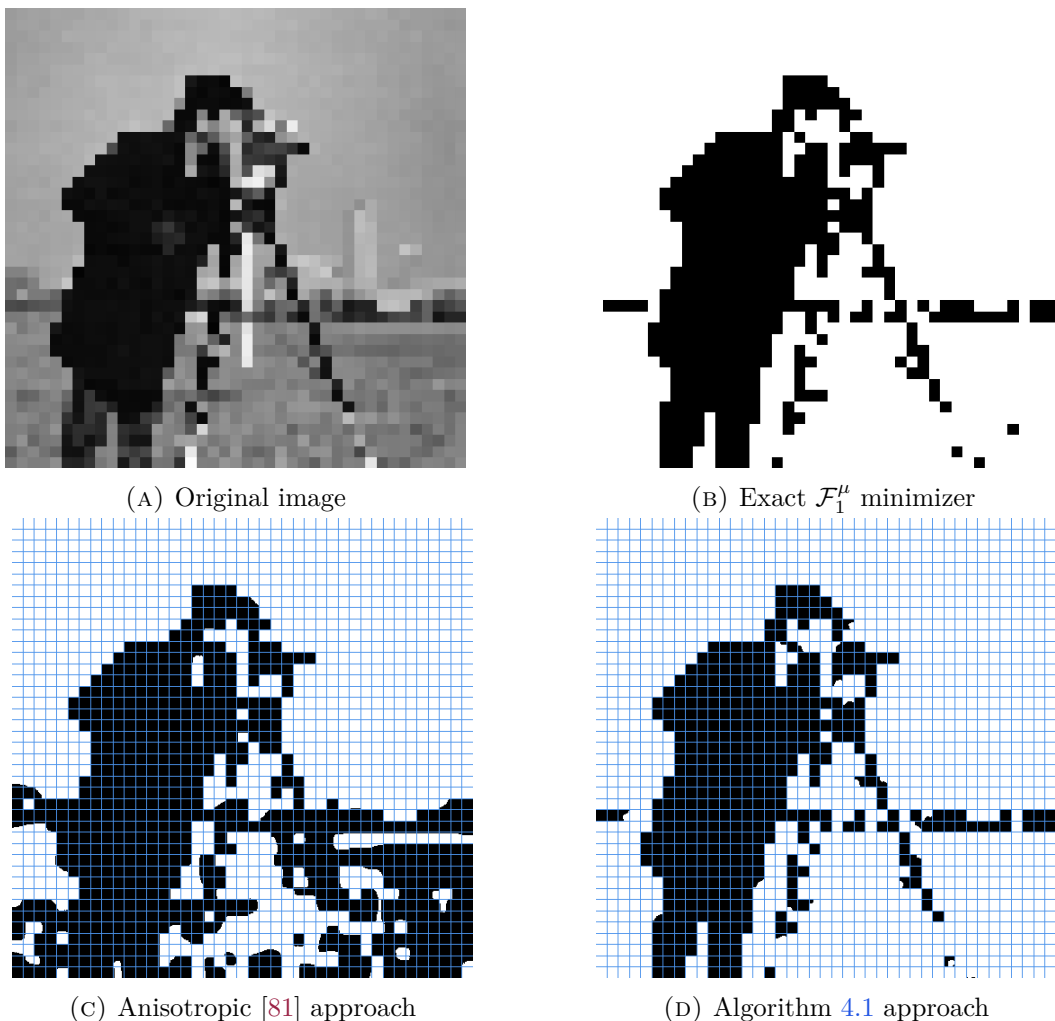


FIGURE 4.10. Comparison between two \mathcal{F}_1^μ approaches, for $\mu = 10$. Segmentations and input image grid are in black-white and blue, respectively.

Consequently, this application can be interpreted as a *parameter setup test* that identifies incorrect parameter initializations without necessitating a comparison with the exact minimizer. Similarly, while computing the exact minimizer of \mathcal{F}_1^μ via brute force (i.e., minimizing

the functional over the sets in $R(G_f)$ is computationally prohibitive, it serves as a valuable benchmark for evaluating the performance of segmentation algorithms on simpler images, provided the associated grid size of the image is sufficiently small.

Manifold-constrained ROF model

In this chapter we study the regularity of the minimizer of the ROF model in a manifold-constrained setting, i.e. when the image to be denoised is modeled by a map between manifolds $f : \Sigma \rightarrow \mathcal{N}$. More precisely, we consider that (Σ, g) is a compact surface (possibly with smooth boundary $\partial\Sigma \neq \emptyset$), and (\mathcal{N}, h) is a complete connected smooth n -dimensional Riemannian manifold. Regarding the target space, we mention that we treat it as an embedded submanifold in some Euclidian space \mathbb{R}^N by Nash theorem [137]. Therefore, for a given $f \in L^2(\Sigma, \mathcal{N})$, we study the regularity of minimizers of the next functional

$$(5.1) \quad \mathcal{E}_{\mathcal{N}}^{\lambda, \sigma}(u) := \int_{\Sigma} |du| \, d\mathbf{m}_g + \frac{\lambda}{2} \int_{\Sigma} d_g^2(u, f) \, d\mathbf{m}_g + \frac{\sigma}{2} \int_{\Sigma} |du|^2 \, d\mathbf{m}_g,$$

where $\lambda > 0$, $\sigma \geq 0$, d_g denotes the geodesic distance on \mathcal{N} , and $d\mathbf{m}_g$ is the volume element corresponding to g . In particular, if $\sigma = 0$ we define the ROF functional in a manifold-constrained setting. Thus we denote $\mathcal{E}_{\mathcal{N}}^{\lambda, 0}$ as $\mathcal{E}_{\mathcal{N}}^{\lambda}$ to unify the notation along the thesis.

In this context, the results of this chapter can be regarded as the manifold-constrained counterpart of [43, 143], with the additional technical challenge of considering non-flat surfaces Σ (abstract 2-manifolds) as domains. This minimization problem leads to the following nonlinear elliptic system with Neumann boundary conditions:

$$(5.2) \quad \begin{cases} \tau_1(u) + \sigma \tau(u) &= -\lambda \exp_u^{-1} f & \text{in } \Sigma, \\ \nu \cdot du &= 0 & \text{on } \partial\Sigma, \end{cases}$$

where $\tau_1(u)$ and $\tau(u)$ denote the 1-tension and the usual tension (or rough Laplacian), i.e.

$$\tau_1(u) := \operatorname{div}_g \left(\frac{du}{|du|} \right), \quad \tau(u) := \operatorname{div}_g(du).$$

Notice that these tensions can be defined in intrinsic and extrinsic ways, as we comment in Section 2.4.1. Moreover, the precise definition of the function $du|du|^{-1}$ and the Neumann condition are provided in Section 2.4.1 and Definition 5.16.

In Section 1.2.4, we show, in the scalar case, that the inheritance of the regularity of f by a minimizer of \mathcal{E}^{λ} is established under certain constraints. Thus it is natural that, in our manifold case, the regularity results can be derived under natural curvature restrictions on \mathcal{N} and a topological condition on the range of f . In this sense, let us recall the following bound of the convexity radius:

$$(5.3) \quad R_{\kappa} := \begin{cases} \frac{1}{2} \min \left\{ \operatorname{inj}_p \mathcal{N}, \frac{\pi}{\sqrt{\kappa}} \right\} & \text{if } \kappa > 0 \\ \frac{\operatorname{inj}_p \mathcal{N}}{2} & \text{if } \kappa \leq 0 \end{cases}$$

This chapter contains the results of [30, 31].

where κ is an upper bound for all sectional curvatures of the geodesic ball $B_h(p, R)$ for $R < R_\kappa$, and $\text{inj}_p(\mathcal{N})$ denotes the injectivity radius of the ball at its center p . In particular, we note that $B_h(p, R)$ is strongly convex provided that $R \leq R_\kappa$ (cf. [57, Theorem IX.6.1]).

In this framework, we establish an analogous version of Theorem 1.3, which addresses Lipschitz regularity of the minimizers of $\mathcal{E}_\mathcal{N}^\lambda$, following the approach of [143] and employing a Bernstein-type technique. We adapt this method to curved spaces, both in the domain and the target. More precisely, we prove the following result:

THEOREM 5.1. *Suppose that $f(\bar{\Sigma}) \subseteq B_h(p, R)$ for some $R < R_\kappa$. Then there exists a minimizer of the functional $\mathcal{E}_\mathcal{N}^\lambda$, which also satisfies $u(\bar{\Sigma}) \subseteq B_h(p, R)$. If we further assume that $\kappa \leq 0$, then*

- (a) $\mathcal{E}_\mathcal{N}^\lambda$ is geodesically convex, and hence the minimizer is unique.
- (b) If $f \in C^{0,1}(\bar{\Sigma}, \mathcal{N})$, then u inherits the same Lipschitz regularity.

Additionally, we provide a local version of previous result, extending Theorem 1.2:

COROLLARY 5.2. *Assume that \mathcal{N} is non-positively curved and $f(\bar{\Sigma})$ is contained in a strongly convex geodesic ball of \mathcal{N} . If we further assume that $f \in C_{loc}^{0,1}(\bar{\Sigma}, \mathcal{N})$, then u is also locally Lipschitz in $\bar{\Sigma}$.*

In the case $\sigma > 0$, we expect better regularity properties (see some Euclidean results in this direction, but without boundary conditions, in [90] for convex scalar solutions, and [155] in the vector-valued situation, removing the convexity assumption). This intuitive expectation comes from the extra diffusivity (and thus ellipticity) coming from the Dirichlet energy. In our case, it allows regularity results for $\kappa > 0$. Indeed, we prove the next result:

THEOREM 5.3. *Suppose that $f(\bar{\Sigma}) \subseteq B_h(p, R)$ for some $R < R_\kappa$. Then for any fixed $\sigma > 0$, there exists a minimizer of the functional $\mathcal{E}_\mathcal{N}^{\lambda, \sigma}$, which also satisfies $u(\bar{\Sigma}) \subseteq B_h(p, R)$.*

- (a) *If we further assume that $\kappa \leq 0$, then $\mathcal{E}_\mathcal{N}^{\lambda, \sigma}$ is geodesically convex, and hence the minimizer is unique. If $f \in C^{0,1}(\bar{\Sigma}, \mathcal{N})$, then u inherits the same Lipschitz regularity.*
- (b) *In the case $\kappa > 0$ there exists a minimizer of $\mathcal{E}_\mathcal{N}^{\lambda, \sigma}$ which is also Lipschitz continuous, provided that $f(\bar{\Sigma}) \subseteq B_h(p, R)$ for some $R < \min \left\{ \frac{\text{inj}_p \mathcal{N}}{2}, \frac{\pi}{4\sqrt{\kappa}} \right\}$.*

Notice that the stronger small range condition in (b) was also required in [79] to prove regularity of stationary p -harmonic maps for $p \geq 2$.

To motivate this chapter, we discuss the interest of the aforementioned regularity study from different perspectives in Section 5.1.

Once the framework is well-defined, we dedicate two sections to establish some technical tools for the study of regularity: in Section 5.2 we provide a formal definition of total variation on manifolds; relating it with the 1-energy (see Section 2.4.1). Then, we prove the convexity of a family of energies. Then, the convexity of total variation is obtained. Complementarily, in Section 5.3 we obtain a partial regularity result for weak solutions of a quasilinear elliptic system, where, as intermediate step, we obtain an intrinsic Cacciopoli inequality of independent interest (see Proposition 5.11).

Applying these technical tools, in Section 5.4 we prove a range invariance result, as well as the existence and uniqueness of minimizers of $\mathcal{E}_\mathcal{N}^{\lambda, \sigma}$. To obtain the regularity results, we first prove, from an extrinsic viewpoint, the Hölder regularity of a relaxed version of (5.2) in Section 5.5. Then, to obtain the Lipschitz regularity and complete the above results, we work intrinsically using the Bernstein technique in Section 5.6. We note that this interplay of extrinsic–intrinsic approaches is essential to achieve the results.

To complete this chapter, we establish the corresponding regularity result for manifold-valued signal denoising, i.e. in the case where the domain is one-dimensional. More precisely, in Section 5.7 we prove the next result

THEOREM 5.4. *Let $f : \Gamma \rightarrow \mathcal{N}$, where Γ is either an interval $[0, 1]$ or \mathbb{S}^1 , so that $f(\Gamma)$ is contained in $B(p, R)$ with $R < \text{inj}_p(\mathcal{N})/2$. If $f \in C^{0,1}(\Gamma, \mathcal{N})$, then there exists a minimizer u of $\mathcal{E}_{\mathcal{N}}^{\lambda, \sigma}$ such that $u \in C^{0,1}(\Gamma, \mathcal{N})$.*

REMARK 5.5. *Hereafter, we adopt the Einstein convention that an index appearing as both a subscript and a superscript in a product implies summation over its range. Specifically, Latin indices range over $\{1, \dots, m\}$, while Greek letters take values in $\{1, \dots, n\}$.*

5.1. Related works

At this stage, we place the results of this chapter in the context of existing regularity studies on the ROF model. As far as we are aware, even the vectorial case (i.e., when $\mathcal{N} = \mathbb{R}^n$) has not been addressed, due to the lack of classical regularity results up to boundary. For the case $\sigma > 0$, if we think of the vector in the right-hand side of the PDE (5.2) as the gradient of some pressure function, then this equation models an isothermal steady laminar flow of a non-Newtonian (Bingham) incompressible fluid, where u stands for the flow velocity and Σ is the cross-section of an infinitely long duct. In this context, our Neumann condition means that there is no flux across the boundary, and the parameter σ controls the plastic viscosity. This is usually referred to as Mosolov problem [132]. We note that the manifold-constrained setting represents a more versatile way to describe real data and apply ROF and Mosolov models, as we explain in Section 1.1.

First, we notice that the regularity of (5.2) appears to be significantly more challenging than its parabolic counterpart (see [84] for the case $\lambda = 0$), as evidenced by the lack of prior references. In a somewhat different spirit, (5.2) can be regarded as a shifted spectral problem for the so-called 1-tension or rough 1-Laplacian (see Section 2.4.1 for technicalities of the definition) associated with the eigenvalue λ and with respect to a model map f , which was introduced by Jost and Kourouma in [105] for the standard 2-tension or rough Laplacian.

In this sense, the $\lambda = 0$ case corresponds to 1-harmonic maps, and hence our results can be regarded as the extension to $p = 1$ of the regularity theory for p -harmonic maps, which is extensively studied mainly for $1 < p < \infty$, and which is itself an extension of the classical harmonic map theory ($p = 2$) started by Eells and Sampson [70] or Schoen and Uhlenbeck [151, Corollary, p. 310], where the non-positive curvature (NPC) hypothesis and the small range condition naturally pops up. Indeed, it is well-known (see e.g. examples in [145]) that there is no hope of getting full regularity results unless we impose some topologic and/or geometric restrictions on the target. Notice that our $p = 1$ case is analytically much harder because the anisotropic diffusivity of the 1-tension, meaning that its ellipticity degenerates in the direction of du , while it becomes singular in directions where u is constant.

Indeed, our small range condition was first considered for $p = 2$ by Hildebrandt, Kaul and Widman [99] to prove some regularity results, which were extended to $p \geq 2$ and $1 < p < 2$ in [75] and [96], respectively. All of these papers deal with the friendlier scenario of Dirichlet boundary conditions. Certainly, our results are closer to the Neumann problem for harmonic maps studied by Hamilton in the NPC case [97]. Let us point out that the lack of any harmonicity hypothesis is a constant technical difficulty in our work, because we have to control tension terms overall which in these previous references did not play any role because they vanish.

5.2. Properties of manifold total variation

5.2.1. Formal definition. In this subsection we extend the energy functional \mathcal{E}_1 for functions not necessarily in $W^{1,1}(\Sigma, \mathcal{N})$. In particular, with the notation in (2.14), we define for $u \in L^1(\Sigma, \mathcal{N})$,

$$(5.4) \quad \mathcal{TV}(u) := \sum_{\ell} \int_{\Omega_{\ell}} \varrho_{\ell} |\nabla u| dx + \int_{\Omega_{\ell}} \varrho_{\ell} d|D^c u| + \int_{\Omega_{\ell} \cap J_u} \varrho_{\ell} d_h(u_{\ell}^+, u_{\ell}^-) d\mathcal{H}^1,$$

if $u \circ \phi_{\ell}^{-1} \in BV(\Omega_{\ell}, \mathcal{N})$; and $+\infty$ otherwise. Note that, again, we are omitting composition with ϕ_{ℓ}^{-1} inside the integrals. Here we are using an extrinsic concept of BV functions; i.e. for an open set $\Omega \subseteq \mathbb{R}^m$,

$$BV(\Omega, \mathcal{N}) := \{u \in BV(\Omega, \mathbb{R}^N) : u(x) \in \mathcal{N} \text{ for a.e. } x \in \Omega\},$$

and J_u denotes the jump set of $u \in BV(\Omega, \mathbb{R}^N)$ while $D^c u$ is the Cantor part of the Radon measure Du . Moreover, we say that $u \in BV(\Sigma, \mathcal{N})$ if $u \circ \phi_{\ell}^{-1} \in BV(\Omega_{\ell}, \mathcal{N})$ for all $\ell \in I$.

Next we prove the next result, concerning functions valued on a geodesic ball $B_h(p, R)$ around a point $p \in \mathcal{N}$ with radius R smaller than R_{κ} defined as in (5.3).

THEOREM 5.6. *Let $R < R_{\kappa}$. The following holds:*

- (a) *Let $\{u_k\}_k \subseteq W^{1,1}(\Sigma, B_h(p, R))$ for some $p \in \mathcal{N}$ such that $u_k \rightarrow u$ in $L^1(\Sigma, B_h(p, R))$, then*

$$\mathcal{TV}(u) \leq \liminf_{k \rightarrow \infty} \mathcal{E}_1(u_k).$$

- (b) *Given $u \in BV(\Sigma, B_h(p, R))$ for some $p \in \mathcal{N}$, there exists $u_k \in C^1(\Sigma, \overline{B_h(p, R)})$, such that $\mathcal{E}_1(u_k) \rightarrow \mathcal{TV}(u)$ as $k \rightarrow \infty$.*

PROOF. First, we observe that it is enough to prove the result for $\Omega := \Omega_{\ell}$. For simplicity of notation, we also omit composition with ϕ_{ℓ}^{-1} . Under this notation, we observe that the result is very close (except for the fact of the different open and closed geodesic balls) to prove that \mathcal{TV} is the relaxed functional with respect to the L^1 -convergence of

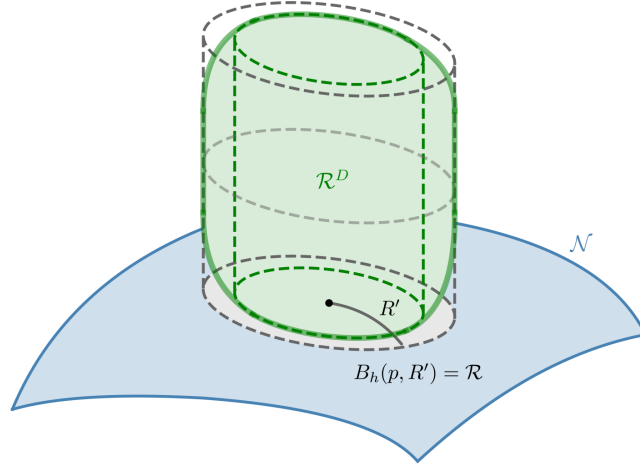
$$u \mapsto \begin{cases} \mathcal{E}_1(u) & \text{if } u \in C^1(\Omega, \overline{B_h(p, R)}) \\ +\infty & \text{otherwise} \end{cases}.$$

In the case that the target space \mathcal{Y} is a compact, orientable manifold without boundary with trivial homotopy group, we could apply directly the results in [89] to obtain that this is the case. However, in our case $\mathcal{Y} = \overline{B_h(p, R)}$ has a nonempty boundary. To fix this issue, we argue as follows.

We take $R < R' < R_{\kappa}$. Since $\partial B_h(p, R')$ is geodesically convex, we can consider the double of $\mathcal{R} := B_h(p, R')$, \mathcal{R}^D , which is typically constructed by gluing together two copies of \mathcal{R} along the identity map of the boundary, but this will not give a regular enough closed submanifold in general. To overcome this, we can argue as in [32, Proposition 4.1], that is, roughly speaking, by perturbing the metric in a small inner neighborhood of the boundary $\partial \mathcal{R}$ to form a cylindrical end so that the gluing is well defined and the submanifold $(\mathcal{R}^D, \tilde{g})$ preserves its degree of regularity and the modified metric \tilde{g} satisfies $\tilde{g}|_{B_g(p, R)} = g$.

Then, [89, Corollary 6.4] applies with $\mathcal{Y} = \mathcal{R}^D$ and the lower semicontinuity in (a) is a direct consequence. For (b), we observe that [89, Corollary 6.4] yields the existence of a sequence $\{u_k\}_k \subseteq C^1(\Omega, \mathcal{R}^D)$ such that $u_k \rightarrow u \in L^1(\Omega, \mathcal{R}^D)$ and $\mathcal{E}_1(u_k) \rightarrow \mathcal{TV}(u)$ as $k \rightarrow \infty$. By the strong convergence, we can take a subsequence, not relabeled, such that $u_k \rightarrow u$ a.e. Therefore, we can suppose that $u_k \in C^1(\Omega, B_h(p, R))$. \square

Now we accurately define the functionals in (5.1) as

FIGURE 5.1. Schematic view of the doubling $(\mathcal{R}^D, \tilde{g})$

$$\mathcal{E}_{\mathcal{N}}^{\lambda, \sigma}(u) := \begin{cases} \mathcal{TV}(u) + \frac{\lambda}{2} \int_{\Sigma} d_h^2(u, f) dm_g + \frac{\sigma}{2} \int_{\Sigma} |du|^2 dm_g & \text{if } u \in H^1(\Sigma, \mathcal{N}) \\ +\infty & \text{if } u \in L^2(\Sigma, \mathcal{N}) \setminus H^1(\Sigma, \mathcal{N}) \end{cases}$$

for $\sigma > 0$ and the ROF case $\sigma = 0$ as

$$\mathcal{E}_{\mathcal{N}}^{\lambda}(u) := \begin{cases} \mathcal{TV}(u) + \frac{\lambda}{2} \int_{\Sigma} d_h^2(u, f) dm_g & \text{if } u \in BV(\Sigma, \mathcal{N}) \\ +\infty & \text{if } u \in L^1(\Sigma, \mathcal{N}) \setminus BV(\Sigma, \mathcal{N}) \end{cases}$$

5.2.2. Geodesic convexity of a generalized p -energy. Let us work for this part in a more general framework: (\mathcal{M}^m, g) a compact m -dimensional manifold with $\partial\mathcal{M} \neq \emptyset$. Consider a C^2 function $F : \mathcal{M} \times \mathbb{R} \rightarrow \mathbb{R}^+$ such that

- $F(x, \cdot)$ is non-decreasing for all $x \in \mathcal{M}$.
- $|F(x, s)| \leq C(1 + |s|^{\frac{p}{2}})$ for $(x, s) \in \mathcal{M} \times \mathbb{R}$ and some $p \geq 1$
- $F(x, |\cdot|^2/2)$ is convex for all $x \in \mathcal{M}$.

