Universitat de València
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COMPOSITION OPERATORS ACTING ON GELFAND-SHILOV CLASSES

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Resumen

En este trabajo investigamos los operadores de composición que actúan en espacios de funciones ultra-diferenciables, definidos por estimaciones globales, que son invariantes bajo la transformada de Fourier, también conocidos como *espacios de Gelfand-Shilov* $\mathcal{S}_{(\omega)}(\mathbb{R})$.

En el Capítulo 1 se introducen conceptos y resultados que utilizamos en el resto del trabajo. En particular, se definen los *espacios de Gelfand-Shilov* $\mathcal{S}_{(\omega)}(\mathbb{R})$ y se trata la sobreyectividad de la aplicación de Borel actuando en ellos, algo que será importante en los capítulos siguientes.

En el Capítulo 2 estudiamos condiciones suficientes y necesarias sobre ψ para que el correspondiente operador de composición C_ψ, actuando entre ciertos espacios de Gelfand-Shilov, sea continuo. También probamos que, para pesos fuertes, los operadores de composición C_ψ actuando en $\mathcal{S}_{(\omega)}(\mathbb{R})$ nunca son compactos.

En el Capítulo 3 investigamos la dinámica de los operadores de composición asociados con polinomios entre espacios adecuados de Gelfand-Shilov. En particular, estudiamos cuándo las iteradas de operadores de composición asociados con polinomios ψ forman una familia equicontinua de operadores continuos entre ciertos espacios de Gelfand-Shilov.

En el Capítulo 4 continuamos la investigación del Capítulo 3. En particular, estudiamos la topologizabilidad y la m-topologizabilidad de los operadores de composición asociados con polinomios ψ actuando sobre espacios de Gelfand-Shilov, propiedades intermedias entre ser de potencia acotada y la continuidad.

En el Capítulo 5 se estudian ciertos operadores de composición entre espacios de modulación.

En el Capítulo 6 se explicitan los problemas abiertos y las líneas de investigación futuras.

Resum

En este treball investiguem els operadors de composició que actuen en espais de funcions ultres-diferenciables definits per estimacions globals que són invariants sota la transformada de Fourier, també coneguts com *espais de Gelfand-Shilov* $\mathcal{S}_{(\omega)}(\mathbb{R})$.

En el Capítol 1 s'introduïxen conceptes i resultats que utilitzem en la resta del treball. En particular, es definixen els *espais de Gelfand-Shilov* $\mathcal{S}_{(\omega)}(\mathbb{R})$ i es tracta la sobreyectividad de l'aplicació de Borel actuant en ells, alguna cosa que serà important en els capítols següents.

En el Capítol 2 estudiem condicions suficients i necessàries sobre ψ perquè el corresponent operador de composició C_ψ, actuant entre uns certs espais de Gelfand-Shilov, siga continu. També provem que, per a pesos forts, els operadors de composició C_ψ actuant en $\mathcal{S}_{(\omega)}(\mathbb{R})$ mai són compactes.

En el Capítol 3 investiguem la dinàmica dels operadors de composició associats amb polinomis entre espais adequats de Gelfand-Shilov. En particular, estudiem quan les iterades d'operadors de composició associats amb polinomis ψ formen una família equicontinua d'operadors continus entre uns certs espais de Gelfand-Shilov.

En el Capítol 4 continuem la investigació del Capítol 3. In particular, estudiem the topologizabilidad i la m-topologizabilidad dels operadors de composició associats amb polinomis ψ actuant sobre espais de Gelfand-Shilov, propietats intermèdies entre ser de potència delimitada i la continuïtat.

En el Capítol 5 s'estudien uns certs operadors de composició entre espais de modulació.

En el Capítol 6 s'expliciten els problemes oberts i les línies d'investigació futures.

Summary

In this work, we investigate composition operators acting on ultradifferentiable function spaces defined by global estimates that are invariant under the Fourier transform, also known as *Gelfand–Shilov spaces* $\mathcal{S}_{(\omega)}(\mathbb{R})$.

In Chapter 1, we introduce concepts and results that are used throughout the rest of the work. In particular, we define the *Gelfand–Shilov spaces* $\mathcal{S}_{(\omega)}(\mathbb{R})$ and address the surjectivity of the Borel mapping acting on them, a topic that will be important in the following chapters.

In Chapter 2, we study necessary and sufficient conditions on ψ for the corresponding composition operator C_ψ, acting between certain Gelfand–Shilov spaces, to be continuous. We also prove that, for strong weights, the composition operators C_ψ acting on $\mathcal{S}_{(\omega)}(\mathbb{R})$ are never compact.

In Chapter 3, we investigate the dynamics of composition operators associated with polynomials between suitable Gelfand–Shilov spaces. In particular, we study when the iterates of composition operators associated with polynomials ψ form an equicontinuous family of continuous operators between certain Gelfand–Shilov spaces.

In Chapter 4, we continue the investigation from Chapter 3. In particular, we study the topologizability and m-topologizability of composition operators associated with polynomials ψ acting on Gelfand–Shilov spaces, which are intermediate properties between being power-bounded and being continuous.

In Chapter 5, certain composition operators between modulation spaces are studied.

In Chapter 6, we outline the open problems and future lines of research.

Chapter 1

Preliminary results

1.1 Introduction

The purpose of this work is to study the behavior of composition operators acting on classes of ultra-differentiable functions, defined by global estimates that ensure the Fourier transform $\mathcal{F}$ is an isomorphism onto its image.

Given a function $\psi : \mathbb{K}^N \to \mathbb{K}^N$ and a suitable family of functions X defined on $\mathbb{K}^N$, the composition operator associated with ψ on X is defined as $C_\psi f = f \circ \psi$ for every $f \in X$. A fundamental and nontrivial problem is to determine necessary and sufficient conditions on ψ for which $C_\psi(X) \subset X$ and $C_\psi : X \to X$ is continuous. Such functions ψ are called symbols for X. For instance, if $X = C^\infty(\mathbb{R})$, then ψ is a symbol for X if and only if $\psi \in X$. This follows from the chain rule and the fact that the identity function $Id(x) = x$ for all $x \in \mathbb{R}$ belongs to X, and hence, $f = f \circ \psi \in X$. Composition operators acting on $C^m(\mathbb{R})$, where $m \in \mathbb{N} \cup \{+\infty\}$, have been studied extensively (see, e.g., [6, 59, 74, 88]).

Composition operators first appeared, albeit implicitly, in Schröder's work [94] in 1871. Schröder was interested in solutions to what is now known as Schröder's equation: given a symbol φ, determine the nontrivial functions f and constants λ such that

$$f \circ \varphi = \lambda f. \tag{1.1}$$

This equation can be reformulated in modern terms as finding the point spectrum of C_φ (i.e. $\sigma_p(C_\varphi)$) in a given function space, such as the space of holomorphic functions $H(\mathbb{D})$ on the unit disk or the space of real-analytic functions $\mathcal{A}(J)$ on an open interval J.

Koenigs [75] solved this problem in 1884 for the unit disk $\mathbb{D}$ under the conditions that φ has a fixed point in $\mathbb{D}$ and its derivative at such point has modulus less than one. Further progress has been made since then (see [27] for a detailed and updated survey on this topic).

Another classical problem involving composition operators is Abel's equation, which seeks functions f in a given function space satisfying

$$C_\varphi(f) = f + 1. \tag{1.2}$$

Although Equations (1.1) and (1.2) may initially appear unrelated, they are deeply connected under specific assumptions. Specifically, if $\varphi(0) = 0$, applying log to both sides of Equation (1.1), dividing by $\log(\lambda)$, and assuming these operations are well-defined, we obtain

$$h \circ \varphi = h + 1, \tag{1.3}$$

which is precisely Abel's equation with $h = \frac{\log(f)}{\log(\lambda)}$. Additionally, if F_1 and F_2 are two real-analytic solutions of Abel's equation, then their difference $f := F_2 - F_1$ is a real-analytic fixed point of the composition operator $C_\varphi : \mathcal{A}(\mathbb{R}) \to \mathcal{A}(\mathbb{R})$, further linking both problems (see [27, Section 3]).

Abel's equation was first mentioned by Abel in [1], and extensive literature exists on its solutions in various function spaces (e.g., [79, Chapter 7], [15, Section 9], and more recent works [98, 104]). In the space of holomorphic functions, see [32, 41, 42, 45]. Abel's equation has been solved in the space of real-analytic functions for $\varphi(x) = \exp(x)$ (see [73, p. 64]). In a series of papers [107, 108, 109, 110], it has been studied for symbols such as $\varphi(x) = \exp(x) - 1$ or $\varphi(x) = \exp(bx)$. Moreover, real-analytic diffeomorphisms φ that admit solutions to Abel's equation have been characterized (see [17, Main Theorem]).