Now we take $\mathcal{F} : L^2(\mathcal{M}, \mathcal{N}) \rightarrow \mathbb{R}$ defined as

$$\mathcal{F}(u) := \begin{cases} \int_{\mathcal{M}} F\left(\cdot, \frac{|du|^2}{2}\right) dm_g & \text{if } u \in W^{1,p}(\mathcal{M}, \mathcal{N}), u(\mathcal{M}) \subseteq B_h(p, R) \\ +\infty & \text{otherwise.} \end{cases}$$

In the above setting, the F -tension field is defined by

$$\tau_F(u) := \sum_i \left(\tilde{\nabla}_{e_i} \left(F' \left(x, \frac{|du|^2}{2} \right) u_* e_i \right) - F' \left(x, \frac{|du|^2}{2} \right) u_*(\nabla_{e_i} e_i) \right),$$

being $\{e_i\}_{i=1}^m$ any local orthonormal basis of $T\mathcal{M}$. Hereafter F' and F'' will denote first and second partial derivatives of $F(x, s)$ with respect to s .

To prove the convexity of the functional \mathcal{F} , one typically needs to derive formulas for its first and second variation. However, unlike in the previous literature, we have no sort of harmonicity condition which cancels out terms involving the F -tension and we have to keep

track all extra boundary terms. Taking these difficulties into account, after a tedious but routine computation (which can be reproduced following the lines of [10, 156]), we obtain

$$\frac{d}{dt}\Big|_{t=0} \mathcal{F}(u_t) = - \int_{\mathcal{M}} \langle V, \tau_F(u) \rangle_h \, dm_g + \int_{\partial\mathcal{M}} F' \left(x, \frac{|du|^2}{2} \right) \langle u_*\nu, V \rangle_h \, dm_{\tilde{g}},$$

where $U : \mathcal{M} \times [0, 1] \rightarrow \mathcal{N}$ is a one-parameter variation of $u \in C^\infty(\mathcal{M}, \mathcal{N})$ so that $U(\cdot, 0) = u$, $u_t := U(\cdot, t)$ and $V = U_* \partial_t \Big|_{t=0}$. Moreover, \tilde{g} is the induced Riemannian metric on $\partial\mathcal{M}$. In turn, the second variation formula reads as

$$(5.5) \quad \begin{aligned} \frac{d^2}{dt^2}\Big|_{t=0} \mathcal{F}(u_t) = & - \int_{\mathcal{M}} \left(\langle \tilde{\nabla}_{\partial_t} V, \tau_F(u) \rangle_h + F' \left(x, \frac{|du|^2}{2} \right) (|\tilde{\nabla} V|_h^2 - R(V, du)) \right) dm_g \\ & + \int_{\mathcal{M}} \left(F'' \left(x, \frac{|du|^2}{2} \right) \langle \tilde{\nabla} V, du \rangle_h^2 + F' \left(x, \frac{|du|^2}{2} \right) \langle \tilde{\nabla}_{\partial_t} V, u_*\nu \rangle_h \right) dm_{\tilde{g}}, \end{aligned}$$

with

$$R(V, du) = \sum_{i=1}^m {}^h\mathcal{R}(V, u_*e_i, u_*e_i, V) = \sum_{i=1}^m \text{Sec}_h(V, u_*e_i) \Big| V \wedge u_*e_i \Big|_h^2.$$

To prove the convexity of the energy functional we need the following result about density of smooth functions into manifolds with boundary.

LEMMA 5.7. *Let $R < R_\kappa$. Then, for any $q \geq 1$ the space $C^\infty(\mathcal{M}, \overline{B_h(p, R)})$ is dense in $W^{1,q}(\mathcal{M}, B_h(p, R))$.*

PROOF. Recall that we have an isometric embedding $\iota : N \hookrightarrow \mathbb{R}^N$ by Nash. Then for any $R' \in (R, R_\kappa)$ we can modify the metric outside $\iota(B_h(p, R')) \subseteq \mathbb{R}^N$ by flattening it out (see Figure 5.2) via a bi-Lipschitz ambient diffeomorphism which keeps the Euclidean metric outside a larger ball so that we get a Riemannian metric on all of \mathbb{R}^n for which the squared distance to the origin is smooth and still strictly convex (for a proof, see [142, Theorem 5]).

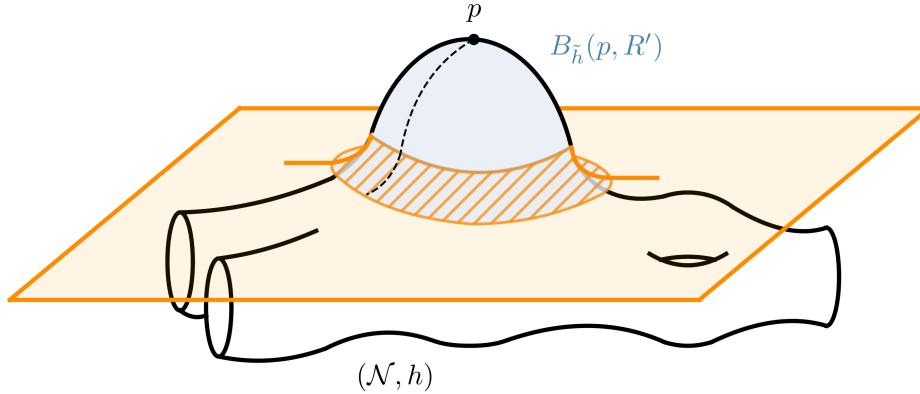


FIGURE 5.2. Schematic view of the idea of flattening the ends to construct a global coordinate system outside a geodesic ball.

The latter implies that (\mathcal{N}, h) can be covered by a global, normal coordinate chart centered at p , meaning that (\mathcal{N}, h) can be regarded as \mathbb{R}^n endowed with a metric that can be expressed globally in normal Cartesian coordinates, and hence the extrinsic definition (2.16) of $W^{1,p}(\mathcal{M}, \mathcal{N})$ coincides with the intrinsic one $W_{\text{in}}^{1,p}(\mathcal{M}, \mathcal{N})$ given by (2.17), cf. [142, Theorem 4]. In this setting, given $u \in W^{1,q}(\mathcal{M}, \mathcal{N})$ with $u(\mathcal{M}) \subseteq B_h(p, R)$, by [142, Corollary 7], we can find a sequence $u_k \in C^\infty(\mathcal{M}, \mathcal{N})$ such that $u_k \rightarrow u$ in $W^{1,q}(\mathcal{M}, \mathcal{N})$ as $k \rightarrow \infty$.

Accordingly, up to extracting a subsequence, we can assume that the convergence is pointwise a.e. and hence that $u_k(\mathcal{M}) \subseteq \overline{B_h(p, R)}$. \square

THEOREM 5.8. *If $\kappa \leq 0$ and $R < R_\kappa$, the functional \mathcal{F} is geodesically convex.*

PROOF. We split the proof into several steps:

Step 1. Second variation estimate. In the case that $\kappa \leq 0$ and U is a variation along a geodesic, then using (5.5) we deduce that

$$\frac{d^2}{dt^2} \Big|_{t=0} \mathcal{F}(u_t) \geq \int_{\mathcal{M}} \left(F' \left(x, \frac{|du|^2}{2} \right) |\tilde{\nabla} V|_h^2 + F'' \left(x, \frac{|du|^2}{2} \right) \langle \tilde{\nabla} V, du \rangle_h^2 \right) dm_g \geq 0,$$

since $F(x, |\cdot|^2/2)$ is convex.

Step 2. Convexity for smooth functions. Given $u, v \in C^\infty(\mathcal{M}, \mathcal{N})$ with $u(\mathcal{M}), v(\mathcal{M}) \subseteq B_h(p, R)$ for $R < R_\kappa$, one can always construct (see [150, proof of Theorem 2.9]) a geodesic homotopy $U : \mathcal{M} \times [0, 1] \rightarrow B_h(p, R)$ with $U(\cdot, 0) = u$ and $U(\cdot, 1) = v$. Let us denote for brevity $U(t) = U(\cdot, t)$ and recall that $t \mapsto U(t)$ are geodesics in \mathcal{N} for each fixed point in \mathcal{M} . Then, by the previous computations, since $t \mapsto \mathcal{F}(U(t))$ is a convex function, it holds:

$$(5.6) \quad \mathcal{F}(U(t)) \leq t\mathcal{F}(u) + (1-t)\mathcal{F}(v) \quad \text{for all } t \in [0, 1].$$

Step 3. Convexity for $W^{1,q}(\mathcal{M}, \mathcal{N})$ functions. We now observe that, since $B_h(p, R)$ is a NPC-space for $R < R_\kappa$, then $L^2(\mathcal{M}, B_h(p, R))$ is also a NPC-space [91, §4.1.1]. Therefore, given $u, v \in L^2(\mathcal{M}, B_h(p, R))$ there exists a unique geodesic $U(t)$ with $U(0) = u, U(1) = v$. In order to prove that (5.6) holds for u, v , it suffices to suppose that both belong to $W^{1,q}(\mathcal{M}, \mathcal{N})$ since, otherwise, the right-hand side is infinite.

Accordingly, we apply Lemma 5.7 to ensure that we can take $u_k, v_k \in C^\infty(\mathcal{M}, \overline{B_h(p, R)})$ such that $u_k, v_k \rightarrow u, v$ strongly in $W^{1,q}(\mathcal{M}, B_h(p, R))$. Consider now U_k the geodesic such that $U_k(0) = u_k$ and $U_k(1) = v_k$. Then, since we are in a NPC space we have that

$$\begin{aligned} \int_{\mathcal{M}} d_h(U(t), U_k(t))^2 dm_g &\leq (1-t) \int_{\mathcal{M}} d_h(U(t), u_k)^2 dm_g \\ &\quad + t \int_{\mathcal{M}} d_h(U(t), v_k)^2 dm_g - t(1-t) \int_{\mathcal{M}} d_h(u_k, v_k)^2 dm_g \end{aligned}$$

and then, as $k \rightarrow \infty$, we obtain that

$$(1-t) \int_{\mathcal{M}} d_h(U(t), u)^2 dm_g + t \int_{\mathcal{M}} d_h(U(t), v)^2 dm_g - t(1-t) \int_{\mathcal{M}} d_h(u, v)^2 dm_g = 0$$

Therefore, by lower semicontinuity with respect to the L^2 -convergence of the energy and strong convergence of the approximations, we can pass to the limit in (5.6) and show that (5.6) is satisfied by $u, v \in W^{1,q}(\mathcal{M}, B_h(p, R))$. \square

5.2.3. Geodesic convexity of manifold total variation. This result permits to show that the convexity also holds for the functional \mathcal{TV} given in (5.4).

THEOREM 5.9. *If $\kappa \leq 0$ and $R < R_\kappa$, then the functional \mathcal{TV} is geodesically convex with \mathcal{N} replaced by $B_h(p, R)$.*

PROOF. First, we define the following approximation of \mathcal{F} :

$$\mathcal{F}_\varepsilon(u) := \begin{cases} \int_{\Sigma} \sqrt{\varepsilon + |du|^2} dm_g & \text{if } u \in W^{1,1}(\Sigma, B_h(p, R)) \\ +\infty & \text{otherwise.} \end{cases}$$

Then, by Sobolev embedding and Theorem 5.8, we have that \mathcal{F}_ε is geodesically convex in $L^2(\Sigma, \mathcal{N})$. Note that due to boundedness of both domain and target in the case when the functional is finite, L^1 and L^2 topologies are equivalent.

Now we reason as in the last part of the proof of Theorem 5.8. By Theorem 5.6, if we take $u, v \in BV(\Sigma, B_h(p, R))$, there exist $\{u_k, v_k\}_k \subseteq C^1(\Sigma, \overline{B_h(p, R)})$ such that $u_k, v_k \rightarrow u, v$ in $L^1(\Omega)$ and $\mathcal{TV}(u_k) \rightarrow \mathcal{TV}(u)$ as $k \rightarrow \infty$.

As before, consider U_k the geodesic in $L^2(\Sigma, B_h(p, R))$ such that $U_k(0) = u_k$ and $U_k(1) = v_k$. Then, again since $L^2(\Sigma, B_h(p, R))$ is a NPC space, we obtain that $U_k \rightarrow U$ in the L^2 -topology, with U being the geodesic joining u and v . Therefore, letting $\varepsilon = \varepsilon_k \rightarrow 0$ as $k \rightarrow \infty$, by the lower semicontinuity given by Theorem 5.6 and by convexity of \mathcal{F}_ε , we have

$$\begin{aligned} \mathcal{TV}(U) &\leq \liminf_{k \rightarrow \infty} \mathcal{TV}(U_k(t)) \leq \limsup_{k \rightarrow \infty} (\mathcal{F}_{\varepsilon_k}(U_k(t))) \leq \\ &\limsup_{k \rightarrow \infty} (t\mathcal{F}_{\varepsilon_k}(u_k) + (1-t)\mathcal{F}_{\varepsilon_k}(v_k)) = t\mathcal{TV}(u) + (1-t)\mathcal{TV}(v). \end{aligned}$$

□

5.3. Partial regularity results for systems

In this section we provide some results of partial regularity for manifold constrained quasi-linear elliptic systems under some assumptions. Let us consider (\mathcal{N}, h) as before (possibly with $\partial\mathcal{N} \neq \emptyset$ and not necessarily compact). Given $\Omega \subseteq \mathbb{R}^m$ ($m \geq 2$) an open bounded Lipschitz domain, we study the regularity of weak solutions $u : \Omega \rightarrow \mathcal{N}$ to the following system of PDEs in extrinsic form:

$$(5.7) \quad -\operatorname{div}(a(x, u, \nabla u)\nabla u) = B(x, u, \nabla u).$$

Setting $A(x, v, z) := a(x, v, z)z$, we have two measurable vector fields A, B defined on $\Omega \times \mathbb{R}^N \times \mathbb{R}^{mN}$ and taking values in \mathbb{R}^{mN} and \mathbb{R}^N , respectively, while a is a scalar function on the same domain. We remark that the interest in these systems of diagonal type comes from their ubiquitous appearance in differential geometry and physics: isothermal coordinates, Plateau and capillarity problems, minimal surfaces and harmonic maps on Riemannian manifolds, or elliptic Monge-Ampère equations (see [101] for details).

It is well-known, since De Giorgi's counterexample [65], that there is no hope to get regularity of weak solutions in general. Accordingly, the goal is to focus on partial regularity, including estimates on the size of the singular set. The point of this section is to prove that a theorem in this spirit still holds in a suitable manifold constrained setting, which in fact allows us to relax the so-called classical smallness assumption. Furthermore, we obtain an intrinsic Caccioppoli type inequality (see Proposition 5.11), which is a tool of independent interest.

In the Euclidean framework (cf. [69, 86, 102]), the following structural and regularity conditions are often assumed:

$$(H1). \quad 0 < \ell \leq a(x, v, z) \leq L.$$

(H2). A has continuous derivatives with respect to z such that

$$\vartheta|\xi|^2 \leq D_z A(x, v, z)\xi \otimes \xi \leq \Theta|\xi|^2.$$

$$(H3). \quad |A(x+y, v+w, z) - A(x, v, z)| \leq L(|y| + |w|)^\alpha(1 + |z|) \text{ for some } \alpha \in (0, 1).$$

$$(H4). \quad B(x, u, \nabla u) \text{ is measurable for all } u \in H^1(\Omega, \mathbb{R}^N) \text{ and } |B(x, v, z)| \leq C_1|z|^2 + D_1.$$

where the above conditions hold for some $\ell, L, \vartheta, \Theta, C_1, D > 0$ and for all $x, y \in \Omega, v, w \in \mathbb{R}^N, z, \xi \in \mathbb{R}^{mN}$. For any weak solution $u \in H^1(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$ of (5.7), under the smallness condition

$$(5.8) \quad \|u\|_{L^\infty(\Omega, \mathcal{N})} < \frac{\vartheta}{2C_1},$$

it is known that u is partially regular, in the sense that ∇u is Hölder continuous on an open subset $\Omega_0 \subseteq \Omega$ of full measure. We stress that condition (5.8) is crucial to get a Caccioppoli inequality, which usually is a starting point for the regularity study.

In our manifold constrained setting, B is decomposed into two terms; the tangent and normal components to \mathcal{N} :

$$B(x, v, z) = \mathbf{t}(x, v, z) + \mathbf{n}(x, v, z) \in T_v \mathcal{N} \oplus T_v^\perp \mathcal{N} \quad \text{for all } (x, v, z) \in \Omega \times \mathcal{N} \times \mathbb{R}^{mN}.$$

By using intrinsic methods, we show that assumption H4 can be relaxed (see Section 5.3.1 for details) by asking instead that there exist some constants $C_{\mathbf{t}}, D \geq 0$ so that

$$(H4^*). \quad |\mathbf{t}(x, v, z)| \leq C_{\mathbf{t}}|z|^2 + D(1 + |z|), \quad \text{for all } (x, v, z) \in \Omega \times \mathcal{N} \times \mathbb{R}^{mN},$$

and we just require a smallness condition for $C_{\mathbf{t}}$, i.e., about the growth of the leading order term in the tangent part, cf. (5.9). We point out that the use of intrinsic methods allows to impose no restriction on the size of the extrinsic \mathbf{n} . As before, $\mathbf{t}(x, u, \nabla u)$ and $\mathbf{n}(x, u, \nabla u)$ are assumed to be measurable for all $u \in H^1(\Omega, \mathbb{R}^N)$.

Considering the function co_κ as follows,

$$\text{co}_\kappa(t) = \begin{cases} \sqrt{\kappa} \cot(\sqrt{\kappa} t), & \kappa > 0, \\ 1/t, & \kappa = 0, \\ \sqrt{|\kappa|} \coth(\sqrt{|\kappa|} t), & \kappa < 0, \end{cases}$$

we prove the next statement:

THEOREM 5.10. *Let $u \in H^1(\Omega, \mathcal{N})$ be a weak solution of (5.7) such that $u(\Omega) \subseteq B_h(p, R)$ for some $p \in \mathcal{N}$ and $R > 0$ with $R < R_\kappa/2$, and let assumptions H1-H3 and H4* hold, as well as the following smallness condition*

$$(5.9) \quad C_{\mathbf{t}} < \ell \min \left\{ \frac{1}{2R}, \text{co}_\kappa(2R) \right\}.$$

Then the first derivatives of u are Hölder continuous on an open set Ω_0 with $\mathcal{L}^m(\Omega \setminus \Omega_0) = 0$. Moreover, $\Omega_0 = \Omega$ for $m = 2$.

The above partial regularity result may be regarded as a natural extension of the already classical regularity statements by Eells and Sampson [70], Hildebrandt, Kaul and Widman [99] or Schoen and Uhlenbeck [151, Corollary] for harmonic mappings into Riemannian manifolds to the wider framework of F - or V -harmonic maps. In Section 5.3.3 we provide an interpretation of (5.7) as the Euler-Lagrange equation associated to an F -energy.

About the assumptions on $u(\Omega)$, we point out that for F - and V -harmonic maps, there holds a comparison principle, for instance, when coupled with Dirichlet boundary conditions (see [103] and [58, Theorem 3]). In this case, if the Dirichlet constraint lies in a geodesic ball with smaller radius than a threshold, then the solution does so too. Moreover, we highlight that in case that $A(x, v, z) = a(x, v)z$ our smallness condition (5.9) recovers that in [100] for the case $\kappa = 0$. Despite the condition $u(\Omega) \subseteq B_h(p, R)$ imposes certainly a restriction, as it amounts to remove any (local) topological defects of the map u , such a requirement

is nevertheless necessary in light of e.g. the construction by T. Rivière [145] of everywhere discontinuous weakly harmonic maps into \mathbb{S}^2 .

Concerning the strategy of the proof, under condition (5.9) and within half of the range of strict convexity, we prove an intrinsic Caccioppoli inequality in Proposition 5.11. This result is, to the best of our knowledge, the first inequality of this type where the so-called excess function (that is, the right-hand side) is fully intrinsic. This is novel even for the previous literature working on manifold constrained settings (compare with e.g. [64, Lemma 8] or [77, Theorem 5.2]). Using this intrinsic inequality, we show that weak solutions of the system belong to a Sobolev space $W^{1,q}(\Omega, \mathbb{R}^N)$ with a $q > 2$ in Proposition 5.12. As pointed out in [86, Remark 2.4], we remark that the smallness condition (5.8) is only needed to provide the integrability result. Therefore, applying [86, Theorem 2, Corollary 4.4] (in which no smallness condition is used), we immediately get the proof of Theorem 5.10.

5.3.1. How to get hypothesis H4 from the weaker H4*. In this section, we aim to show that the assumption H4 follows from the weaker requirement H4*. After proving this, as a by-product of Theorem 5.10, we recover the classical regularity result, but with a milder smallness condition.

Let us thus assume that H4* holds. First, if we split the right-hand side of (5.7) into a tangent and a normal part, the extrinsic part \mathbf{n} is completely characterized in terms of the second fundamental form \mathcal{A}_v of $\mathcal{N} \hookrightarrow \mathbb{R}^N$ at $v \in \mathcal{N}$ as

$$\mathbf{n}(x, u, \nabla u) = a(x, u, \nabla u) \sum_{\alpha=1}^m \mathcal{A}_{u(x)}(\partial_\alpha u, \partial_\alpha u), \quad \text{in } T_{u(x)}^\perp \mathcal{N}.$$

Next, we extend this expression to the whole of $\Omega \times \mathbb{R}^N \times \mathbb{R}^{mN}$ in a way that H4 holds. For this, we follow the ideas in [97, p. 108]. For a sufficiently small $\delta > 0$, we consider a tubular neighborhood of $\overline{B_h(p, R)}$:

$$T_\delta(\overline{B_h(p, R)}) := \left\{ x \in \mathbb{R}^{N+1} : \text{dist}(x, \overline{B_h(p, R)}) < \delta \right\},$$

where dist denotes the Euclidean distance. Let $0 \leq \psi \leq 1$ be a test function such that $\psi(\overline{B_h(p, R)}) = 1$ and $\text{supp}(\psi) \subseteq T_\delta(\overline{B_h(p, R)})$. Observe that any point in $T_\delta(\overline{B_h(p, R)})$ can be projected onto $\overline{B_h(p, R)}$ uniquely by a submersion π_δ (e.g. see [161]). In this setting, we can define the extensions

$$\tilde{\mathbf{n}}(x, v, z) := \begin{cases} \psi(v) a(x, v, z) \sum_{\alpha=1}^m \mathcal{A}_{\pi_\delta(v)}(z_\alpha, z_\alpha), & \text{if } (x, v, z) \in \Omega \times T_\delta(\overline{B_h(p, R)}) \times \mathbb{R}^{mN} \\ 0 & \text{otherwise.} \end{cases}$$

$$\tilde{\mathbf{t}}(x, v, z) := \begin{cases} \psi(v) \mathbf{t}(x, v, z), & \text{if } (x, v, z) \in \Omega \times T_\delta(\overline{B_h(p, R)}) \times \mathbb{R}^{mN}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, by imposing just H4*, it follows that H4 holds for $\tilde{B} = \tilde{\mathbf{t}} + \tilde{\mathbf{n}}$ with

$$C_1 := C_t + \frac{D}{2} + L \|\mathcal{A}\|_{L^\infty(B_h(p, R))} \quad \text{and} \quad D_1 := 3D/2$$

Here we have used the notation

$$\|\mathcal{A}\|_{L^\infty(B_h(p, R))} := \inf \left\{ C \in \mathbb{R}^+ : \mathcal{A}_q(X, X) \leq C|X|^2 \text{ for all } q \in B_h(p, R) \text{ and } X \in T_q \mathcal{N} \right\}.$$

Finally, let us remark that u also solves (5.7) with \tilde{B} on the right-hand side, since f and \tilde{B} coincide on $\Omega \times \mathcal{N} \times \mathbb{R}^{mN}$. Therefore, without loss of generality, we will work with the extension \tilde{B} in the sequel (but we keep the notation B for clarity).

5.3.2. Caccioppoli inequality and integrability of solutions. Given $x_0 \in \Omega$ and $r < \text{dist}(x_0, \partial\Omega)$ we define the barycenter of u in the ball $B_r(x_0)$ as the barycenter of the particular measure

$$\mathfrak{m} := u \# \mathcal{L}^m|_{\overline{B_r(x_0)}},$$

whose support by the assumption about $u(\Omega)$ lies on $\overline{B_h(p, R)}$. Namely, the barycenter can be regarded simply as the minimizer of

$$\mathfrak{b} \mapsto \int_{B_r(x_0)} d_h^2(u(x), \mathfrak{b}) dx.$$

PROPOSITION 5.11. *Under the assumptions of Theorem 5.10, the following intrinsic Caccioppoli inequality holds:*

$$(5.10) \quad \int_{B_r(x_0)} |\nabla u|^2 dx \leq \frac{C}{r^2} \int_{B_{2r}(x_0)} (1 + d_h^2(u, \mathfrak{b}_{2r})) dx,$$

for some $C > 0$, for all $x_0 \in \Omega$ and for r satisfying

$$r < \min \left\{ \frac{\text{dist}(x_0, \partial\Omega)}{2}, 1 \right\};$$

where \mathfrak{b}_{2r} denotes the barycenter of u in the ball $B_{2r}(x_0)$.

PROOF. First of all, we observe that the condition $u(\Omega) \subseteq B_h(p, R)$ permits to say that \mathfrak{b}_{2r} exists, and that $\mathfrak{r}_{\mathfrak{b}} := d_h(u, \mathfrak{b}_{2r}) < R_\kappa$ for all $x_0 \in \Omega$ and $r < \text{dist}(x_0, \partial\Omega)/2$. Unless otherwise stated, B_r denotes hereafter a ball centered in x_0 with radius r .

Let us consider x_0 and r as above, and let $\eta \in C_0^\infty(\Omega)$ be such that $0 \leq \eta \leq 1$, $\eta|_{B_r} \equiv 1$, $\text{supp}(\eta) \subseteq B_{2r}$ and $|\nabla \eta| < c/r$ for a some $c > 0$. We take

$$\varphi = -\eta^2 \exp_u^{-1} \mathfrak{b}_{2r} =: \eta^2 X_{\mathfrak{b}} \in T_u \mathcal{N}.$$

Note that $X_{\mathfrak{b}}$ inherits the regularity of the map u , and by [104, Theorem 6.6.1] it holds

$$(5.11) \quad \begin{aligned} \text{Hess} \left(\frac{\mathfrak{r}_{\mathfrak{b}}^2}{2} \right) (\nabla u, \cdot)^\# &= \nabla_{\nabla u} \left(\frac{\nabla \mathfrak{r}_{\mathfrak{b}}^2}{2} \right) = -\nabla_{\nabla u} (\exp_x^{-1} \mathfrak{b}_{2r}) \\ &= -\sum_{\alpha=1}^m \nabla_{e_\alpha} (\exp_{u(x)}^{-1} \mathfrak{b}_{2r}) = \nabla X_{\mathfrak{b}}, \end{aligned}$$

where $\{e_\alpha\}_{\alpha=1}^m$ is the canonical orthonormal basis and $\#$ is the musical isomorphism i.e. $\omega^\#$ is the vector field given by $g(\omega^\#, \cdot) = \omega(\cdot)$ for any 1-form ω .

Now, if we multiply (5.7) by φ , we can write

$$-\int_{B_{2r}} \eta^2 \text{div}(A(x, u, \nabla u)) \cdot X_{\mathfrak{b}} dx = \int_{B_{2r}} \eta^2 \mathfrak{t}(x, u, \nabla u) \cdot X_{\mathfrak{b}} dx.$$

Integrating by parts and using (5.11), we get

$$\begin{aligned}
(5.12) \quad & \int_{B_{2r}} \eta^2 a(x, u, \nabla u) \operatorname{Hess} \left(\frac{\mathbf{r}_b^2}{2} \right) (\nabla u, \nabla u) \, dx \\
& = \int_{B_{2r}} \eta^2 \mathbf{t}(x, u, \nabla u) \cdot X_b \, dx - 2 \int_{B_{2r}} \eta a(x, u, \nabla u) \nabla \eta \cdot \nabla u \cdot X_b \, dx =: \mathcal{J}.
\end{aligned}$$

Concerning the left-hand side of equation (5.12), by means of the comparison theorem (see [104, Theorem 6.6.1] or [147, p. 154]), we get

$$\begin{aligned}
(5.13) \quad & \int_{B_{2r}} \eta^2 a(x, u, \nabla u) \operatorname{Hess} \left(\frac{\mathbf{r}_b^2}{2} \right) (\nabla u, \nabla u) \, dx \\
& \geq \ell \int_{B_{2r}} \eta^2 \mathbf{r}_b \operatorname{co}_\kappa(\mathbf{r}_b) |\nabla u|^2 \, dx \geq \ell \min \{1, 2R \operatorname{co}_\kappa(2R)\} \int_{B_{2r}} \eta^2 |\nabla u|^2 \, dx.
\end{aligned}$$

On the other hand, using Young's inequality, we have, for any $\varepsilon > 0$,

$$\begin{aligned}
\mathcal{J} & \leq 2RC_t \int_{B_{2r}} \eta^2 |\nabla u|^2 \, dx + D \int_{B_{2r}} \eta^2 (1 + |\nabla u|) \mathbf{r}_b \, dx \\
& \quad + \varepsilon \int_{B_{2r}} \eta^2 |\nabla u|^2 \, dx + \frac{L^2 c^2}{r^2 \varepsilon} \int_{B_{2r}} \mathbf{r}_b^2 \, dx \\
& \leq 2RC_t \int_{B_{2r}} \eta^2 |\nabla u|^2 \, dx + \frac{D}{2} \int_{B_{2r}} (1 + \mathbf{r}_b^2) \, dx \\
& \quad + \varepsilon(1 + D) \int_{B_{2r}} \eta^2 |\nabla u|^2 \, dx + \frac{1}{\varepsilon} \left(\frac{L^2 c^2}{r^2} + \frac{D}{4} \right) \int_{B_{2r}} \mathbf{r}_b^2 \, dx.
\end{aligned}$$

Under the smallness condition, combining the expressions (5.12), (5.13), (5.3.2), and having in mind that $r < 1$, we easily get (5.10). \square

PROPOSITION 5.12. *Let $u \in H^1(\Omega, \mathcal{N})$ be such that $u(\Omega) \subseteq B_h(p, R)$ for some $p \in \mathcal{N}$ and $R < \operatorname{inj}_p(\mathcal{N})/2$, and satisfying (5.10). Then, there exists $q > 2$ and $C > 0$ such that $u \in W^{1,q}(\Omega, \mathbb{R}^N)$ and*

$$\left(\int_{B_r(x_0)} (1 + |\nabla u|)^q \, dx \right)^{2/q} \leq C \int_{B_{2r}(x_0)} (1 + |\nabla u|^2) \, dx,$$

for every $x_0 \in \Omega$ and $0 < r < \operatorname{dist}(x_0, \partial\Omega)/2$.

PROOF. First, observe that the range condition $u(\Omega) \subseteq B_h(p, R)$, with $R < \operatorname{inj}_p(\mathcal{N})/2$ yields the existence of \mathbf{b}_{2r} and the injectivity of the exponential map centered at \mathbf{b}_{2r} in the range of u . Therefore, we can choose polar coordinates centered at \mathbf{b}_{2r} , and we find that

$$d_h(u, \mathbf{b}_{2r})^2 = |u^\rho|^2 = \left| u^\rho - \int_{B_{2r}} u^\rho \, dx \right|^2,$$

where u^ρ is the radial component of u . Consequently, using Sobolev–Poincaré inequality we note that

$$\begin{aligned}
& \frac{1}{r^2} \int_{B_{2r}} d_h^2(u, \mathbf{b}_{2r}) \, dx \leq Cr^{m-2} \int_{B_{2r}} |u^\rho|^2 \, dx \\
& \leq Cr^{m-2} r^2 \left(\int_{B_{2r}} |\nabla u^\rho|^{\frac{2m}{m+2}} \, dx \right)^{\frac{m+2}{m}} \leq Cr^m \left(\int_{B_{2r}} |\nabla u|^{\frac{2m}{m+2}} \, dx \right)^{\frac{m+2}{m}},
\end{aligned}$$

where, in the last inequality we use that $|\nabla u^\rho| \leq |\nabla u|$. Therefore, from (5.10), we find

$$\int_{B_r} |\nabla u|^2 dx \leq C \left(\int_{B_{2r}} (1 + |\nabla u|)^{\frac{2m}{m+2}} dx \right)^{\frac{m+2}{m}}.$$

Finally, the result is obtained by [87, Proposition 5.1]. \square

5.3.3. Examples. Quasilinear elliptic systems of constrained PDEs of type (5.7) naturally arise when one minimizes the following energy functionals in $H^1(\Omega, \mathcal{N})$:

$$\mathcal{F}(u) := \begin{cases} \int_{\Omega} F\left(x, \frac{|du|^2}{2}\right) dx, & \text{if } u \in H^1(\Omega, \mathcal{N}) \\ +\infty, & \text{otherwise.} \end{cases}$$

In fact, the Euler-Lagrange equation associated to \mathcal{F} can be written in terms of the tension field τ as

$$(5.14) \quad 0 = \tau_F(u) = F'\left(x, \frac{|du|^2}{2}\right) \tau(u) + \nabla u \cdot \nabla \left(F'\left(x, \frac{|du|^2}{2}\right) \right),$$

where $\tau(u) = P_u(\Delta u)$ such that $P_u : \mathbb{R}^N \rightarrow T_u \mathcal{N}$ is the orthogonal projection and F' the partial derivative of $F(x, s)$ with respect to s . Notice that (5.14) includes the well studied cases of F -harmonic maps ([10]), if there is no x -dependence, as well as \mathbf{f} -harmonic maps (introduced by Lichnerowicz in [115]) with $F(x, s) = \mathbf{f}(x)s$. We point out that (5.14) can be written as

$$-\operatorname{div}\left(F'\left(x, \frac{|du|^2}{2}\right) \nabla u\right) = F'\left(x, \frac{|du|^2}{2}\right) \sum_{\alpha=1}^m \mathcal{A}_u(\partial_\alpha u, \partial_\alpha u).$$

Therefore, as explained in Section 5.3.1, (5.14) can be put into the form (5.7) with right-hand side satisfying H4.

Another example in which our study can be applied is the case of V -harmonic maps, with V a given smooth vector field on Ω ([58]). Then the system of PDEs reads as

$$\tau(u) - \langle V, \nabla u \rangle = 0.$$

In this case, $\mathbf{t}^i(x, z) := V^\alpha(x) z_\alpha^i$, satisfies H4* with $C_t = 0$. Notice that, in the case that $F \in C^2(\Omega \times \mathbb{R})$, such that $0 < \ell \leq F'(x, s) \leq L$ and the function $G(x, z) := F(x, |z|^2/2)$ is uniformly convex in z , assumptions H1-H3 are satisfied.

5.4. Existence and uniqueness of minimizers

We begin this section with the following invariance principle for minimizers, which guarantees that the small range condition of the source is inherited by any minimizer.

PROPOSITION 5.13. *Given $\sigma \geq 0$, let u be a minimizer of $\mathcal{E}_N^{\lambda, \sigma}$. If f satisfies $f(\overline{\Sigma}) \subseteq B_h(p, R)$ for some $p \in \mathcal{N}$ and $R < R_\kappa$, then $u(\overline{\Sigma}) \subseteq B_h(p, R)$.*

PROOF. By assumption, we can introduce geodesic polar coordinates (r, θ) in $B_h(p, 2R)$ centered at p . We consider the following Lipschitz retraction: $\pi : \mathcal{N} \rightarrow B_h(p, R)$, with

$$\pi(q) := \begin{cases} (r, \theta) & \text{if } q = (r, \theta), r < R \\ (2R - r, \theta) & \text{if } q = (r, \theta), R \leq r < 2R, \\ p & \text{if } q \in \mathcal{N} \setminus B_h(p, 2R) \end{cases}$$

We will show that π is length decreasing; i.e. $d_h(\pi(q), \pi(\tilde{q})) \leq d_h(q, \tilde{q})$ for any $q, \tilde{q} \in \mathcal{N}$ as in the proof of [104, Lemma 10.2.4]. First, observe that it is enough to prove

$$(5.15) \quad d_h(\pi(r, \theta), \pi(\tilde{r}, \tilde{\theta})) \leq d_h((r, \theta), (\tilde{r}, \tilde{\theta})) \quad \text{for } q, \tilde{q} \in B_h(p, 2R).$$

By construction, π is clearly length decreasing in r ; i.e. (5.15) holds with θ fixed. Therefore, we will only prove the case that $r = \tilde{r}$, for which we take a curve $\gamma(s) := (r, \theta(s))$. Now, for each s fixed, $c_s(t) := (t, \theta(s))$ is a radial geodesic with $c_s(0) = p$ and $c_s(r) = \gamma(s)$. Accordingly, $J_s(t) := \frac{\partial}{\partial s} c_s(t)$ is a Jacobi field with $J_s(0) = 0$, $J_s(r) = \gamma'(s)$ and $d\pi(\gamma'(s)) = J_s(r_\pi)$, with $(r_\pi, \theta) = \pi(\gamma(s))$ ($r_\pi = 2R - r < R < r \leq 2R$).

Thus, assuming without loss of generality that $\gamma'(s) \neq 0$, by Rauch comparison theorem and a straightforward computation, we obtain that

$$\frac{|\gamma'(s)|}{|d\pi(\gamma'(s))|} = \frac{|J_s(r)|}{|J_s(r_\pi)|} \geq \frac{\sin_\kappa(r)}{\sin_\kappa(2R - r)} > 1,$$

where $\sin_\kappa(\cdot) = \sin(\sqrt{\max\{\kappa, 0\}} \cdot)$. Hence π is also length decreasing in the θ directions.

Consequently, $\mathcal{E}_\mathcal{N}^{\lambda, \sigma}(\pi \circ u) \leq \mathcal{E}_\mathcal{N}^{\lambda, \sigma}(u)$, with strict inequality unless $\pi \circ u = u$. Thus, by the minimality of u , we conclude that $u(\Sigma) \subseteq B_h(p, R)$. \square

As a by-product, under the hypothesis of the above statement, we can restrict the search of admissible minimizers of $\mathcal{E}_\mathcal{N}^{\lambda, \sigma}$ to the following set:

$$\Lambda_R^\sigma := \begin{cases} \{u \in H^1(\Sigma, \mathcal{N}) : u(\Sigma) \subseteq B_h(p, R)\} & \text{if } \sigma > 0 \\ \{u \in BV(\Sigma, \mathcal{N}) : u(\Sigma) \subseteq B_h(p, R)\} & \text{if } \sigma = 0 \end{cases}$$

PROPOSITION 5.14. *Given $\sigma \geq 0$, if f satisfies $f(\Sigma) \subseteq B_h(p, R)$ for some $p \in \mathcal{N}$ and $R < R_\kappa$, then there exists a minimizer of $\mathcal{E}_\mathcal{N}^{\lambda, \sigma}$ in Λ_R^σ .*

PROOF. We follow the direct method in the calculus of variations. However, we will only give the proof in the case $\sigma = 0$, the other case being similar, but easier.

We note that $\mathcal{E}_\mathcal{N}^\lambda(u) \geq 0$. Thus there exists a minimizing sequence $\{u_k\}_k$ such that $\mathcal{E}_\mathcal{N}^\lambda(u_k) \rightarrow \inf \mathcal{E}_\mathcal{N}^\lambda$ as $k \rightarrow \infty$. Moreover, by Proposition 5.13, we can assume that $u_k(\Sigma) \subseteq B_h(p, R)$. Now, by (2.15) and the definition of \mathcal{TV} , we have that $\mathcal{E}_\mathcal{N}^\lambda$ is coercive. Therefore, there is a subsequence, not relabeled, and $u \in BV(\Sigma, \mathbb{R}^N)$ such that $u_k \rightarrow u \in L^1(\Sigma, \mathbb{R}^N)$. Hence we may further assume that $u_k \rightarrow u$ a.e. in Σ . This yields that $u \in \Lambda_R^0$. Finally, by lower semicontinuity, we conclude that u is a minimizer of $\mathcal{E}_\mathcal{N}^\lambda$. \square

As a direct consequence of the convexity results in Theorems 5.8, 5.9 and the fact that d_h^2 is strictly convex in $B_h(p, R)$ for $R < R_\kappa$, we obtain the following result.

PROPOSITION 5.15. *Under the hypothesis of Proposition 5.14, if $\kappa \leq 0$, then $\mathcal{E}_\mathcal{N}^{\lambda, \sigma}$ has a unique minimizer in Λ_R^σ .*

5.5. Regularity results for an approximate problem

For $\sigma \geq 0$, a minimizer u of $\mathcal{E}_\mathcal{N}^{\lambda, \sigma}$ is a weak solution to the corresponding Euler-Lagrange system:

$$(5.16) \quad \begin{cases} \operatorname{div}_g Z_u = -\lambda \exp_u^{-1} f & \text{in } \Sigma, \\ \nu \cdot Z_u = 0 & \text{on } \partial\Sigma, \end{cases}$$

where ν represents the outer unit normal of $\partial\Sigma$, and Z_u is given by

$$Z_u := \left(\frac{1}{|du|} + \sigma \right) du.$$

Even in the smooth case; i.e $u \in H^1(\Sigma, \mathcal{N})$, we need to clarify the meaning of $\frac{du}{|du|}$. We understand it as a multivalued function defined as

$$(5.17) \quad \frac{du}{|du|} : x \mapsto \begin{cases} \frac{du(x)}{|du(x)|}, & \text{if } du(x) \neq 0, \\ B_{g \boxtimes h}(0, 1) \subseteq T_x^* \Sigma \otimes T_{u(x)} \mathcal{N}, & \text{if } du(x) = 0. \end{cases}$$

Our definition of a (regular) solution to (5.16) is the following one:

DEFINITION 5.16. *Given $\sigma \geq 0$, we say that $u \in C^{0,1}(\bar{\Sigma}, \mathcal{N})$ is a (regular) solution of (5.16) if there exists $Z \in T^* \Sigma \otimes u^* T \mathcal{N}$ with $\operatorname{div}_g Z \in L^2(\Sigma, \mathbb{R}^n)$ satisfying*

$$\operatorname{div}_g Z = -\lambda \exp_u^{-1} f \quad \text{with} \quad Z - \sigma du \in \frac{du}{|du|} \quad \mathcal{L}^2\text{-a.e. in } \Sigma,$$

and it fulfills the homogeneous Neumann condition

$$\nu \cdot Z^\alpha = 0 \quad \mathcal{H}^1\text{-a.e. on } \partial \Sigma, \quad \text{for all } \alpha = 1, \dots, n.$$

5.5.1. Setup of the regularized functional/system. In this section, we will study the regularity of the minimizers of $\mathcal{E}_{\mathcal{N}}^{\lambda, \sigma}$. Firstly, we consider the next approximation to $\mathcal{E}_{\mathcal{N}}^{\lambda, \sigma}$:

$$\mathcal{E}_{\mathcal{N}, \varepsilon}^{\lambda, \sigma}(u) := \begin{cases} \int_{\Sigma} \sqrt{|du|^2 + \varepsilon^2} \, dm_g + \frac{\lambda}{2} \int_{\Sigma} d_h^2(u, f) \, dm_g + \frac{\sigma}{2} \int_{\Sigma} |du|^2 \, dm_g & \text{if } u \in H^1(\Sigma, \mathcal{N}) \\ +\infty & \text{otherwise.} \end{cases}$$

with $\varepsilon, \sigma > 0$. As in Proposition 5.14, the existence of a minimizer for $\mathcal{E}_{\mathcal{N}, \varepsilon}^{\lambda, \sigma}$ is guaranteed.