Numerous classical problems are related to composition operators. For instance, one of the earliest examples of hypercyclic operators are the translation operators on $H(\mathbb{C})$, due to Birkhoff ([20]) in 1929:

Theorem. *On the space $H(\mathbb{C})$ of entire functions equipped with the usual topology given by uniform convergence on compact sets, the composition operators $(C_{T_{-a}}f)(z) = f(z + a)$, for all $z \in \mathbb{C}$, $a \neq 0$ and $f \in H(\mathbb{C})$, are hypercyclic on $H(\mathbb{C})$, i.e. there is $f \in H(\mathbb{C})$ such that $\{f(\bullet + na) : n \in \mathbb{N}_0\}$ is dense in $H(\mathbb{C})$.*

See for instance [64, Sections 4.1, 4.3, 4.5]) and the references therein for more literature on hypercyclic operators. The famous backward shifts in linear chaos are also composition operators. So, it is not so surprising that the properties of composition operators have been thoroughly studied in various function spaces, e.g. holomorphic function space, real-analytic function space (the standard references on this topic are [46, 95, 96]), smooth function space (see for instance [6, 88, 89, 55]), and the Schwartz class $\mathcal{S}(\mathbb{R})$ (see [13, 53, 54]) and the space of its multipliers $\mathcal{O}_M(\mathbb{R})$ (see [6, 72]). In [71], the (weighted) composition operators are studied on both L_p-spaces and weighted spaces of continuous functions.

Inspired by the Schwartz class $\mathcal{S}(\mathbb{R})$ and the Gevrey class $G^s(\mathbb{R})$ (see Definition 1.4.2), Gelfand and Shilov introduced in 1968 ([56]) the spaces $\Sigma_d(\mathbb{R})$:

Definition 1.1.1. *Let $d > 1$. The Gelfand-Shilov space $\Sigma_d(\mathbb{R})$ consists of those functions $f \in C^\infty(\mathbb{R})$ such that, for each $h > 0$,*

$$\sup_{x\in\mathbb{R}} \sup_{j,\ell\in\mathbb{N}_0} \frac{|x^\ell f^{(j)}(x)|}{h^{j+\ell}(j!\,\ell!)^d} < +\infty.$$

Since then, broader families of ultra-differentiable classes have been introduced, such as the Carleman classes $\mathcal{E}_{(M_p)}(\mathbb{R})$ of Beurling type (Definition 1.4.6), $\mathcal{S}_{(M_p)}(\mathbb{R})$ (Definition 1.5.2) or $\mathcal{E}_{(\omega)}(\mathbb{R})$ (Definition 1.4.10), $\mathcal{S}_{(\omega)}(\mathbb{R})$ introduced by Beurling and Björck. We highlight [29, 33, 76] as some of the most influential works on unifying and developing the various approaches to ultra-differentiable function theory.

As far as we know, there has been no comprehensive study of the composition operators acting on $\Sigma_d(\mathbb{R})$. Thus, our research initiates from scratch, though guided by previous works in $\mathcal{S}(\mathbb{R})$. While some results align with previous findings from [53, 54], others show notable differences, leaving several open problems (see Chapter 7).

It is straightforward to prove that if φ is a non-constant polynomial, then $f \circ \varphi \in \mathcal{S}(\mathbb{R})$ whenever $f \in \mathcal{S}(\mathbb{R})$. Our first objective was to determine whether this property also holds in $\Sigma_d(\mathbb{R})$. Unfortunately, it turns out that this is not the case. In fact, the property fails even for the simplest nontrivial polynomial of degree greater than one that we examined:

Theorem. *Let $1 < d \leq d' < \frac{3d}{2}$, and consider $\psi(x) = x^2$. Then, there exists $f \in \Sigma_d(\mathbb{R})$ such that $f \circ \psi \notin \Sigma_{d'}(\mathbb{R})$.*

After some detour, we generalised the result (see Corollary 2.3.8). In particular, we proved:

Theorem. *Let* ψ *be a polynomial of degree* $N > 1$. *For every* $d \leq d' < \frac{2N-1}{N} d$, *we have that* $C_\psi(\Sigma_d(\mathbb{R})) \not\subset \Sigma_{d'}(\mathbb{R})$.