LEMMA 5.17. *Given $f \in L^2(\Sigma, \mathcal{N})$, there exists $u \in H^1(\Sigma, \mathcal{N})$ which minimizes $\mathcal{E}_{\mathcal{N}, \varepsilon}^{\lambda, \sigma}$.*

PROOF. Let $\{u_k\}_k \subseteq H^1(\Sigma, \mathcal{N})$ be a minimizing sequence. Then, identifying again \mathcal{N} -valued functions with their extension by Nash embedding, we obtain a subsequence, not relabeled, and a function $u \in H^1(\Sigma, \mathbb{R}^N)$ such that $u_k \rightarrow u$ in $H^1(\Sigma, \mathbb{R}^N)$ and $u_k \rightarrow u$ a.e. in Σ . Therefore, by lower semicontinuity, we conclude that $u \in H^1(\Sigma, \mathcal{N})$ is a minimizer. \square

Any minimizer of $\mathcal{E}_{\mathcal{N}, \varepsilon}^{\lambda, \sigma}$ is a weak solution to the corresponding Euler-Lagrange system

$$(\mathcal{S}_{\varepsilon, \sigma}^f) \quad \begin{cases} \operatorname{div}_g \mathcal{Z}_{\varepsilon, \sigma} = -\lambda \exp_u^{-1} f & \text{in } \Sigma \\ \nu \cdot du = 0 & \text{on } \partial \Sigma \end{cases}$$

such that $\mathcal{Z}_{\varepsilon, \sigma}$ is defined as

$$\mathcal{Z}_{\varepsilon, \sigma} = \mathcal{Z}_{\varepsilon, \sigma}(x, du) := \left(\frac{1}{\sqrt{|du|^2 + \varepsilon^2}} + \sigma \right) du.$$

Now the goal is to study the regularity of weak solutions to $(\mathcal{S}_{\varepsilon, \sigma}^f)$. In order to do this, we write the system locally using extrinsic coordinates in the target, by Nash theorem and isothermal coordinates in the domain, as described in Sections 2.4.1 and 2.4.2. Moreover, since we want to obtain regularity estimates up to the boundary of Σ , we will focus on

neighborhoods around any point in $\partial\Sigma$, the proofs for interior neighborhoods being similar, but easier than the boundary case.

We now fix some notation. For $x \in \Sigma$ and $r > 0$, $B_r(x)$ denotes the open ball in \mathbb{R}^2 with center $x = (x_1, x_2)$ and radius $r > 0$. We consider the following sets:

$$\begin{aligned} B_r^+(\bar{x}) &= B_r(x) \cap \{x_2 > \bar{x}_2\} \\ \Omega_r(\bar{x}) &= B_r(\bar{x}) \cap B_1^+(0) \end{aligned}$$

The first step towards regularity is to flatten $\partial\Sigma$ around a fixed point $z \in \partial\Sigma$. As $\partial\Sigma$ is smooth, there exists a smooth transformation Ψ_z for a some neighborhood \mathcal{U} of z such that

$$\Psi_z(\mathcal{U} \cap \Sigma) = B_1(0) \cap \{x_2 > 0\} =: B_1^+, \quad \Psi_z(\mathcal{U} \cap \partial\Sigma) = B_1(0) \cap \{x_2 = 0\} =: \Gamma_1.$$

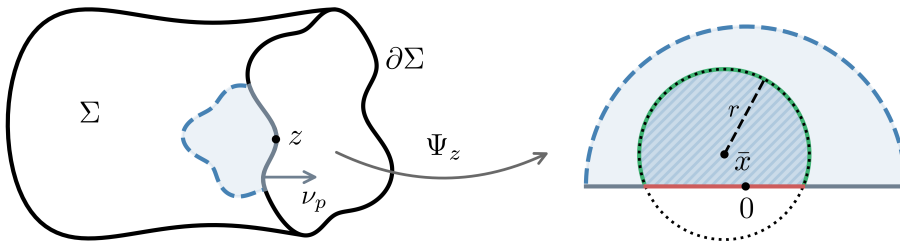


FIGURE 5.3. Setup for the problem after locally flattening the boundary. The blue area denotes the neighborhood \mathcal{U} in the left figure and B_1^+ in the right figure. The striped area represents $\Omega_r(\bar{x})$, the red segment indicates $\Gamma_1 \cap \partial\Omega_r$, and the green segment marks the region where $u = v$.

Without loss of generality, we may assume that $\mathcal{U} = U_\ell$ and $\Psi_z = \phi_\ell$ for some ℓ . For simplicity we write $\varrho := \varrho_\ell$ (recall the notation from Section 2.4.2). Therefore, any weak solution to $(\mathcal{S}_{\varepsilon, \sigma}^f)$ must satisfy

$$(5.18) \quad \begin{cases} \sum_i \partial_i A_i^\alpha(x, \nabla u) = B^\alpha(x, u, \nabla u) & \text{for } x \in B_1^+ \\ A_2^\alpha(x, \nabla u)|_{\Gamma_1} = 0 \end{cases}$$

for $\alpha \in \{1, \dots, N\}$, with

$$(5.19) \quad A_i^\alpha(x, \xi) = (a(x, \xi) + \sigma)\xi_i^\alpha,$$

whereas a and B are defined as

$$(5.20) \quad \begin{aligned} a(x, \xi) &:= \frac{\varrho(x)}{\sqrt{|\xi|^2 + \varepsilon^2 \varrho^2(x)}}, \\ B^\alpha(x, p, \xi) &:= -(a(x, \xi) + \sigma) \sum_i \mathcal{A}_p^\alpha(\xi_i, \xi_i)(x) - \lambda \varrho^2(x) (\exp_p^{-1} f(x))^\alpha. \end{aligned}$$

In particular, we note that A satisfies by definition that

$$(5.21) \quad |A(x, \xi)| \leq (C/\varepsilon + \sigma)|\xi|.$$

Moreover, the matrix $A_{ij}^{\alpha\beta}(x, \xi) := \frac{\partial A_i^\alpha(x, \xi)}{\partial \xi_j^\beta}$ satisfies the following ellipticity condition:

$$\sum_{i,j,\alpha,\beta} A_{ij}^{\alpha\beta}(x, \xi) \eta_i^\alpha \eta_j^\beta = \frac{-a(x, \xi)}{\xi^2 + \varrho^2(x)\varepsilon^2} \sum_{i,\alpha} (\xi_i^\alpha \eta_i^\alpha)^2 + (a(x, \xi) + \sigma)|\xi|^2 \geq \sigma|\eta|^2$$

for all $\eta \in \mathbb{R}^{2N}$. Furthermore, for every pair $p, q \in \mathbb{R}^{2N}$ the next inequalities hold:

$$(5.22) \quad \sum_{i,\alpha} (A_i^\alpha(x, \xi) - A_i^\alpha(x, \eta))(\xi_i^\alpha - \eta_i^\alpha) \geq \sigma|\xi - \eta|^2,$$

$$\sum_{i,\alpha} A_i^\alpha(x, \xi) \xi_i^\alpha \geq \sigma|\xi|^2,$$

$$(5.23) \quad |A(x, \xi) - A(y, \xi)| \leq \frac{C}{\min \varrho} |x - y|,$$

$$(5.24) \quad |A(x, \xi) - A(x, \eta)| \leq \left(\frac{C}{\min \varrho} + \sigma \right) |\xi - \eta|.$$

Now, we observe that

$$B(x, p, \xi) = \mathbf{n}(x, p, \xi) + \mathbf{t}(x, p) \in T_p \mathcal{N} \oplus T_p^\perp \mathcal{N},$$

with $\mathbf{t}(x, p) = -\lambda \varrho^2(x) \exp_p^{-1}(f(x))$.

Therefore, we can unconstrain the problem by extending \mathbf{n} and \mathbf{t} to $B_1^+ \times \mathbb{R}^N \times \mathbb{R}^{2N}$ as in Section 5.3.1. Thus, we assume that \mathbf{n} , \mathbf{t} (and therefore B) have already been extended in such a way that the following holds:

$$(5.25) \quad |\mathbf{t}(x, p)| \leq C|p|, \quad |B(x, p, \xi)| \leq C_1 + C_2|\xi|^2.$$

We will also need a generalized Poincaré inequality for hemispheres. The proof of the following result, which we omit, is an easy adaptation of the Poincaré inequality for balls in [73, Theorem 2, p.291], combined with Lemma 2.1.

LEMMA 5.18. *Let $\bar{x} \in B_1^+$ and $r_0 < \text{dist}(\bar{x}, \partial B_1^+ \setminus \Gamma_1)$. Then, there exists $C > 0$ such that, for all $r \leq r_0$, it holds*

- (a) $\|u\|_{L^1(\Omega_r)} \leq Cr \|\nabla u\|_{L^1(\Omega_r)}$, for any $u \in W_{\partial\Omega_r \setminus \Gamma_1}^{1,1}(\Omega_r)$,
- (b) $\|u - (u)_{\bar{x}, r}\|_{L^1(\Omega_r)} \leq Cr \|\nabla u\|_{L^1(\Omega_r)}$, for any $u \in W^{1,1}(\Omega_r)$

5.5.2. Higher integrability of solutions. Let $\bar{x} \in B_1^+$ and $r < \text{dist}(\bar{x}, \partial B_1^+ \setminus \Gamma_1)$ and suppose that $u(\overline{\Omega}_r(\bar{x})) \subseteq \overline{B_g(p, R)}$ with $R < R_\kappa$. We define the barycenter of u in $\Omega_r(\bar{x})$ as the barycenter of the measure $\mu := u \# \mathcal{L}^m|_{\overline{B_r(x_0)}}$; that is, the minimizer of

$$\mathbf{b} \mapsto \int_{\Omega_r(\bar{x})} d_g^2(u(x), \mathbf{b}) dx.$$

using the notation of Section 5.3.2. Moreover, we set the next radius

$$(5.26) \quad R_\kappa^* := \begin{cases} \frac{\text{inj}_p \mathcal{N}}{2} & \text{if } \kappa \leq 0 \\ \min \left\{ \frac{\text{inj}_p \mathcal{N}}{2}, \frac{\pi}{4\sqrt{\kappa}} \right\} & \text{if } \kappa > 0 \end{cases}.$$

Thus, using Propositions 5.11 and 5.12, we deduce the following result:

PROPOSITION 5.19. *Let $f \in L^2(\Sigma, \mathcal{N})$ be such that $f(\Sigma) \subseteq B_h(p, R)$ with some $p \in \mathcal{N}$ and $R < R_\kappa^*$ and let $u \in \Lambda_R^\sigma$ be a weak solution of $(\mathcal{S}_{\varepsilon, \sigma}^f)$. Then the next intrinsic Caccioppoli inequality is satisfied*

$$(5.27) \quad \int_{\Omega_r(\bar{x})} |\nabla u|^2 dx \leq \frac{C}{r^2} \int_{\Omega_{2r}(\bar{x})} (1 + d_g^2(u, \mathbf{b}_{2r})) dx$$

for all $\bar{x} \in \overline{B_1^+}$ and $r < \frac{\text{dist}(\bar{x}, \partial B_1^+ \setminus \Gamma_1)}{2}$, where $C > 0$ and \mathbf{b}_{2r} denotes the barycenter of u in $\Omega_{2r}(\bar{x})$. Consequently, there exist $q > 2$ and $\tilde{C} > 0$ such that $u \in W^{1,q}(B_1^+, \mathbb{R}^N)$ and

$$(5.28) \quad \left(\int_{\Omega_r(\bar{x})} (1 + |\nabla u|^q) dx \right)^{1/q} \leq \tilde{C} \left(\int_{\Omega_{2r}(\bar{x})} (1 + |\nabla u|^2) dx \right)^{1/2}$$

for all $\bar{x} \in \overline{B_1^+}$ and $r < \frac{\text{dist}(\bar{x}, \partial B_1^+ \setminus \Gamma_1)}{2}$.

From now on, we will assume that $f(\Sigma) \subseteq B_h(p, R)$ for some $p \in \mathcal{N}$ and $R < R_\kappa^*$.

5.5.3. Hölder regularity of the approximate solutions. The next step is to prove Hölder regularity of weak solutions to $(\mathcal{S}_{\varepsilon, \sigma}^f)$.

THEOREM 5.20. *Let $u \in H^1(\Sigma, \mathcal{N})$ be a weak solution of $(\mathcal{S}_{\varepsilon, \sigma}^f)$. Then $u \in C^{0,\beta}(\overline{\Sigma}, \mathcal{N})$ for all $\beta \in (0, 1)$.*

PROOF. Fix $\bar{x} \in \overline{B_{1-\delta}^+(0)}$, with $0 < \delta < 1$, take $r < \frac{\delta}{2} < 1$, and consider v a solution to the system

$$(5.29) \quad \begin{cases} \sum_i \partial_i A_i^\alpha(\bar{x}, \nabla v) = 0, & \text{in } \Omega_r(\bar{x}), \quad \alpha = 1, \dots, N, \\ A_2^\alpha(\bar{x}, \nabla v)|_{\partial\Omega_r(\bar{x}) \cap \Gamma_1} = 0, \quad v|_{\partial\Omega_r(\bar{x}) \cap \Gamma_1} = u. \end{cases}$$

We note that it is easy to adapt [39, Theorem 3.I] to our case, which leads to the inequality

$$(5.30) \quad \int_{\Omega_\rho(\bar{x})} |\nabla v|^2 dx \leq C \left(\frac{\rho}{r} \right)^2 \int_{\Omega_r(\bar{x})} |\nabla v|^2 dx$$

for all $\rho \leq r$ and some $C > 0$. Hereinafter, we will omit \bar{x} if it is not needed. As v is a weak solution of (5.29), it satisfies

$$\sum_{i,\alpha} \int_{\Omega_r} A_i^\alpha(\bar{x}, \nabla v) \partial_i \varphi^\alpha dx = 0$$

for all $\varphi \in H_{\partial\Omega_r \setminus \Gamma_1}^1(\Omega_r) \cap L^\infty(\Omega_r)$. If $\varphi = v - u$, by (5.21) and (5.22) one can estimate

$$\sigma \int_{\Omega_r} |\nabla v|^2 dx \leq \sum_{i,\alpha} \int_{\Omega_r} A_i^\alpha(\bar{x}, \nabla v) \partial_i u^\alpha dx \leq \left(\frac{C}{\varepsilon} + \sigma \right) \int_{\Omega_r} |\nabla v| |\nabla u| dx,$$

which by means of Young's inequality yields

$$(5.31) \quad \int_{\Omega_r} |\nabla v|^2 dx \leq C \int_{\Omega_r} |\nabla u|^2 dx,$$

for some $C > 0$ which depends on σ and ε . On the other hand, from equations (5.18) and (5.29), we deduce that u and v fulfill

$$\begin{aligned}
(5.32) \quad & \sum_{i,\alpha} \int_{\Omega_r} (A_i^\alpha(\bar{x}, \nabla u) - A_i^\alpha(\bar{x}, \nabla v)) \partial_i \varphi^\alpha dx \\
& = \sum_{\alpha} \int_{\Omega_r} B^\alpha(x, u, \nabla u) \varphi^\alpha dx + \sum_{i,\alpha} \int_{\Omega_r} (A_i^\alpha(\bar{x}, \nabla u) - A_i^\alpha(x, \nabla u)) \partial_i \varphi^\alpha dx,
\end{aligned}$$

for all $\varphi \in H^1_{\partial\Omega_r \setminus \Gamma_1}(\Omega_r) \cap L^\infty(\Omega_r)$. In particular, set $w = u - v$ and take $M > 0$ so that

$$(5.33) \quad \|w\|_{L^\infty(\Omega_{r/2}, \mathbb{R}^N)} \leq M.$$

Let us consider $\varphi = w \cdot (T^s - (|w| + M)^s)_+$ with $s \in (0, 1)$ and $T > 0$ to be specified later on. Then, by (5.22) and (5.24) the left-hand side of (5.32) can be estimated as follows

$$\begin{aligned}
& \sum_{i,\alpha} \int_{\Omega_r} (A_i^\alpha(\bar{x}, \nabla u) - A_i^\alpha(\bar{x}, \nabla v)) \partial_i \varphi^\alpha dx \\
& \geq \sigma \int_{\Omega_r} |\nabla w|^2 (T^s - (|w| + M)^s)_+ dx - \left(\frac{C}{\min \varrho} + \sigma \right) s \int_{K_T} |\nabla w|^2 (|w| + M)^s dx,
\end{aligned}$$

where $K_T := \{x \in \Omega_r : |w(x)| < T - M\}$. Concerning the right-hand side of (5.32), from (5.25) we deduce that

$$\sum_{\alpha} \int_{\Omega_r} |B^\alpha(x, u, \nabla u) \varphi^\alpha| dx \leq \int_{\Omega_r} (C_1 + C_2 |\nabla u|^2) |w| (T^s - (|w| + M)^s)_+ dx,$$

while (5.23) implies

$$\begin{aligned}
& \sum_{i,\alpha} \int_{\Omega_r} (A_i^\alpha(\bar{x}, \nabla u) - A_i^\alpha(x, \nabla u)) \partial_i \varphi^\alpha dx \leq \frac{C}{\min \varrho} \int_{\Omega_r} |x - \bar{x}| |\nabla \varphi| dx \\
& \leq C \int_{\Omega_r} |x - \bar{x}| |\nabla w| (T^s - (|w| + M)^s)_+ dx + Cs \int_{K_T} |x - \bar{x}| |\nabla w| (|w| + M)^s dx.
\end{aligned}$$

Now, plugging the above estimates into (5.32), taking into account that $|w| + M < T$ on K_T and $|x - \bar{x}| < r$, we derive

$$\begin{aligned}
& \sigma \int_{\Omega_r} |\nabla w|^2 (T^s - (|w| + M)^s)_+ dx \\
& \leq (C + \sigma) s T^s \int_{K_T} |\nabla w|^2 dx + \int_{\Omega_r} (C_1 + C_2 |\nabla u|^2) |w| (T^s - (|w| + M)^s)_+ dx \\
& \quad + Cr \int_{\Omega_r} |\nabla w| (T^s - (|w| + M)^s)_+ dx + Crs T^s \int_{K_T} |\nabla w| dx,
\end{aligned}$$

Young's inequality applied to the last two terms, joint with $r, s < 1$, yields

$$\begin{aligned}
& \frac{\sigma}{2} \int_{\Omega_r} |\nabla w|^2 (T^s - (|w| + M)^s)_+ dx \\
& \leq Cs T^s \int_{K_T} |\nabla w|^2 dx + CT^s r^2 + C_1 T^s \int_{\Omega_r} |w| dx + C_2 T^s \int_{K_T} |\nabla u|^2 |w| dx.
\end{aligned}$$

Now we take $T = 2^{\frac{1}{s}+1} M$ so that, by definition of M in (5.33), one gets

$$\sigma \int_{\Omega_r} |\nabla w|^2 (T^s - (|w| + M)^s)_+ dx \geq \frac{\sigma T^s}{2} \int_{\Omega_{r/2}} |\nabla w|^2 dx.$$

Then we deduce that

$$(5.34) \quad \int_{\Omega_{r/2}} |\nabla w|^2 dx \leq Cs \int_{\Omega_r} |\nabla w|^2 dx + C_1 \int_{\Omega_r} |w| dx + C_2 \int_{K_T} |w| |\nabla u|^2 dx + Cr^2,$$

for some $C > 0$. In the sequel we will estimate the integrals of the right-hand side. For the first one, from (5.31) we reach

$$\int_{\Omega_r} |\nabla w|^2 dx \leq \int_{\Omega_r} (|\nabla u|^2 + |\nabla v|^2) dx \leq C \int_{\Omega_r} |\nabla u|^2 dx.$$

To estimate the second integral, by means of Poincaré's inequality, Lemma 5.18(a), Hölder and the previous inequality, we observe that

$$\int_{\Omega_r} |w| dx \leq Cr \int_{\Omega_r} |\nabla w| dx \leq \tilde{C} r^2 \left(\int_{\Omega_r} |\nabla u|^2 dx \right)^{1/2} \leq C_R r^2,$$

where we have also applied the Caccioppoli inequality in (5.27). Finally, for the third integral we have that

$$\begin{aligned} \int_{K_T} |w| |\nabla u|^2 dx &\leq Cr^2 \left(\int_{\Omega_r} |\nabla u|^q dx \right)^{\frac{2}{q}} \left(\int_{\Omega_r} |w|^{\frac{q}{q-2}} dx \right)^{\frac{q-2}{q}} \\ &\leq CT^{\frac{2}{q}} r^2 \int_{\Omega_{2r}} (1 + |\nabla u|^2) dx \left(\int_{\Omega_r} |w| dx \right)^{\frac{q-2}{q}} \\ &\leq CT^{\frac{2}{q}} \left(\int_{\Omega_r} |\nabla u|^2 dx \right)^{\frac{q-2}{2q}} \int_{\Omega_{2r}} (1 + |\nabla u|^2) dx, \end{aligned}$$

using Hölder's and (5.28) inequalities and the definition of K_T . Thus, we deduce from (5.34) that

$$(5.35) \quad \int_{\Omega_{r/2}} |\nabla w|^2 dx \leq C \left(\left(s + T^{\frac{2}{q}} \left(\int_{\Omega_r} |\nabla u|^2 dx \right)^{\frac{q-2}{2q}} \right) \int_{\Omega_{2r}} |\nabla u|^2 dx + (1 + T^{\frac{2}{q}}) r^2 \right).$$

If we define $\phi(\rho) := \int_{\Omega_\rho} |\nabla u|^2 dx$, we deduce that

$$\begin{aligned} \phi(\rho) &\leq C \int_{\Omega_{r/2}} |\nabla w|^2 dx + C \left(\frac{\rho}{r} \right)^2 \int_{\Omega_{r/2}} |\nabla v|^2 dx \\ &\leq C \int_{\Omega_{r/2}} |\nabla w|^2 dx + C \left(\frac{\rho}{r} \right)^2 \phi(r) \\ &\leq C \left(s + \left(\frac{\rho}{r} \right)^2 + T^{\frac{2}{q}} \phi(r)^{\frac{q-2}{2q}} \right) \phi(2r) + C(1 + T^{\frac{2}{q}}) r^2, \end{aligned}$$

for all $\rho < r/2$, applying (5.30), (5.31) and (5.35) inequalities. Possibly increasing C we deduce, for $\rho < r/4$,

$$\phi(\rho) \leq C \left(s + \left(\frac{\rho}{r} \right)^2 + T^{\frac{2}{q}} \phi(r)^{\frac{q-2}{2q}} \right) \phi(r) + C(1 + T^{\frac{2}{q}}) r^2.$$

Observe that for $r/4 < \rho < r$, we obviously have $\phi(\rho) \leq \phi(r) \leq 16 \left(\frac{\rho}{r}\right)^2 \phi(r)$. Then, the previous inequality holds for any $\rho < r$. Setting $\rho = \tau r$, $\beta \in (0, 1)$ and $s = \tau^2$, with $3C\tau^2 = \tau^{2\tilde{\beta}}$, $\tilde{\beta} \in (\beta, 1)$. Since $T^{\frac{2}{q}}\phi(r)^{\frac{q-2}{q}} \leq \tau^2$, for $\phi(r) \leq \varepsilon_1$, we get

$$\phi(\tau r) \leq \tau^{2\tilde{\beta}}\phi(r) + C(1 + T^{\frac{2}{q}})r^2 \leq \tau^{2\tilde{\beta}}\phi(r) + C(1 + T^{\frac{2}{q}})r^{2\beta}.$$

Therefore, by induction, we get

$$\begin{aligned} \phi(\tau^k r) &\leq \tau^{2\tilde{\beta}k}\phi(r) + C(1 + T^{\frac{2}{q}})(\tau^{k-1}r)^{2\beta} \sum_{s=0}^{\infty} (\tau^{2(\tilde{\beta}-\beta)})^s \\ &= \left(\phi(r) + C(1 + T^{\frac{2}{q}}) \frac{r^{2\beta}}{\tau^{2\beta} - \tau^{2\tilde{\beta}}} \right) \tau^{2\beta k} < \varepsilon_1 \tau^{2\beta k}, \end{aligned}$$

if we choose $\phi(r) \leq \frac{\varepsilon_1}{2}$ and $C(1 + T^{\frac{2}{q}}) \frac{r^{2\beta}}{\tau^{2\beta} - \tau^{2\tilde{\beta}}} \leq \frac{\varepsilon_1}{2}$, which holds for $r \leq r_0$ for some $r_0 < \frac{\delta}{2}$. Then,

$$\phi(\rho) \leq C \left(\frac{\rho}{r_0} \right)^{2\beta}, \quad \text{for every } \rho \leq r_0.$$

Applying Poincaré’s inequality, Lemma 5.18(b), it leads to $u \in \mathcal{L}^{(2,2+2\beta)}(\Omega_r, \mathbb{R}^N)$, and thus $C^{0,\beta}(\Omega_{r_0}, \mathcal{N})$ for all $\beta \in (0, 1)$ thanks to [38, Theorem I.2]. Therefore, $u \in C^{0,\beta}(\bar{\Sigma}, \mathcal{N})$ for all $\beta \in (0, 1)$. \square

5.5.4. Hölder regularity of the gradient.

THEOREM 5.21. *Let u be a weak solution of $(\mathcal{S}_{\varepsilon,\sigma}^f)$. Then $u \in C^{1,\beta_0}(\bar{\Sigma}, \mathcal{N})$ for some $\beta_0 \in (0, 1)$.*

PROOF. Again, for $0 < \delta < 1$, we fix $\bar{x} \in \overline{B_{1-\delta}^+(0)}$, and take $r < \frac{\delta}{2}$. Recall that Theorem 5.20 ensures that $u \in C^{0,\beta}(\bar{\Sigma}, \mathcal{N})$ for every $\beta \in (0, 1)$ and thus, u satisfies

$$(5.36) \quad \text{osc}_{\Omega_r(\bar{x})} u \leq Cr^\beta, \quad \int_{\Omega_r(\bar{x})} |\nabla u|^2 \leq Cr^{2\beta},$$

for $r < \frac{\delta}{2}$ and $\beta \in (0, 1)$. Let v be again a solution to (5.31), and set $w := u - v$. Now we will apply (5.32) with $\varphi = w \min\{1, \ell/|w|\}$ for $\ell \geq 1$. From the left-hand side of (5.32), we deduce that

$$\begin{aligned} &\sum_{i,\alpha} \int_{\Omega_r} (A_i^\alpha(\bar{x}, \nabla u) - A_i^\alpha(\bar{x}, \nabla v)) \partial_i \varphi^\alpha \, dx \\ &\geq \sigma \int_{\Omega_r} |\nabla w|^2 \min \left\{ 1, \frac{\ell}{|w|} \right\} \, dx - (C + \sigma) \int_{K_\ell} |\nabla w|^2 \, dx, \end{aligned}$$

where $K_\ell := \{x \in \Omega_r : |w(x)| > \ell\}$. From the right-hand side of (5.32), by means of (5.25), we have that

$$\begin{aligned} &\sum_{\alpha} \int_{\Omega_r} |B^\alpha(x, u, \nabla u) \varphi^\alpha| \, dx \leq \int_{\Omega_r} (C_1 + C_2 |\nabla u|^2) |\varphi| \, dx \\ &\leq C_1 \int_{\Omega_r} |w| \, dx + C_2 \int_{\Omega_r} |\nabla u|^2 \min \{|w|, \ell\} \, dx. \end{aligned}$$

Moreover, using (5.23) and Young’s inequality, we can write

$$\begin{aligned}
& \sum_{i,\alpha} \int_{\Omega_r} (A_i^\alpha(\bar{x}, \nabla u) - A_i^\alpha(x, \nabla u)) \partial_i \varphi^\alpha dx \\
& \leq C \int_{\Omega_r} |x - \bar{x}| |\nabla \varphi| dx \leq Cr \left(\int_{\Omega_r} |\nabla w| \min \left\{ 1, \frac{\ell}{|w|} \right\} dx + \int_{K_\ell} |\nabla w| dx \right) \\
& \leq \tilde{C}r \int_{\Omega_r} |\nabla w| dx \leq \frac{\sigma}{4} \int_{\Omega_r} |\nabla w|^2 dx + C_\sigma r^4.
\end{aligned}$$

Consequently, plugging the above estimates into (5.32), we reach the next inequality

$$\begin{aligned}
(5.37) \quad & \sigma \int_{\Omega_r} |\nabla w|^2 \min \left\{ 1, \frac{\ell}{|w|} \right\} dx \\
& \leq C_1 \int_{\Omega_r} |w| dx + C_2 \int_{\Omega_r} |\nabla u|^2 \min\{|w|, \ell\} dx \\
& \quad + C \int_{K_\ell} |\nabla w|^2 dx + \frac{\sigma}{4} \int_{\Omega_r} |\nabla w|^2 dx + C_\sigma r^4.
\end{aligned}$$

Now we note that, applying Lemma 5.18(a), Hölder and Young inequalities for any $\theta > 0$, it holds

$$(5.38) \quad \int_{\Omega_r} |w| dx \leq Cr \int_{\Omega_r} |\nabla w| dx \leq \theta \int_{\Omega_r} |\nabla w|^d dx + C_\theta r^4.$$

On the other hand, notice that

$$\int_{\Omega_r} |\nabla w|^2 \min \left\{ 1, \frac{\ell}{|w|} \right\} dx \geq \int_{K_\ell^c} |\nabla w|^2 dx = \int_{\Omega_r} |\nabla w|^2 dx - \int_{K_\ell} |\nabla w|^2 dx.$$

Substituting this and (5.38) into (5.37) and taking $C_1\theta = \frac{\sigma}{4}$, we attain

$$(5.39) \quad \frac{\sigma}{2} \int_{\Omega_r} |\nabla w|^2 dx \leq (C + \sigma) \int_{K_\ell} |\nabla w|^2 dx + C_2 \int_{\Omega_r} |\nabla u|^2 \min\{|w|, \ell\} dx + Cr^4.$$

Parallel to this, we recall that u satisfies

$$\sum_{i,\alpha} \int_{\Omega_r} A_i^\alpha(x, \nabla u) \partial_i \varphi^\alpha dx = \sum_\alpha \int_{\Omega_r} B^\alpha(x, u, \nabla u) \varphi^\alpha(x) dx,$$

for $\varphi \in H^1(\Omega_r \cup \gamma_r) \cap L^\infty(\Omega_r)$. By setting $\varphi := (u - u(\bar{x})) \min\{|w|, \ell\}$ and using the notation $\partial_i w = (\partial_i w^1, \dots, \partial_i w^N)$, we have the equality

$$\begin{aligned}
& \sum_{i,\alpha} \int_{\Omega_r} A_i^\alpha(x, \nabla u) \partial_i u^\alpha \min\{|w|, \ell\} dx + \sum_{i,\alpha} \int_{K_\ell^c} A_i^\alpha(x, \nabla u) (u^\alpha - u^\alpha(\bar{x})) \frac{1}{|w|} w \cdot \partial_i w dx \\
& = \sum_\alpha \int_{\Omega_r} B^\alpha(x, u, \nabla u) (u^\alpha - u^\alpha(\bar{x})) \min\{|w|, \ell\} dx \\
& \leq \operatorname{osc}_{\Omega_r} u \left(C_1 \int_{\Omega_r} |w| dx + C_2 \int_{\Omega_r} |\nabla u|^2 \min\{|w|, \ell\} dx \right),
\end{aligned}$$

where the latter follows from (5.25). Thanks to (5.22) and (5.21), applied to the first and second term on the left-hand side above, we reach

$$\begin{aligned} & \sum_{i,\alpha} \int_{\Omega_r} A_i^\alpha(x, \nabla u) \partial_i u^\alpha \min\{|w|, \ell\} dx \geq \sigma \int_{\Omega_r} |\nabla u|^2 \min\{|w|, \ell\} dx, \\ & - \sum_{i,\alpha} \int_{K_\ell^c} A_i^\alpha(x, \nabla u) (u^\alpha - u^\alpha(\bar{x})) \frac{w \cdot \partial_i w}{|w|} dx \leq \left(\frac{C}{\varepsilon} + \sigma \right) \operatorname{osc} u \int_{\Omega_r} |\nabla u| |\nabla w| dx. \end{aligned}$$

Then we deduce that there exists $C > 0$ such that

$$(5.40) \quad (\sigma - C_2 \operatorname{osc} u) \int_{\Omega_r} |\nabla u|^2 \min\{|w|, \ell\} dx \leq \operatorname{osc} u \left(C_1 \int_{\Omega_r} |w| dx + C \int_{\Omega_r} |\nabla u| |\nabla w| dx \right).$$

Applying Young's inequality for any $\zeta > 0$ and both estimates in (5.36), we find for the last term that

$$\operatorname{osc} u \int_{\Omega_r} |\nabla u| |\nabla w| dx \leq \zeta \int_{\Omega_r} |\nabla w|^2 dx + C_\zeta r^{4\beta}.$$

Consequently, (5.40) leads us to the next inequality

$$(\sigma - C_2 \operatorname{osc} u) \int_{\Omega_r} |\nabla u|^2 \min\{|w|, \ell\} dx \leq C_1 \operatorname{osc} u \int_{\Omega_r} |w| dx + \zeta \int_{\Omega_r} |\nabla w|^2 dx + C_\zeta r^{4\beta},$$

for all $\zeta > 0$. Let us choose $r_0 < \delta/2 < 1$ such that $C_2 \operatorname{osc}_{\Omega_{r_0}} u \leq \sigma/2$. Then, arguing as in (5.38) and using $r < 1$, we conclude that

$$(5.41) \quad \frac{\sigma}{2} \int_{\Omega_r} |\nabla u|^2 \min\{|w|, \ell\} dx \leq 2\zeta \int_{\Omega_r} |\nabla w|^2 dx + C_\zeta r^{4\alpha},$$

for all $\zeta > 0$ and $r < r_0$.

Combining (5.39) and (5.41) with $\zeta = \frac{\sigma^2}{16C_2}$, we have

$$\frac{\sigma}{4} \int_{\Omega_r} |\nabla w|^2 dx \leq (C + \sigma) \int_{K_\ell} |\nabla w|^2 dx + Cr^{4\beta},$$

where C does not depend on ℓ . Thus, when $\ell \rightarrow \infty$ we obtain

$$\int_{\Omega_r} |\nabla w|^2 dx \leq Cr^{4\beta} \quad \text{for } r \leq r_0 < \frac{\delta}{2}.$$

Adapting [39, Theorem 3.I] to our case, we deduce that

$$\int_{\Omega_\rho} |\nabla v - (\nabla v)_{\bar{x}, \rho}|^2 dx \leq C \left(\frac{\rho}{r} \right)^{2+2\beta_0} \int_{\Omega_r} |\nabla v - (\nabla v)_{\bar{x}, r}|^2 dx,$$

for some $\beta_0 > 0$ and $\rho < r$. Thus,

$$\begin{aligned} \int_{\Omega_\rho} |\nabla u - (\nabla u)_{\bar{x}, \rho}|^2 dx & \leq 2 \int_{\Omega_r} |\nabla w - (\nabla w)_{\bar{x}, r}|^2 dx + 2 \int_{\Omega_r} |\nabla v - (\nabla v)_{\bar{x}, r}|^2 dx \\ & \leq C\rho^{4\beta} + C \left(\frac{\rho}{r} \right)^{2+2\beta_0} \int_{\Omega_r} |\nabla v - (\nabla v)_{\bar{x}, r}|^2 dx \\ & \leq C\rho^{4\beta} + 2C \left(\frac{\rho}{r} \right)^{2+2\beta_0} \left(\int_{\Omega_r} |\nabla u - (\nabla u)_{\bar{x}, r}|^2 dx + r^{4\beta} \right). \end{aligned}$$

Set $\beta = \frac{1+\beta_0}{2}$. Accordingly, we get that

$$\frac{1}{\rho^{2+2\beta_0}} \int_{\Omega_\rho} |\nabla u - (\nabla u)_{\bar{x},\rho}|^2 dx \leq 3C + \frac{2C}{r^{2+2\beta_0}} \int_{\Omega_r} |\nabla u - (\nabla u)_{\bar{x},r}|^2 dx,$$

so $\nabla u \in \mathcal{L}^{(2,2+2\beta_0)}(\Omega_{r/2})$. By [38, Theorem I.2], $\nabla u \in C^{0,\beta_0}(\Omega_{r/2})$. Therefore, we can conclude that $u \in C^{1,\beta_0}(\bar{\Sigma})$. \square

5.5.5. Higher regularity via continuity and bootstrapping methods. In order to obtain a higher order of regularity, we need to smooth out the function f and to use classical results in regularity theory. Since f is a Lipschitz function, we can approximate it by a mollified sequence, which we call f_δ , as in Section 2.4.4. Thanks to the extra regularity of this approximation, we can obtain the next statement.

PROPOSITION 5.22. *Let u be a weak solution of $(\mathcal{S}_{\varepsilon,\sigma}^f)$ with $f = f_\delta$. Then $u \in C^\infty(\bar{\Sigma}, \mathcal{N})$.*

PROOF. To avoid technicalities, we assume that Σ is an open bounded set in \mathbb{R}^2 (otherwise, we argue by partition of unity subordinate to a cover by isothermal coordinates, as usual). We note that u is a $C^{1,\beta_0}(\bar{\Sigma}, \mathbb{R}^N)$ solution to

$$(5.42) \quad \begin{cases} \sum_i \partial_i A_i^\alpha(x, \nabla w) = B_\delta^\alpha(x) & \text{in } \Sigma \\ \nabla w \cdot \nu = 0 & \text{on } \partial\Sigma, \end{cases}$$

for $\alpha \in \{1, \dots, n\}$, with

$$B_\delta^\alpha(x) := -(a(x, \nabla u) + \sigma) \sum_i A_u^\alpha(\partial_i u, \partial_i u)(x) - \lambda \varrho^2(x) (\exp_{u(x)}^{-1} f_\delta(x))^\alpha \in C^{0,\beta_0}(\bar{\Sigma}).$$

and A and a as in (5.19) and (5.20), respectively. Moreover, by standard comparison technique, we can show that the solution to (5.42) is unique. We rewrite (5.42) as

$$(5.43) \quad \begin{cases} Lw := \sum_{i,j} A_{ij}(x, \nabla w) \partial_{ij}^2 w + \Xi(x, \nabla w) = B_\delta^\alpha & \text{in } \Sigma \\ \nabla w \cdot \nu = 0 & \text{on } \partial\Sigma, \end{cases}$$

with

$$A_{ij}(x, P) = \frac{\varrho}{\sqrt{\varepsilon^2 \varrho^2 + |P|^2}} \left[\delta_{ij} I^{N \times N} - \left(\frac{P_i}{\sqrt{\varepsilon^2 \varrho^2 + |P|^2}} \otimes \frac{P_j}{\sqrt{\varepsilon^2 \varrho^2 + |P|^2}} \right) \right] + \sigma \delta_{ij} I^{N \times N},$$

$$\Xi[x, P] = \frac{\langle \nabla \varrho, P \rangle |P|^2}{(\varepsilon^2 \varrho^2 + |P|^2)^{\frac{3}{2}}}.$$

It is easy to see that the system (5.43) satisfies the classical Agmon-Douglis-Nirenberg conditions [2]. Note that the system has at most one solution in $C^{2,\beta_0}(\bar{\Sigma})$ due to uniqueness for (5.42). Therefore, the following Schauder estimates hold for solutions [2, Remark 2]:

$$\|w\|_{C^{2,\beta_0}(\bar{\Sigma}, \mathbb{R}^N)} \leq C \|B\|_{C^{0,\beta_0}(\bar{\Sigma}, \mathbb{R}^N)}.$$

As before, one can check that $w = 0$ is the unique solution to the homogeneous Neumann problem for the system

$$t\Delta w + (1-t)Lw = 0.$$

Hence, by the continuity method (e.g. [85, Theorem 5.25]), we can prove that there exists a solution to (5.43) $w \in C^{2,\beta_0}(\bar{\Sigma})$. By uniqueness, $u = w$. Finally, a bootstrap technique shows that $u \in C^\infty(\bar{\Sigma})$. \square

5.6. Lipschitz regularity

In this section and afterwards, we will use repeatedly the following notations:

$$s_\kappa(t) = \begin{cases} \frac{\sin(\sqrt{\kappa}t)}{\sqrt{\kappa}} & \kappa > 0, \\ t & \kappa = 0, \\ \frac{\sinh(\sqrt{|\kappa}|t)}{\sqrt{|\kappa|}} & \kappa < 0, \end{cases} \quad c_\kappa = s'_\kappa, \quad \text{ta}_\kappa = \frac{s_\kappa}{c_\kappa} \quad \text{and} \quad \text{co}_\kappa = \frac{c_\kappa}{s_\kappa}.$$

5.6.1. Energy density as a subsolution of an elliptic PDE. As in [143] we get Lipschitz regularity for general (non-necessarily convex) domains, but we have the extra difficulty of dealing with maps into a target manifold (instead of scalar-valued) and on a curved domain. Typically, regularity follows from integral estimates as in [84], but with the drawback of having to require convexity of the boundary. To overcome this, we adapt to our setting the Bernstein technique used in [143]; let us highlight that this approach only works arguing with the intrinsic version of our elliptic system $(\mathcal{S}_{\varepsilon,\sigma}^{f_\delta})$, namely, set $w := |du|^2$ where $u := u_{\varepsilon,\sigma}^\delta$ solves the system

$$(\mathcal{S}_{\varepsilon,\sigma}^{f_\delta}) \quad \begin{cases} \operatorname{div}_g \left[\left(\frac{1}{\sqrt{w + \varepsilon^2}} + \sigma \right) du \right] = -\lambda \exp_u^{-1} f_\delta & \text{in } \Sigma \\ \nu \cdot du = 0 & \text{on } \partial\Sigma \end{cases}$$

with f_δ the mollified version of the initial f defined as in (2.18).

The goal is to get uniform bounds for w , which are independent of the parameters ε, σ and δ in order to allow a limiting process when any of these values goes to 0. As a prior step, we derive an elliptic PDE for w to which we will apply a maximum principle argument. With this aim, first notice that the system at the first line in $(\mathcal{S}_{\varepsilon,\sigma}^{f_\delta})$ can be rewritten as

$$(5.44) \quad \left(\frac{1}{\sqrt{w + \varepsilon^2}} + \sigma \right) \tau(u)^\alpha - \frac{1}{2(w + \varepsilon^2)^{3/2}} \langle \nabla w, \nabla u^\alpha \rangle_g = -\lambda (\exp_u^{-1} f_\delta)^\alpha$$

for $\alpha = 1, \dots, n$ and τ be the 2-tension (recall Section 2.4.1).

We define the operator

$$(5.45) \quad \mathcal{L}w := -\left(\frac{1}{\sqrt{w + \varepsilon^2}} + \sigma \right) \Delta_g w + \frac{1}{(w + \varepsilon^2)^{3/2}} (h_{\alpha\beta} \circ u) \nabla^2 w (\nabla u^\alpha, \nabla u^\beta).$$

Observe that taking normal coordinates around a fixed but arbitrary point $x \in \Sigma$ and correspondingly around $u(x) \in \mathcal{N}$, it is easy to check \mathcal{L} is an elliptic operator.

LEMMA 5.23. *Suppose that the supremum of the sectional curvatures of \mathcal{N} within $B_h(p, R)$ is non-positive. Then w as above satisfies*

$$(5.46) \quad \begin{aligned} & \mathcal{L}w + \lambda w + 2 \left(\frac{1}{\sqrt{w + \varepsilon^2}} + \sigma \right) \left(|\nabla^{g \boxtimes h} du|_{g \boxtimes h}^2 - \mathcal{R} \right) \\ & \leq \frac{3}{2} \frac{\langle dw \otimes \langle \nabla w, \nabla u \rangle_g, du \rangle_{g \boxtimes h}}{(w + \varepsilon^2)^{5/2}} + \lambda |df_\delta|^2 - \frac{\langle dw \otimes \tau(u), du \rangle_{g \boxtimes h}}{(w + \varepsilon^2)^{3/2}} - \frac{|\nabla w|_g^2}{2(w + \varepsilon^2)^{3/2}}, \end{aligned}$$

where the curvature term \mathcal{R} is given by

$$(5.47) \quad \begin{aligned} \mathcal{R} &:= \sum_{i,j=1}^2 \langle \mathcal{R}^{u^*h}(e_i, e_j)e_j, e_i \rangle_{u^*h} - \text{Ric}_\Sigma(du, du), \quad \text{with} \\ \mathcal{R}^{u^*h}(X, Y)Z &:= \mathcal{R}^h(u_*X, u_*Y)u_*Z, \end{aligned}$$

for a local orthonormal frame $\{e_i\}_{i=1,2}$ of Σ .

PROOF. Recall that on $T^*\Sigma \otimes u^*T\mathcal{N}$ we have the natural bundle metric $g \boxtimes h := g^{-1} \otimes u^*h$ so that $\langle A, B \rangle_{g \boxtimes h} := g^{ij}(h_{\alpha\beta} \circ u)A_i^\alpha B_j^\beta$, while $\tilde{\nabla}$ denotes the linear connection on $u^*T\mathcal{N}$ given by (2.12). Differentiating (5.44) and multiplying with du , using the inner product induced by $g \boxtimes h$, we reach

$$\begin{aligned} & \left(\frac{1}{\sqrt{w + \varepsilon^2}} + \sigma \right) \langle \tilde{\nabla} \tau(u), du \rangle_{g \boxtimes h} \\ & - \frac{1}{2} \frac{\langle dw \otimes \tau(u), du \rangle_{g \boxtimes h}}{(w + \varepsilon^2)^{3/2}} + \frac{3}{4} \frac{\langle dw \otimes \langle \nabla w, \nabla u \rangle_g, du \rangle_{g \boxtimes h}}{(w + \varepsilon^2)^{5/2}} \\ & - \frac{1}{2(w + \varepsilon^2)^{3/2}} \left((h_{\alpha\beta} \circ u) \nabla^2 w (\nabla u^\alpha, \nabla u^\beta) + \frac{1}{2} |\nabla w|_g^2 \right) = -\lambda \langle \tilde{\nabla}(\exp_u^{-1} f_\delta), du \rangle_{g \boxtimes h}. \end{aligned}$$

Replacing the first term on the left-hand side by means of Bochner-Weitzenböck formula (see e.g. [139, p. 128]), we have

$$(5.48) \quad \begin{aligned} & \left(\frac{1}{\sqrt{w + \varepsilon^2}} + \sigma \right) \left(\frac{\Delta g w}{2} - |\nabla^{g \boxtimes h} du|_{g \boxtimes h}^2 + \mathcal{R} \right) \\ & - \frac{\langle dw \otimes \tau(u), du \rangle_{g \boxtimes h}}{2(w + \varepsilon^2)^{3/2}} + \frac{3}{4} \frac{\langle dw \otimes \langle \nabla w, \nabla u \rangle_g, du \rangle_{g \boxtimes h}}{(w + \varepsilon^2)^{5/2}} \\ & - \frac{1}{2(w + \varepsilon^2)^{3/2}} \left((h_{\alpha\beta} \circ u) \nabla^2 w (\nabla u^\alpha, \nabla u^\beta) + \frac{1}{2} |\nabla w|_g^2 \right) = -\lambda \langle \tilde{\nabla}(\exp_u^{-1} f_\delta), du \rangle_{g \boxtimes h}, \end{aligned}$$

where \mathcal{R} is defined as in (5.47).

On the other hand, for any $X \in T\Sigma$, by the chain rule we have

$$\tilde{\nabla}_X(\exp_u^{-1} f_\delta) = {}^h \nabla_{u_*X}(\exp_{(\cdot)}^{-1} f_\delta) + D_{f_{\delta*}X} \exp_u^{-1}(\cdot).$$

Here notice that for each $x \in \Sigma$, D stands for the standard directional derivative in the vector space $T_{u(x)}\mathcal{N}$. Hence, for the term on the right-hand side of (5.48), we can write

$$\begin{aligned} -\langle \tilde{\nabla}(\exp_u^{-1} f_\delta), du \rangle_{g \boxtimes h} &= -\sum_{i=1}^2 \left(\langle {}^h \nabla_{u_*e_i}(\exp_{(\cdot)}^{-1} f_\delta), u_*e_i \rangle_h + \langle D_{f_{\delta*}e_i} \exp_u^{-1}(\cdot), u_*e_i \rangle_h \right) \\ &= \nabla^2 \left(\frac{d_h^2(\cdot, f_\delta)}{2} \right) (du, du) - \sum_{i=1}^2 \langle D_{f_{\delta*}e_i} \exp_u^{-1}(\cdot), u_*e_i \rangle_h \\ &\geq w - \sum_{i=1}^2 \langle D_{f_{\delta*}e_i} \exp_u^{-1}(\cdot), u_*e_i \rangle_h, \end{aligned}$$

which follows from the Hessian comparison theorem for the squared distance function for $\kappa \leq 0$ (see e.g. (10.47) in [158, p. 266]). Moreover, the first equality is obtained as in (5.11).

Now, to further estimate the second term in the right-hand side, consider the geodesic on \mathcal{N} given by $\gamma_i(t) := \exp_{f_\delta(x)}(t f_{\delta*x} e_i)$. Then it is well-known (see, for instance, [106, p. 540]) that

$$\frac{d}{dt} \Big|_{t=0} \exp_{u(x)}^{-1} (\gamma_i(t)) = J'(0) \in T_{u(x)}\mathcal{N},$$

where $J(s)$ is the Jacobi field along the geodesic $\zeta(s) = \exp_{u(x)}(s \exp_{u(x)}^{-1} f_\delta(x))$ determined by $J(0) = 0$ and $J(1) = \gamma_i'(0) = f_{\delta*x}e_i \in T_{f_\delta(x)}\mathcal{N}$ (see Figure 5.4). Then one can reproduce almost verbatim the proof of Jacobi field comparison in [112, Theorem 11.2], but for a non-unit speed geodesic, as in our case $|\zeta'|_h = d_h(u(x), f_\delta(x))$, and a non-necessarily normal Jacobi field, to deduce that $t|J'(0)| \leq |J(t)|$. In particular, for $t = 1$, we reach

$$|J'(0)| \leq |J(1)| = |f_{\delta*x}e_i|_h.$$

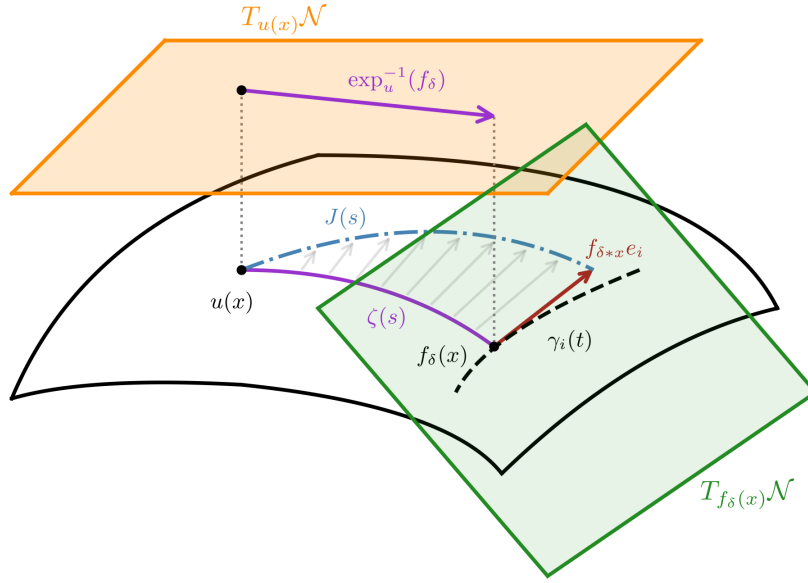


FIGURE 5.4. Setting to apply Jacobi field comparison.

Taking all the above considerations into account and using Young's inequality, we get

$$-\langle \tilde{\nabla}(\exp_u^{-1} f_\delta), du \rangle_{g \boxtimes h} \geq w - |df_\delta| \cdot |du| \geq \frac{1}{2}(w - |df_\delta|^2).$$

Substituting into (5.48) and rearranging terms, we can write

$$\begin{aligned} & \frac{3}{4} \frac{\langle dw \otimes \langle \nabla w, \nabla u \rangle_g, du \rangle_{g \boxtimes h}}{(w + \varepsilon^2)^{5/2}} - \frac{\langle dw \otimes \tau(u), du \rangle_{g \boxtimes h}}{2(w + \varepsilon^2)^{3/2}} - \frac{|\nabla w|_g^2}{4(w + \varepsilon^2)^{3/2}} \\ & \geq \frac{\lambda}{2}(w - |df_\delta|^2) + \frac{1}{2}\mathcal{L}w + \left(\frac{1}{\sqrt{w + \varepsilon^2}} + \sigma \right) \left(|\nabla^{g \boxtimes h} du|_{g \boxtimes h}^2 - \mathcal{R} \right). \end{aligned}$$

Further rearrangement of the terms and multiplication by 2 leads to the inequality in the statement. \square

Let us remark that, as Σ is a compact surface, there is always a constant C_Σ so that $\text{Ric}_\Sigma \geq -C_\Sigma g$, which hence does not introduce any further restriction in our current setting. We keep the subindex to identify this constant. We conclude that w is a subsolution of an elliptic equation. More precisely,

COROLLARY 5.24. *Assume that the upper bound κ for the sectional curvatures of \mathcal{N} within $B_h(p, R)$ is non-positive. Then $w = |du|^2$, for u the unique solution of $(\mathcal{S}_{\varepsilon, \sigma}^f)$, satisfies the inequality*

$$\mathcal{L}w + \lambda w - 2\left(\frac{1}{\sqrt{w + \varepsilon^2}} + \sigma\right)C_\Sigma w \leq \frac{3}{2} \frac{|\nabla w|_g^2}{(w + \varepsilon^2)^{3/2}} + \lambda |df_\delta|^2.$$

PROOF. Notice that, as $\nabla^{g \boxtimes h} du$ is a symmetric $T\mathcal{N}$ -valued 2-tensor, for any orthonormal basis $\{e_1, e_2\}$ of Σ , we have

$$|\nabla^{g \boxtimes h} du|_{g \boxtimes h}^2 = \sum_{i,j=1}^2 |\nabla^{g \boxtimes h} du(e_i, e_j)|_h^2 \geq \frac{1}{2} \sum_{i=1}^2 |\nabla^{g \boxtimes h} du(e_i, e_i)|_h^2 = \frac{1}{2} |\tau(u)|_h^2.$$

On the other hand, by means of Cauchy-Schwarz and Young's inequality, we can estimate

$$\begin{aligned} \frac{\langle dw \otimes \tau(u), du \rangle_{g \boxtimes h}}{(w + \varepsilon^2)^{3/2}} &\leq \frac{|dw \otimes \tau(u)|_{g \boxtimes h} \sqrt{w}}{(w + \varepsilon^2)^{3/2}} \leq \frac{|\tau(u)|_h}{(w + \varepsilon^2)^{1/4}} \cdot \frac{|dw|_g \sqrt{w}}{(w + \varepsilon^2)^{5/4}} \\ &\leq \frac{|\tau(u)|_h^2}{2(w + \varepsilon^2)^{1/2}} + \frac{|\nabla w|_g^2 w}{2(w + \varepsilon^2)^{5/2}} \leq \frac{|\nabla^{g \boxtimes h} du|_{g \boxtimes h}^2}{(w + \varepsilon^2)^{1/2}} + \frac{|\nabla w|_g^2}{2(w + \varepsilon^2)^{3/2}}. \end{aligned}$$

Similarly, for the first term on the right-hand side of (5.46), we get

$$(5.49) \quad \frac{\langle dw \otimes \langle \nabla w, \nabla u \rangle_g, du \rangle_{g \boxtimes h}}{(w + \varepsilon^2)^{5/2}} \leq \frac{|\nabla w|_g^2 w}{(w + \varepsilon^2)^{5/2}} \leq \frac{|\nabla w|_g^2}{(w + \varepsilon^2)^{3/2}}.$$

Accordingly, we reach

$$\begin{aligned} &\mathcal{L}w + \lambda w + 2\left(\frac{1}{\sqrt{w + \varepsilon^2}} + \sigma\right)\left(|\nabla^{g \boxtimes h} du|_{g \boxtimes h}^2 - \mathcal{R}\right) \\ &\leq \frac{3}{2} \frac{|\nabla w|_g^2}{(w + \varepsilon^2)^{3/2}} + \lambda |df_\delta|^2 + \frac{|\nabla^{g \boxtimes h} du|_{g \boxtimes h}^2}{(w + \varepsilon^2)^{1/2}}. \end{aligned}$$

Finally, discarding the non-negative term $\left(\frac{1}{\sqrt{w + \varepsilon^2}} + 2\sigma\right)|\nabla^{g \boxtimes h} du|_{g \boxtimes h}^2$ on the left-hand side, and using the curvature assumptions, we deduce the inequality in the statement. \square

5.6.2. Baby case: Lipschitz regularity in the convex case. At this stage, let us point out that the above estimates can be applied directly to provide a uniform bound on the Lipschitz constants of the approximating solutions in the convex case. With this goal, let us denote by $r_\partial := d_g(\cdot, \partial\Sigma)$ the Riemannian distance to the boundary of our domain surface. As $\partial\Sigma$ is smooth, the function r_∂ is known to be smooth in the interior of Σ out of the cut locus of $\partial\Sigma$ (see [148, Proposition 3.10]). In addition, $\nabla r_\partial|_{\partial\Sigma} = -\nu$, where ν is the unit outward normal to $\partial\Sigma$, and hence for any $X \in T\partial\Sigma$ we have

$$(5.50) \quad \nabla^2 r_\partial(X, X) = \langle \nabla_X \nabla r_\partial, X \rangle_g = -\langle \nabla_X \nu, X \rangle_g = -k_\partial |X|_g^2,$$

being k_∂ the geodesic curvature of the boundary curve $\partial\Sigma$. Consequently, for a convex boundary, the distance function r_∂ is concave.

PROPOSITION 5.25. *Let \mathcal{N} be a complete n -manifold with a non-positive upper bound κ for the sectional curvatures within $B_h(p, R)$. Assume that f is a Lipschitz function with $f(\bar{\Sigma}) \subseteq B_h(p, R)$ for some $R < R_\kappa$. Then the unique solution u of $(\mathcal{S}_{\varepsilon, \sigma}^f)$ is Lipschitz with*

constant only depending on the Lipschitz constant of f , under the extra assumption that the boundary curve $\partial\Sigma$ is convex.

PROOF. A routine computation in local coordinates gives

$$\partial_k w = 2g^{ij} h_{\alpha\beta}|_u \left(\partial_{ik}^2 u^\alpha - \partial_\ell u^\alpha \Gamma_{ik}^\ell + {}^h \Gamma_{\delta\gamma}^\alpha|_u \partial_i u^\gamma \partial_k u^\delta \right) \partial_j u^\beta,$$

where the shortcut $F|_u$ means composition with the map u .

Now choose a chart $\{x^1, x^2\}$ so that $x^2 = r_{\partial}$, and thus for the normal derivative we have $\nabla_\nu = -\partial_2$. By normalization of the first vector, we can assume that $\{\partial_1, \partial_2\}$ is a local orthonormal frame. Then, taking $k = 2$ in the above formula and using the Neumann boundary condition to remove the last term, we reach

$$\frac{\partial w}{\partial \nu} = -\partial_2 w = -2g^{ij} h_{\alpha\beta}|_u \nabla_{i2}^2 u^\alpha \partial_j u^\beta = -2h_{\alpha\beta}|_u \nabla_{12}^2 u^\alpha \partial_1 u^\beta.$$

On the other hand, notice that

$$\nabla_{12}^2 u^\alpha = \langle \nabla_{\partial_1} \nabla u^\alpha, \partial_2 \rangle = \partial_1 \langle \nabla u^\alpha, \partial_2 \rangle - \langle \nabla u^\alpha, \nabla_{\partial_1} \partial_2 \rangle.$$

Moreover, by the Neumann boundary condition, $\langle \nabla u^\alpha, \partial_2 \rangle$ vanishes on the boundary, and thus its gradient points in the normal direction. In addition, $\nabla u^\beta = \partial_1 u^\beta \partial_1 \in T\partial\Sigma$. Accordingly, we can write

$$(5.51) \quad \begin{aligned} \frac{\partial w}{\partial \nu} &= -\partial_2 w = -2h_{\alpha\beta}|_u \left\langle \nabla \langle \nabla u^\alpha, \partial_2 \rangle, \nabla u^\beta \right\rangle + 2h_{\alpha\beta}|_u \partial_1 u^\alpha \partial_1 u^\beta \langle \partial_1, \nabla_{\partial_1} \partial_2 \rangle \\ &= 0 - w \langle \partial_1, \nabla_{\partial_1} \nu \rangle = -k_{\partial} w \leq 0, \end{aligned}$$

which follows by applying the convexity hypothesis.

The latter estimate, by means of Hopf's maximum principle, ensures that the maximum of w cannot be attained at the boundary $\partial\Sigma$. Let us then evaluate the inequality from Corollary 5.24 at a maximum interior point x , taking into account that $\nabla w = 0$ and $\mathcal{L}w \geq 0$. Then discarding also the squared norm on the left-hand side, we reach

$$(\lambda - 2\sigma C_\Sigma)w \leq 2 \frac{1}{\sqrt{w} + \varepsilon^2} C_\Sigma w + \lambda |df_\delta|^2 \leq 2C_\Sigma \sqrt{w} + \lambda |df_\delta|^2.$$

Without loss of generality, we can take σ small enough so that $\sigma C_\Sigma < \lambda/4$ and assume that $w > 1$ (otherwise we are done). This leads to

$$\frac{\lambda}{2} \sqrt{w} \leq 2C_\Sigma + \frac{\lambda}{\sqrt{w}} |df_\delta|^2 \leq 2C_\Sigma + \lambda C \operatorname{Lip}(f)^2,$$

which implies the stated Lipschitz regularity of u in $\bar{\Sigma}$. \square

5.6.3. Modification of the argument for a weighted energy density. To achieve our main result removing the extra condition about convexity of the boundary, one needs to work harder. The key idea comes from [113, Lemma 2.4] and amounts to multiply w by a suitable test function so that for the product the nonnegativity in (5.51) still holds and we can argue at a maximum interior point with an appropriate version of Corollary 5.24, which we deduce in the following lines.

LEMMA 5.26. *Set $\tilde{w} := \varphi w$ for some positive function $\varphi \in C^2(\bar{\Sigma})$. Then, under the same assumptions and notation as in Lemma 5.23 and at a maximum interior point of Σ , it holds*

$$\begin{aligned} & \lambda \tilde{w} - 2 \left(\frac{1}{\sqrt{w + \varepsilon^2}} + \sigma \right) C_\Sigma \tilde{w} \\ & \leq \lambda \varphi |df_\delta|^2 + \sigma w \left(2 \frac{|\nabla \varphi|_g^2}{\varphi} - \Delta \varphi \right) + \frac{1}{2} \sqrt{w} \left(11 \frac{|\nabla \varphi|_g^2}{\varphi} + 5 |\nabla^2 \varphi|_g \right). \end{aligned}$$

PROOF. Taking into account that the Hessian of a product reads as

$$\nabla_{ij}^2 \tilde{w} = \varphi \nabla_{ij}^2 w + \partial_i w \partial_j \varphi + \partial_j w \partial_i \varphi + w \nabla_{ij}^2 \varphi,$$

we can write

$$\begin{aligned} (5.52) \quad \mathcal{L} \tilde{w} &= \varphi \mathcal{L} w - \left(\frac{1}{\sqrt{w + \varepsilon^2}} + \sigma \right) \left(w \Delta \varphi + 2 \langle \nabla w, \nabla \varphi \rangle_g \right) \\ &+ \frac{w}{(w + \varepsilon^2)^{3/2}} (h_{\alpha\beta} \circ u) \nabla^2 \varphi (\nabla u^\alpha, \nabla u^\beta) \\ &+ \frac{2}{(w + \varepsilon^2)^{3/2}} (h_{\alpha\beta} \circ u) \langle \nabla w, \nabla u^\alpha \rangle_g \langle \nabla \varphi, \nabla u^\beta \rangle_g. \end{aligned}$$

Then, we note that

$$\mathcal{L} \tilde{w} \leq \varphi \mathcal{L} w - \left(\frac{1}{\sqrt{w + \varepsilon^2}} + \sigma \right) \left(w \Delta \varphi + 2 \langle \nabla w, \nabla \varphi \rangle_g \right) + \frac{|\nabla^2 \varphi|_g w}{(w + \varepsilon^2)^{3/2}} + 2 \frac{|\nabla w|_g |\nabla \varphi|_g w}{(w + \varepsilon^2)^{3/2}},$$

which follows by applying Cauchy-Schwarz arguing with normal coordinates both in the domain and target, around a point $x \in \Sigma$ and $u(x) \in \mathcal{N}$, respectively. Now multiply by φ the inequality in Corollary 5.24 and substitute the above estimate to reach

$$\begin{aligned} (5.53) \quad & \mathcal{L} \tilde{w} + \lambda \tilde{w} - 2 \left(\frac{1}{\sqrt{w + \varepsilon^2}} + \sigma \right) C_\Sigma \tilde{w} \\ & \leq \frac{3\varphi |\nabla w|_g^2}{2(w + \varepsilon^2)^{3/2}} + \lambda \varphi |df_\delta|^2 - w \left(\frac{1}{\sqrt{w + \varepsilon^2}} + \sigma \right) \Delta \varphi \\ & + 2 \left(\frac{1}{\sqrt{w + \varepsilon^2}} + \sigma \right) |\nabla w|_g |\nabla \varphi|_g + |\nabla^2 \varphi|_g \sqrt{w} + 2 \frac{|\nabla w|_g |\nabla \varphi|_g}{\sqrt{w + \varepsilon^2}}. \end{aligned}$$

Evaluating (5.53) at a maximum interior point $x \in \Sigma$, we find that

$$(5.54) \quad 0 = \nabla \tilde{w} = (\nabla \varphi) w + \varphi \nabla w, \quad \text{and hence} \quad \nabla w = -\frac{\nabla \varphi}{\varphi} w.$$

By substitution of the latter, we get

$$\begin{aligned} & \mathcal{L} \tilde{w} + \lambda \tilde{w} - 2 \left(\frac{1}{\sqrt{w + \varepsilon^2}} + \sigma \right) C_\Sigma \tilde{w} \\ & \leq \frac{3|\nabla \varphi|_g^2}{2\varphi} \sqrt{w} + \lambda \varphi |df_\delta|^2 - \sigma w \Delta \varphi + \sqrt{2w} |\nabla^2 \varphi| \\ & + 2 \left(\frac{1}{\sqrt{w + \varepsilon^2}} + \sigma \right) \frac{|\nabla \varphi|_g^2}{\varphi} w + |\nabla^2 \varphi|_g \sqrt{w} + 2 \frac{|\nabla \varphi|_g^2}{\varphi} \sqrt{w}, \end{aligned}$$

where we have also used that $(\Delta \varphi)^2 \leq 2|\nabla^2 \varphi|^2$. Rearranging the right-hand side and taking into account that $\mathcal{L} \tilde{w} \geq 0$ at a maximum interior point, we deduce the estimate in the statement. \square

5.6.4. Lipschitz regularity of the approximated minimizers.

PROPOSITION 5.27. *Let u be the minimizer of $\mathcal{E}_{\mathcal{N},\varepsilon}^{\lambda,\sigma}$ when $\kappa \leq 0$. Then there exist constants $C > 0$ and \tilde{C}_Σ , depending only on the domain surface Σ , such that the following uniform bound holds:*

$$|du| \leq C \text{Lip}(f)^2 + \tilde{C}_\Sigma.$$

PROOF. By the already mentioned regularity properties of the distance to the boundary r_∂ , we can find a constant $\ell > 0$ so that r_∂ is C^∞ on the set $\{x \in \Sigma : r_\partial(x) \leq \ell\}$. Then consider a function η which coincides with r_∂ if $r_\partial \leq \ell/2$, and vanishes for $r_\partial \geq \ell$.

Let us take in particular $\tilde{w} := e^{c\eta}w$ with $c := 2 \max_{\partial\Sigma} |k_\partial|$. With the same conventions as in the proof of Proposition 5.25 and using (5.51), on boundary points we have

$$(5.55) \quad \frac{\partial \tilde{w}}{\partial \nu} = e^{c\eta}w(-k_\partial - c) \leq \tilde{w}(|k_\partial| - c) \leq 0,$$

being the latter true by the choice of c . Consequently, arguing as in the proof of Proposition 5.25, Hopf's maximum principle ensures that the maximum of \tilde{w} occurs at the interior of Σ . By composing r_∂ with any cut-off function in $W^{2,\infty}$, we have the following growth for the derivatives of our concrete test function $\varphi = e^{c\eta}$:

$$(5.56) \quad \frac{|\nabla \varphi|^2}{\varphi} + \frac{1}{2}|\nabla^2 \varphi| \leq \frac{3}{2}c^2\varphi|\nabla \eta|^2 + \frac{c}{2}\varphi|\nabla^2 \eta| \leq C\varphi\left(\frac{3}{2}c^2 + \frac{c^2}{4}\right) \leq \tilde{C}_\Sigma\varphi,$$

where C comes from the bounds of the first and second derivatives of the cut-off function and we have also used (5.50). The notation indicates that the constant only depends on Σ , but is uniform in σ and ε , and its concrete meaning may change from line to line.

We are now in position to apply Lemma 5.26 to reach

$$(\lambda - 2\sigma C_\Sigma)\tilde{w} \leq 2C_\Sigma\varphi\sqrt{w} + \lambda\varphi|df_\delta|^2 + (\sigma w + \sqrt{w})\tilde{C}_\Sigma\varphi,$$

and rearranging terms, it leads to

$$(\lambda - 2\sigma C_\Sigma - \sigma\tilde{C}_\Sigma)\sqrt{w} \leq 2C_\Sigma + \frac{\lambda}{\sqrt{w}}|df_\delta|^2 + \tilde{C}_\Sigma.$$

Finally, taking σ small enough so that $\sigma(2C_\Sigma - \sigma\tilde{C}_\Sigma) < \frac{\lambda}{2}$ and arguing as at the end of the proof of Proposition 5.25, we deduce the estimate and hence the regularity claim in the statement. In fact, notice that we have a bound for $\max \tilde{w}$, but the conclusion follows because $|du|^2 \leq \tilde{w}$. \square

5.6.5. Convergence and Lipschitz regularity of the original problem.

PROPOSITION 5.28. *Let $\sigma \geq 0$ and u be the minimizer of $\mathcal{E}_{\mathcal{N}}^{\lambda,\sigma}$ when $\kappa \leq 0$. Then $u \in C^{0,1}(\bar{\Sigma}, \mathcal{N})$.*

PROOF. Let $\{\varepsilon_k\}_k, \{\delta_k\}_k$ be such that $\varepsilon_k, \delta_k \rightarrow 0$, and let u_k be a weak solution of $(\mathcal{S}_{\varepsilon_k, \sigma}^{f_{\delta_k}})$. We can keep $\sigma > 0$ fixed, because for the case when $\sigma = 0$, we simply take $\sigma_k \rightarrow 0$ and repeat all the computations. By Propositions 5.22 and 5.27, we have that the functions u_k are uniformly bounded in $C^{0,1}(\bar{\Sigma}, \mathcal{N})$.

Therefore, one can find a subsequence (which will not be relabeled for brevity) such that $u_k \rightarrow u$ strongly in $L^\infty(\bar{\Sigma}, \mathcal{N})$ to some $u \in C^{0,1}(\bar{\Sigma}, \mathcal{N})$. Moreover, working again in isothermal coordinates for the domain and in extrinsic by isometric embedding in \mathbb{R}^N for the target, we have that $\nabla u_k \xrightarrow{*} \nabla u$ in $L^\infty(\Omega_\ell, \mathbb{R}^{2N})$ for all ℓ .

Remember from (5.18), that it holds that

$$A_k(x) = \left(\frac{\varrho(x)}{\sqrt{|\nabla u_k(x)|^2 + \varepsilon_k^2 \varrho^2(x)}} + \sigma \right) \nabla u_k(x) =: \varrho(x) \mathbf{a}_k(x) + \sigma \nabla u_k(x).$$

Since $\|\mathbf{a}_k\|_{L^\infty(\Omega_\ell, \mathbb{R}^{2N})} \leq 1$,

$$\mathbf{a}_k \xrightarrow{*} {}^\ell \mathbf{a} \text{ in } L^\infty(\Omega_\ell; \mathbb{R}^{2N}) \text{ and } \|{}^\ell \mathbf{a}\|_{L^\infty(\Omega_\ell)} \leq 1,$$

for a subsequence, not relabeled. Thus

$$A_k \xrightarrow{*} {}^\ell Z := \varrho {}^\ell \mathbf{a} + \sigma \nabla u \text{ in } L^\infty(\Omega_\ell, \mathbb{R}^{2N}).$$

From (5.18), we deduce that $\operatorname{div}(A_k)$ is uniformly bounded in $L^2(\Omega_\ell, \mathbb{R}^N)$. Applying [72, §5, Lemma 4], we deduce that

$$\langle \nabla u_k, A_k \rangle \rightharpoonup \langle \nabla u, \varrho {}^\ell \mathbf{a} \rangle + \sigma |\nabla u|^2 \text{ in } L^2(\Omega_\ell).$$

Furthermore, from the definition of \mathbf{a}_k we note that $\langle \nabla u_k, \mathbf{a}_k \rangle \geq |\nabla u_k| - \varepsilon_k$ and by the lower semicontinuity of the modulus with respect to weak convergence, we have that

$$\varrho |\nabla u| + \sigma |\nabla u|^2 \leq \liminf_{k \rightarrow \infty} \langle \nabla u_k, A_k \rangle \leq \varrho \langle \nabla u, {}^\ell \mathbf{a} \rangle + \sigma |\nabla u|^2,$$

which implies that $\langle \nabla u, {}^\ell \mathbf{a} \rangle = |\nabla u|$. Moreover, by the strong convergence to u we obtain that ${}^\ell Z(x) \in T_{u(x)} \mathcal{N}$, for a.e. $x \in \Omega_\ell$, which in turn leads to ${}^\ell \mathbf{a}(x) \in T_{u(x)} \mathcal{N}$, for a.e. $x \in \Omega_\ell$. We let now $\mathbf{a} := \sum_\ell \chi_\ell \cdot ({}^\ell \mathbf{a} \circ \phi_\ell)$ and define the 1-form \mathcal{Z} as

$$\mathcal{Z}(\partial_i) := \mathbf{a}_i,$$

which satisfies (5.17).

On the other hand, by strong convergence and boundedness of $\operatorname{div}(\mathcal{Z}_{\varepsilon_k})$, we can take limits in the weak formulation on $(\mathcal{S}_{\varepsilon, \sigma}^f)$ and we obtain that u and \mathcal{Z} satisfy the system of equations in (5.16). Finally, as in [84, Theorem 2] we note that, for $\varphi \in C^1(\bar{\Sigma})$,

$$\int_\Sigma \operatorname{div}_g(\varphi(\mathbf{a}^\alpha + \sigma \nabla u^\alpha)) \, \mathrm{d}\mathbf{m}_g = \int_{\partial \Sigma} \varphi \langle \mathbf{a}^\alpha + \sigma \nabla u^\alpha, \nu \rangle \, \mathrm{d}\mathbf{m}_{\tilde{g}},$$

where \tilde{g} is the induced Riemannian metric on $\partial \Sigma$, and

$$\int_\Sigma \operatorname{div}_g(\varphi(A_k^\alpha)) \, \mathrm{d}\mathbf{m}_g = 0.$$

Consequently, we have that

$$\left| \int_{\partial \Sigma} \varphi \langle \mathbf{a}^\alpha + \sigma \nabla u^\alpha, \nu \rangle \, \mathrm{d}\mathbf{m}_{\tilde{g}} \right| = \left| \int_\Sigma (\mathbf{h} - h_k) \, \mathrm{d}\mathbf{m}_g \right|$$

with $\mathbf{h} = \operatorname{div}(\varphi(\mathbf{a}^\alpha + \sigma \nabla u^\alpha))$ and $h_k = \operatorname{div}(\varphi(A_k^\alpha))$. Then, since $h_k \rightarrow \mathbf{h}$ in $L^2(\Sigma)$, we conclude that the boundary condition in (5.16) holds. \square

LEMMA 5.29. Notice that with the same proof, but keeping $\varepsilon > 0$ fixed and letting $\delta, \sigma \rightarrow 0$, we also obtain Lipschitz regularity for minimizers of $\mathcal{E}_{\mathcal{N},\varepsilon}^{\lambda,0}$. Additionally, the local statement in Corollary 5.2 follows by redoing similar computations, but choosing φ in Lemma 5.26 as a suitable cut-off function within a fixed ball.

5.6.6. Lipschitz regularity for the steady Mosolov problem. In this part the goal is to prove claim (b) in Theorem 5.3. As before, the trick is to multiply w by a suitable test function, which now has to depend on the target manifold in order to compensate the quadratic terms in w arising from the positive upper curvature bound. Hereafter we will use the following notations:

$$\tilde{\varphi} = \frac{\varphi}{\psi \circ \rho}, \quad \text{with } \rho = r_h \circ u := d_h(p, u(\cdot)), \quad \text{and } \tilde{w} = \tilde{\varphi}w,$$

where $\varphi = e^{c\eta}$ is the test function coming from the proof of Proposition 5.27 and ψ will be specified later on. Let us first compute the terms appearing in (5.52) with this new test function. Unless otherwise stated, all Laplacians, gradients and inner products are computed with respect to g , and ψ is always evaluated in ρ . Hereafter, C_Σ does not have a specific meaning as before, it just indicates any constant which only depends on the domain surface.

LEMMA 5.30. With the previous notation, let $\kappa > 0$ be an upper bound for the absolute value of the sectional curvatures of $B_h(p, R) \subseteq \mathcal{N}$. Set $w = |du|^2$ for u any solution of $(\mathcal{S}_{\varepsilon,\sigma}^{f,\delta})$. Then \tilde{w} attains its maximum at an interior point of Σ . Moreover, at this point, the elliptic operator \mathcal{L} defined as in (5.45) for $w > 1$ satisfies

$$\begin{aligned} \psi \mathcal{L} \tilde{w} \leq & \varphi \mathcal{L} w + \tilde{w} \left(\frac{1}{\sqrt{w + \varepsilon^2}} + \sigma \right) (\psi' \text{co}_\kappa(\rho) w + \psi'' |\nabla \rho|^2) - 2\psi' \lambda R \tilde{w} + \sigma \varphi |\nabla^{g[\boxtimes]h} du|^2 \\ & + C_\Sigma \left(\frac{w^{3/2}}{\psi^2} (1 - \psi' + (\psi')^2 + |\psi''|) - \psi' \text{co}_{-\kappa}(\rho) \sqrt{w} + \frac{1}{4\sigma} \frac{(\psi')^2}{\psi} \right), \end{aligned}$$

provided that $0 < \psi \leq 1$ is a non-increasing function.

PROOF. After applying the chain rule several times and some algebraic manipulation, we obtain that

$$\Delta \tilde{\varphi} = \frac{\Delta \varphi}{\psi} - \frac{\tilde{\varphi}}{\psi} (\psi' \Delta \rho + \psi'' |\nabla \rho|^2) - 2 \frac{\psi'}{\psi} \langle \nabla \rho, \nabla \tilde{\varphi} \rangle.$$

On the other hand, let us check that (5.55) still holds in the current setting. In fact, choosing again $x^2 = r_\rho$ so that $\nabla_\nu = -\partial_2$, we obtain

$$\frac{\partial \tilde{w}}{\partial \nu} = \frac{\partial}{\partial \nu} \left(\frac{e^{c\eta} w}{\psi \circ \rho} \right) = \varphi w \left(-k_\rho - c + \frac{\psi'}{\psi^2} \partial_2 \rho \right) \leq \tilde{w} \frac{|\psi'|}{\psi} \frac{\partial r_h}{\partial y^\alpha} \partial_2 u^\alpha = 0,$$

where we have applied the Neumann boundary condition and the same choice of c from Proposition 5.27. Once more by virtue of Hopf's maximum principle, we can argue from now on at a maximum interior point for \tilde{w} , at which as in (5.54) we have

$$(5.57) \quad \nabla w = -w \nabla (\log \tilde{\varphi}).$$

Next, using that

$$(5.58) \quad \nabla \tilde{\varphi} = \frac{\nabla \varphi}{\psi} - \frac{\tilde{\varphi}}{\psi} \psi' \nabla \rho,$$

leads to

$$2 \langle \nabla w, \nabla \tilde{\varphi} \rangle = -2w \left\langle \frac{\nabla \varphi}{\psi}, \nabla (\log \tilde{\varphi}) \right\rangle + 2w \frac{\psi'}{\psi} \langle \nabla \rho, \nabla \tilde{\varphi} \rangle.$$

Summing up, we have

$$-\psi \left(\Delta \tilde{\varphi} + \frac{2}{w} \langle \nabla w, \nabla \tilde{\varphi} \rangle \right) = 2 \frac{|\nabla \varphi|^2}{\varphi} - \Delta \varphi + \tilde{\varphi} (\psi' \Delta \rho + \psi'' |\nabla \rho|^2) - 2 \frac{\psi'}{\psi} \langle \nabla \varphi, \nabla \rho \rangle.$$

Doing similar computations for the remaining terms in (5.52), $\mathcal{L}\tilde{w}$ is equal to

$$\begin{aligned} & \tilde{\varphi} \mathcal{L}w + \frac{w}{\psi} \left(\frac{1}{\sqrt{w + \varepsilon^2}} + \sigma \right) \left(2 \frac{|\nabla \varphi|^2}{\varphi} - \Delta \varphi + \tilde{\varphi} (\psi' \Delta \rho + \psi'' |\nabla \rho|^2) - 2 \frac{\psi'}{\psi} \langle \nabla \varphi, \nabla \rho \rangle \right) \\ & + \frac{w/\psi}{(w + \varepsilon^2)^{\frac{3}{2}}} \left((\nabla^2 \varphi - \tilde{\varphi} \psi' \nabla^2 \rho) (\nabla u^\alpha, \nabla u^\alpha) - \tilde{\varphi} \psi'' \langle \nabla \rho, \nabla u^\alpha \rangle^2 - 2 \langle \nabla \varphi, \nabla u^\alpha \rangle \langle \nabla u^\alpha, \nabla \tilde{\varphi} / \tilde{\varphi} \rangle \right), \end{aligned}$$

where we have taken normal coordinates around $u(x) \in \mathcal{N}$, being x the maximum interior point of w , and we understand sum over repeated indices.

By choosing ψ to be a positive non-increasing function, using (5.56) and taking into account that $|\nabla \rho| \leq \sqrt{w}$, we can estimate

$$\begin{aligned} \mathcal{L}\tilde{w} & \leq \tilde{\varphi} \mathcal{L}w + \frac{w}{\psi} \left(\frac{1}{\sqrt{w + \varepsilon^2}} + \sigma \right) \left(C_\Sigma \varphi + \tilde{\varphi} (\psi' \Delta \rho + \psi'' |\nabla \rho|^2) - 2 \frac{\psi'}{\psi} |\nabla \varphi| \sqrt{w} \right) \\ & - \frac{w/\psi}{(w + \varepsilon^2)^{\frac{3}{2}}} \tilde{\varphi} \psi' \nabla^2 \rho (\nabla u^\alpha, \nabla u^\alpha) + \frac{1/\psi}{(w + \varepsilon^2)^{\frac{1}{2}}} \left(C_\Sigma \varphi w + \tilde{\varphi} |\psi''| w^2 + 2\psi \frac{|\nabla \varphi|}{\varphi} |\nabla \tilde{\varphi}| w \right). \end{aligned}$$

On the other hand, by the composition rule and the Hessian comparison theorem as in [147, Lemma 2.9] for sectional curvatures bounded above by κ , we get

$$\psi' \Delta \rho = \psi' \Delta (r_h \circ u) = \psi' \left(\nabla^2 r_h (\nabla u^\alpha, \nabla u^\alpha) + \langle {}^h \nabla r_h|_u, \tau(u) \rangle_h \right) \leq \psi' \text{co}_\kappa(\rho) w - \psi' |\tau(u)|_h.$$

In turn, our equation (5.44) allows us to get a bound for the last term above as follows

$$\left(\frac{1}{\sqrt{w + \varepsilon^2}} + \sigma \right) \tilde{w} |\tau(u)|_h \leq \tilde{w} \left(\frac{1}{2(w + \varepsilon^2)^{\frac{3}{2}}} |\nabla w| \sqrt{w} + \lambda |\exp_u^{-1} f|_h \right) \leq \frac{\tilde{\varphi}}{2} |\nabla w| + 2R\lambda \tilde{w}.$$

It remains to estimate the term with $\nabla^2 \rho$, for which we use that the sectional curvatures of \mathcal{N} are bounded below within the geodesic ball $B_h(p, R) \supset u(\bar{\Sigma})$, say by $-\kappa$. Then the remaining bound of the Hessian comparison theorem leads to

$$\begin{aligned} \frac{|\psi'|}{\psi} \tilde{\varphi} \nabla^2 \rho (\nabla u^\alpha, \nabla u^\alpha) & \leq \frac{|\psi'|}{\psi} \tilde{\varphi} \left(\text{co}_{-\kappa}(\rho) w + |\nabla^{g|_{\mathbb{R}^h}} du| \right) \\ & \leq \frac{|\psi'|}{\psi} \text{co}_{-\kappa}(\rho) \tilde{w} + \sigma \tilde{\varphi} |\nabla^{g|_{\mathbb{R}^h}} du|^2 + \frac{1}{4\sigma} \tilde{\varphi} \left(\frac{\psi'}{\psi} \right)^2, \end{aligned}$$

which follows by Young's inequality.

The last ingredient to bound $\mathcal{L}\tilde{w}$ at an interior maximum is that

$$(5.59) \quad |\nabla w| = \frac{w}{\tilde{\varphi}} |\nabla \tilde{\varphi}| \leq w \left(\frac{|\nabla \varphi|}{\psi \tilde{\varphi}} + \frac{|\psi'|}{\psi} |\nabla \rho| \right) \leq w \frac{|\nabla \varphi|}{\varphi} - \frac{\psi'}{\psi} w^{3/2},$$

where we have applied (5.57) and (5.58).

Gathering all the above inequalities, taking into account that φ is by construction a bounded function on Σ , and in case $w > 1$, after a lengthy but straightforward manipulation we reach the claimed inequality. \square

LEMMA 5.31. *With the previous notation, let $\kappa > 0$ be an upper bound for the absolute value of the sectional curvatures of $B_h(p, R) \subseteq \mathcal{N}$ with $R < R_\kappa^*$. Set $w = |du|^2$ for u any solution of $(\mathcal{S}_{\varepsilon, \sigma}^f)$. For the elliptic operator \mathcal{L} defined as in (5.45) one has*

$$\begin{aligned} \mathcal{L}w + 2\sigma |\nabla^{g_{\boxtimes h}} du|_{g_{\boxtimes h}}^2 &\leq \left(\frac{1}{\sqrt{w + \varepsilon^2}} + \sigma \right) (\kappa w + C_\Sigma) w + 2\lambda\pi |df_\delta| \sqrt{w} \\ &\quad + \frac{|\nabla w|_g^2}{2(w + \varepsilon^2)^{\frac{3}{2}}} + 2\lambda R \frac{|\nabla w|_g}{\sqrt{w + \varepsilon^2}}. \end{aligned}$$

PROOF. From the proof of Lemma 5.23, as we have only used the curvature assumption to estimate the term with λ , we have the equality

$$\begin{aligned} &\mathcal{L}w + 2 \left(\frac{1}{\sqrt{w + \varepsilon^2}} + \sigma \right) \left(|\nabla^{g_{\boxtimes h}} du|_{g_{\boxtimes h}}^2 - \mathcal{R} \right) - 2\lambda \langle \tilde{\nabla}(\exp_u^{-1} f_\delta), du \rangle_{g_{\boxtimes h}} \\ &= \frac{3}{2} \frac{\langle dw \otimes \langle \nabla w, \nabla u \rangle_g, du \rangle_{g_{\boxtimes h}}}{(w + \varepsilon^2)^{5/2}} - \frac{\langle dw \otimes \tau(u), du \rangle_{g_{\boxtimes h}}}{(w + \varepsilon^2)^{3/2}} - \frac{|\nabla w|_g^2}{2(w + \varepsilon^2)^{3/2}}, \end{aligned}$$

A slight variation of the argument in the proof of Lemma 5.23, using Jacobi field comparison, allows the method to be applied in the case of positive curvature bounds. This is achieved after reparametrizing to obtain a unit-speed geodesic and making slight modifications for not necessarily normal fields. Additionally, by employing Hessian comparison for the squared distance function (see [104, Theorem 6.6.1]), we obtain

$$-\langle \tilde{\nabla}(\exp_u^{-1} f_\delta), du \rangle_{g_{\boxtimes h}} \geq d_h(u, f) \operatorname{co}_\kappa(d_h(u, f)) w - 2 \frac{d_h(u, f)}{s_\kappa(d_h(u, f))} |df_\delta| \sqrt{w} \geq -\pi |df_\delta| \sqrt{w},$$

where to discard the first term we have applied that $d_h(u, f) < 2R < \frac{\pi}{2\sqrt{\pi}}$ by the definition of R_κ^* in (5.26).

Combining the latter with (5.49), we can write

$$\begin{aligned} \mathcal{L}w + 2 \left(\frac{1}{\sqrt{w + \varepsilon^2}} + \sigma \right) |\nabla^{g_{\boxtimes h}} du|_{g_{\boxtimes h}}^2 &\leq 2 \left(\frac{1}{\sqrt{w + \varepsilon^2}} + \sigma \right) \mathcal{R} + 2\lambda\pi |df_\delta| \sqrt{w} \\ &\quad - \frac{\langle dw \otimes \tau(u), du \rangle_{g_{\boxtimes h}}}{(w + \varepsilon^2)^{3/2}} + \frac{|\nabla w|_g^2}{2(w + \varepsilon^2)^{3/2}}. \end{aligned}$$

For the curvature term given by (5.47), we have

$$\begin{aligned} \mathcal{R} &\leq \kappa \sum_{i,j} |u_* e_i \wedge u_* e_j|_h^2 + C_\Sigma w \\ &\leq \kappa \sum_{i,j} \left(|u_* e_i|_h^2 |u_* e_j|_h^2 - \langle u_* e_i, u_* e_j \rangle_h^2 \right) + C_\Sigma w \leq \frac{\kappa}{2} w^2 + C_\Sigma w, \end{aligned}$$

which follows thanks to the standard estimates

$$\sum_{i,j} \langle u_* e_i, u_* e_j \rangle_h^2 \geq \sum_i |u_* e_i|_h^4 \geq \frac{1}{2} \left(\sum_i |u_* e_i|_h^2 \right)^2 = \frac{1}{2} w^2.$$

In turn, by applying Kato's inequality, which yields $|\nabla w|_g \leq 2\sqrt{w}|\nabla^{g\boxtimes h} du|_{g\boxtimes h}$, to the last term in (5.6.6), we obtain

$$\mathcal{L}w + 2\sigma|\nabla^{g\boxtimes h} du|_{g\boxtimes h}^2 \leq \left(\frac{1}{\sqrt{w+\varepsilon^2}} + \sigma\right)(w^2 + 2C_\Sigma w) + \lambda\pi|df_\delta|\sqrt{w} + \frac{|\nabla w|_g|\tau(u)|_h}{w+\varepsilon^2}.$$

Finally, our equation (5.44) leads to

$$|\tau(u)|_h \leq \left(\frac{1}{\sqrt{w+\varepsilon^2}} + \sigma\right)^{-1} \left(\frac{|\nabla w|_g\sqrt{w}}{2(w+\varepsilon^2)^{3/2}} + \lambda d_h(u, f)\right) \leq \frac{|\nabla w|_g}{2(w+\varepsilon^2)^{1/2}} + 2\lambda R\sqrt{w+\varepsilon^2},$$

which leads to the inequality in the statement. \square

By combining the estimates in the previous two lemmas, after some extra work and an appropriate choice of the test function ψ , we can prove a Lipschitz regularity result for constants uniform in ε , but depending on σ .

PROPOSITION 5.32. *Let u be a minimizer of $\mathcal{E}_{\mathcal{N},\varepsilon}^{\lambda,\sigma}$ when $\kappa > 0$ is an upper bound for the absolute value of the sectional curvatures in $B_h(p, R) \subseteq \mathcal{N}$ for $R < R_\kappa^*$. Then there exist constants $C > 0$, depending on the domain surface Σ , κ , λ and/or R (but are independent of ε, σ), such that the following bound holds*

$$\sigma^2|du| \leq C(\text{Lip}(f) + 1).$$

PROOF. Setting again $\tilde{w} = \tilde{\varphi}w$ and with the notation at the beginning of this subsection, Lemma 5.30 ensures that \tilde{w} attains its maximum at an interior point x . We perform all the computations at x hereafter. Let us multiply by $\tilde{\varphi}$ the inequality from Lemma 5.31, which joint with the one in Lemma 5.30 and further estimates using (5.59), allow us to write

$$\begin{aligned} 0 \leq \mathcal{L}\tilde{w} + \tilde{\varphi}\sigma|\nabla^{g\boxtimes h} du|_{g\boxtimes h}^2 &\leq \frac{\tilde{\varphi}}{\psi} \left(\frac{1}{\sqrt{w+\varepsilon^2}} + \sigma\right) \left((\psi' \text{co}_\kappa(\rho) + \kappa\psi)w^2 + (C_\Sigma\psi + \psi''|\nabla\rho|^2)w \right) \\ &\quad + 2\lambda\tilde{\varphi}\pi|df_\delta|\sqrt{w} - 4\frac{\psi'}{\psi}\lambda R\tilde{w} - \frac{\psi'}{\psi}\text{co}_{-\kappa}(\rho)C_\Sigma\sqrt{w} + 2\lambda RC_\Sigma\tilde{\varphi}\sqrt{w} \\ &\quad + \frac{C_\Sigma}{\psi^2}w^{3/2} \left(1 - \psi' + (\psi')^2 + |\psi''|\right) + \frac{1}{4\sigma}C_\Sigma\frac{(\psi')^2}{\psi^2}. \end{aligned}$$

By assumption, as $\rho(x) < \frac{\pi}{4\sqrt{\kappa}}$, we set $\psi = c_\kappa^2$. Thus, we may assume that $w > 1$; otherwise, we are done since ψ is bounded from below. Taking into account that $\frac{\psi'}{\psi} = -2\kappa \text{ta}_\kappa$ and $\psi'' \leq 0$, we can estimate

$$\sigma\tilde{\varphi}c_\kappa^2(\rho)w^2 \leq C_{\lambda,\Sigma}|df_\delta|\sqrt{w} + C(w^{3/2} + \sigma^{-1})$$

where C is a constant that may depend on Σ , λ , κ and/or R , but is independent of ε, σ and has a meaning that may change from line to line. Notice that we have used that

$$\frac{|\psi'|}{\psi}\text{co}_{-\kappa}(\rho) \leq 2\sin(\sqrt{\kappa}\rho) \left(1 + \frac{1}{\sqrt{\kappa}\rho}\right) \leq C.$$

Accordingly, we conclude

$$\sigma|du| \leq C_{\lambda,\Sigma}|df_\delta| + C(1 + \sigma^{-1}),$$

which leads to the Lipschitz regularity claimed. \square

The same proof as that of Proposition 5.28, together with lower semicontinuity of \mathcal{E}_σ yields the following regularity result without further restrictions on the curvature of the manifold \mathcal{N} :

PROPOSITION 5.33. *For any $\sigma > 0$ there exists a minimizer u of $\mathcal{E}_{\mathcal{N}}^{\lambda,\sigma}$, such that u belongs to $C^{0,1}(\overline{\Sigma}, \mathcal{N})$.*

5.7. Regularity results for signal denoising

In this section, we establish the Lipschitz regularity of minimizers of $\mathcal{E}_{\mathcal{N},\varepsilon}^{\lambda,\sigma}$ for $\varepsilon, \sigma \geq 0$ and $f \in \text{Lip}(\Gamma)$. We present the proofs explicitly for the case $\Gamma = [0, 1]$, since the case $\Gamma = \mathbb{S}^1$ is simpler due to the absence of boundary conditions. In this setting, the system of Euler–Lagrange equations takes the following form (where, as before, we identify manifold-valued functions with their embedded representation provided by Nash’s theorem):

$$(\mathcal{S}_{\varepsilon,\sigma}^f) \quad \begin{cases} (\mathcal{Z}_{\varepsilon,\sigma}^\alpha(u_x))_x + (\mathcal{A}_u(u_x, \mathcal{Z}_{\varepsilon,\sigma}(u_x)))^\alpha &= -\lambda(\exp_u^{-1} f)^\alpha & \text{in } (0, 1), \\ u_x^\alpha &= 0 & \text{at } \{0, 1\} \end{cases},$$

with

$$\mathcal{Z}_{\varepsilon,\sigma}^\alpha(u_x) = \left(\frac{1}{\sqrt{\varepsilon^2 + |u_x|^2}} + \sigma \right) u_x^\alpha.$$

In this case, we first show that weak solutions have better a-priori regularity than in the case of the domain being a surface Σ .

LEMMA 5.34. *Let u be a bounded weak solution to $(\mathcal{S}_{\varepsilon,\sigma}^f)$ with $\varepsilon, \sigma > 0$. Then $u_x \in C^{0,1}([0, 1])$.*

PROOF. First of all, we observe that since $u \in H^1(0, 1)$, direct from the system, one obtains $\mathcal{Z}_{\varepsilon,\sigma}(u_x) \in W^{1,1}(0, 1)$. We consider now the transformation $A : \mathbb{R}^N \rightarrow \mathbb{R}^N$ given by

$$A(\xi) = \left(\frac{1}{\sqrt{\varepsilon^2 + |\xi|^2}} + \sigma \right) \xi.$$

Since its Jacobian is strictly positive on $(0, 1)$, we can apply the inverse function theorem, and obtain that A^{-1} is well-defined and belongs to $C^1(\mathbb{R}^N)$. Therefore, as $u_x = A^{-1}(\mathcal{Z}_{\varepsilon,\sigma}(u_x))$ we obtain that $u_x \in W^{1,1}(0, 1)$. Hence, by Sobolev embedding, u_x is bounded. Then, once again from the system, we get that $\mathcal{Z}_{\varepsilon,\sigma}(u_x)$ is a Lipschitz function, which implies that u_x is also Lipschitz. \square

We are now in position to prove Theorem 5.3. We only sketch its proof since all the arguments have been already shown in a more complicated framework.

Once the regularity of the solutions of the approximating problems has been obtained, a mollification of the data f as in Proposition 5.22 yields smoothness of the approximating minimizers. Then, the same Bernstein technique as in Proposition 5.25, shows that indeed the Lipschitz constants of the approximations are uniformly bounded. Therefore, a passage to the limit as in Proposition 5.28, together with lower semicontinuity of $\mathcal{E}_{\mathcal{N}}^{\lambda,\sigma}$, proves the result. In this aspect, we remark that the relaxation of the radius condition is given by the simplicity of the 1-dimensional case (see [88]). \square

Finally, we obtain a local estimate as in [82, 83] (see also [95]), which is stated as follows

PROPOSITION 5.35. *Suppose that $f(\Gamma) \subseteq B(p, R)$ with $R \leq R_\kappa^*$. If u is a bounded weak solution of $(\mathcal{S}_{0,0}^f)$, then the next pointwise estimate holds:*

$$(5.60) \quad |u_x| \leq C|f_x| \quad \text{for some } C > 0, \text{ a.e. in } \Gamma$$

PROOF. As in [82], let us consider u to be a weak solution of $(\mathcal{S}_{\varepsilon,\varepsilon^2}^f)$. We will prove that for every $\tilde{\Gamma} \Subset \Gamma$ it is satisfied that

$$(5.61) \quad \int_{\tilde{\Gamma}} |u_x|_\varepsilon^p dx \leq C^p \int_{\tilde{\Gamma}} |f_x|^p dx + p\varepsilon^p |\tilde{\Gamma}| \quad \text{for } p \in [1, 2],$$

where $C > 0$, and

$$(5.62) \quad \limsup_{\varepsilon \rightarrow \varepsilon} \int_{B_r(x_0)} |u_x|_\varepsilon^p dx \leq C^p \int_{B_r(x_0)} |f_x|^p dx \quad \text{for } p \in (1, 2],$$

for $0 < r < R$ and $x_0 \in \tilde{\Gamma}$ such that $B_R(x_0) \Subset \Gamma$. Thus, we define the following function

$$\bar{\varphi} := -\varphi^2 |u_x|_\varepsilon^{p-1} \frac{u_x}{|u_x|_\varepsilon},$$

where φ is a Lipschitz function. From the system $(\mathcal{S}_{\varepsilon,\varepsilon^2}^f)$, we then have that

$$(5.63) \quad \lambda \int_{\tilde{\Gamma}} \bar{\varphi} \cdot (\exp_u^{-1}(f))_x dx = \int_{\tilde{\Gamma}} (\bar{\varphi})_x \cdot ((\mathcal{Z}_{\varepsilon,\varepsilon^2}(u_x))_x + \mathcal{A}_u(u_x, \mathcal{Z}_{\varepsilon,\varepsilon^2}(u_x))) dx.$$

The right-hand side of the above equation is equal to

$$(5.64) \quad - \int_{\tilde{\Gamma}} (\varphi^2 |u_x|_\varepsilon^{p-1})_x \frac{u_x}{|u_x|_\varepsilon} \cdot ((\mathcal{Z}_{\varepsilon,\varepsilon^2}(u_x))_x + \mathcal{A}_u(u_x, \mathcal{Z}_{\varepsilon,\varepsilon^2}(u_x))) dx$$

$$(5.65) \quad - \int_{\tilde{\Gamma}} \varphi^2 |u_x|_\varepsilon^{p-1} \left(\frac{u_x}{|u_x|_\varepsilon} \right)_x \cdot ((\mathcal{Z}_{\varepsilon,\varepsilon^2}(u_x))_x + \mathcal{A}_u(u_x, \mathcal{Z}_{\varepsilon,\varepsilon^2}(u_x))) dx.$$

Firstly, we note that (5.64) can be computed as in [82, Lemma] since $u_x \cdot \mathcal{A}_u(\cdot, \cdot) = 0$. To analyze (5.65), we will use the technique from [84, Lemma 2], obtaining

$$\begin{aligned} & \int_{\tilde{\Gamma}} \varphi^2 |u_x|_\varepsilon^{p-1} \left(\frac{u_x}{|u_x|_\varepsilon} \right)_x \cdot (\mathcal{Z}_{\varepsilon,\varepsilon^2}(u_x))_x dx \\ &= \int_{\tilde{\Gamma}} \varphi^2 |u_x|_\varepsilon^{p-2} (\pi_u + \pi_u^\perp)(u_{xx}) \cdot (\mathcal{Z}_{\varepsilon,\varepsilon^2}(u_x))_x dx + \int_{\tilde{\Gamma}} |u_x|_\varepsilon^{p-4} (u_x \cdot u_{xx})(u_x \cdot (\mathcal{Z}_{\varepsilon,\varepsilon^2}(u_x))_x) dx \\ &\geq \int_{\tilde{\Gamma}} \varphi^2 |u_x|_\varepsilon^{p-2} \pi_u^\perp(u_{xx}) \cdot (\mathcal{Z}_{\varepsilon,\varepsilon^2}(u_x))_x dx = \int_{\tilde{\Gamma}} \varphi^2 |u_x|_\varepsilon^{p-2} \mathcal{A}_u(u_x, u_x) \cdot \mathcal{A}_u(u_x, (\mathcal{Z}_{\varepsilon,\varepsilon^2}(u_x))_x) dx, \end{aligned}$$

Furthermore, we observe that

$$\begin{aligned} & \int_{\tilde{\Gamma}} \varphi^2 |u_x|_\varepsilon^{p-1} \left(\frac{u_x}{|u_x|_\varepsilon} \right)_x \cdot \mathcal{A}_u(u_x, (\mathcal{Z}_{\varepsilon,\varepsilon^2}(u_x))_x) dx = \int_{\tilde{\Gamma}} \varphi^2 |u_x|_\varepsilon^{p-2} u_{xx} \cdot \mathcal{A}_u(u_x, (\mathcal{Z}_{\varepsilon,\varepsilon^2}(u_x))_x) dx \\ &= - \int_{\tilde{\Gamma}} \varphi^2 |u_x|_\varepsilon^{p-2} \mathcal{A}_u(u_x, u_x) \cdot \mathcal{A}_u(u_x, (\mathcal{Z}_{\varepsilon,\varepsilon^2}(u_x))_x) dx, \end{aligned}$$

using that $u_x \cdot \mathcal{A}_u(\cdot, \cdot) = 0$. Thus, we conclude that

$$- \int_{\tilde{\Gamma}} \varphi^2 |u_x|_{\varepsilon}^{p-1} \left(\frac{u_x}{|u_x|_{\varepsilon}} \right)_x \cdot ((\mathcal{Z}_{\varepsilon, \varepsilon^2}(u_x))_x + \mathcal{A}_u(u_x, \mathcal{Z}_{\varepsilon, \varepsilon^2}(u_x))) dx \leq 0.$$

Therefore, as in [82, Lemma] for $\varphi \equiv 1$, we deduce from (5.63) that

$$(5.66) \quad \int_{\tilde{\Gamma}} |u_x|_{\varepsilon}^{p-2} u_x \cdot \partial_u(\exp_u^{-1}(f)) u_x dx \leq \int_{\tilde{\Gamma}} |u_x|_{\varepsilon}^{p-2} |u_x| |\partial_u(\exp_u^{-1}(f)) f_x| dx.$$

Now, noting that for $p \leq 2$ we have $|u_x|_{\varepsilon}^{p-2} |u_x|^2 \geq |u_x|_{\varepsilon}^p - \varepsilon^p$, we employ the Hessian comparison as in Lemma 5.31 and we deduce that

$$(5.67) \quad \begin{aligned} & \int_{\tilde{\Gamma}} |u_x|_{\varepsilon}^p dx - \varepsilon^p |\tilde{\Gamma}| \leq \int_{\tilde{\Gamma}} |u_x|_{\varepsilon}^{p-2} |u_x|^2 dx \\ & \leq \tilde{C} \int_{\tilde{\Gamma}} |u_x|_{\varepsilon}^{p-2} u_x \cdot \partial_u(\exp_u^{-1}(f)) u_x dx, \end{aligned}$$

with $\tilde{C} := \max \left\{ 1, \frac{1}{2R \coth(2R)} \right\}$. Additionally, as in Lemma 5.23, we obtain that

$$|\partial_f(\exp_u^{-1}(f)) f_x| \leq \pi |f_x|.$$

Consequently, we have

$$(5.68) \quad \begin{aligned} & \tilde{C} \int_{\tilde{\Gamma}} |u_x|_{\varepsilon}^{p-2} |u_x| |\partial_f(\exp_u^{-1}(f)) f_x| dx \leq \tilde{C} \pi \int_{\tilde{\Gamma}} |u_x|_{\varepsilon}^{p-2} |u_x| |f_x| dx \\ & \leq \tilde{C} \pi \int_{\tilde{\Gamma}} |u_x|_{\varepsilon}^{p-1} |f_x| dx \leq \frac{p-1}{p} \int_{\tilde{\Gamma}} |u_x|_{\varepsilon}^p dx + \frac{(\tilde{C} \pi)^p}{p} \int_{\tilde{\Gamma}} |f_x|^p dx, \end{aligned}$$

using Young's inequality in the last expression. Combining (5.66), (5.67), and (5.68), we deduce (5.61) with $C = \tilde{C} \pi$. Moreover, we can adapt [82, Lemma] using (5.61) to deduce (5.62). Finally, if we repeat [82, Theorem] using (5.62), we obtain the local estimate (5.60). \square

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