Apparently, this unexpected negative result makes impossible to study the general dynamics of the composition operators C_ψ, with ψ being a polynomial, acting on $\Sigma_d(\mathbb{R})$, as in the series of papers [53, 54] for the Schwartz class $\mathcal{S}(\mathbb{R})$. However, we were able to obtain the following positive result, which allows us to bypass the difficulty:

Theorem. *If* $d > 1$ *and* ψ *is a non-constant polynomial, then* $C_\psi : \Sigma_d(\mathbb{R}) \to \Sigma_{2d}(\mathbb{R})$.

A key observation is that the theorem above does not depend on the degree of the polynomial ψ. Noting that if ψ is a polynomial, then so is its mth iteration ψ_m, and that $C_\psi^m = C_{\psi_m}$ for every $m \in \mathbb{R}$, it allows us to study the dynamics of the sequence of iterates of C_ψ as a set of continuous operators from $\Sigma_d(\mathbb{R})$ to $\Sigma_{d'}(\mathbb{R})$, for $d' \geq d$, when ψ is a polynomial. The index $2d$ is optimal in this sense, since $\frac{2N-1}{N} \to 2$ as the degree N of the polynomial ψ tends to $+\infty$.

The space $\Sigma_d(\mathbb{R})$ is a particular case of a broader family of function spaces $\mathcal{S}_{(\omega)}(\mathbb{R})$ (see Definition 1.5.3). Notably, for a weight function of the following form

$$\omega(t) = \max\{0, \log(t)\}^p,$$

with $p > 1$, polynomials ψ remain symbols for the corresponding space $\mathcal{S}_{(\omega)}(\mathbb{R})$, just as they did for $\mathcal{S}(\mathbb{R})$. This is not so surprising when one takes into account the fact that the corresponding $\mathcal{S}_{(\omega)}(\mathbb{R})$ for the degenerate case $p = 1$ (i.e. for $\omega(t) = \max\{0, \log(t)\}$, which is not, strictly speaking, a valid weight) coincides with the Schwartz class $\mathcal{S}(\mathbb{R})$, both as sets and as Fréchet spaces (see Remark 1.5.4). It also holds that

$$\mathcal{S}_{(\omega)}(\mathbb{R}) \subset \mathcal{S}(\mathbb{R}),$$

for every weight ω, and the inclusion is continuous and has dense range.

There are non-trivial symbols for $\Sigma_d(\mathbb{R})$. For instance, $\psi(x) = \sqrt{1 + x^2}$, for $x \in \mathbb{R}$, is a non-trivial symbol for $\Sigma_d(\mathbb{R})$. Notice also that the derivatives of ψ are bounded.

We also showed that for ψ to be a symbol for $\Sigma_d(\mathbb{R})$, ψ' has to be bounded (Theorem 2.3.11).

In [11], we further provided sufficient conditions on ψ to guarantee that $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\sigma)}(\mathbb{R})$, where $\sigma(t) = \omega(t^{1/a})$ and $a \geq 1$ (see, for instance, Proposition 2.4.1 and Corollary 2.4.10).

As in the case of the Schwartz class $\mathcal{S}(\mathbb{R})$ (see [55, Theorem 3.3]), composition operators are never compact when acting on $\mathcal{S}_{(\omega)}(\mathbb{R})$, whenever ω is a strong weight (see condition (ε)(1.15) and Theorem 2.2.2).

We then established, although using methods quite different from those applied in $\mathcal{S}(\mathbb{R})$, that the following result for $\mathcal{S}(\mathbb{R})$ also holds for $\mathcal{S}_{(\omega)}(\mathbb{R})$ (see Proposition 3.3.15 and Corollary 3.3.16):

Theorem 1.1.2 ([53, Theorem 3.11.])**.** Let φ be a polynomial with degree greater than or equal to two. Then, the following are equivalent:

(1) $C_\varphi : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is power bounded.

(2) $C_\varphi : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is mean ergodic.

(3) φ has no fixed points.

We also proceeded to calculate the spectrum of the composition operator C_ψ, where ψ is either a polynomial of degree one or two or $\psi(x) = \sqrt{1+x^2}$ (see for instance Corollary 3.4.14, Proposition 3.4.18, Proposition 3.4.6 and Proposition 3.4.6). The case where ψ is a polynomial of degree two with exactly one fixed point remains unsolved but some partial results regarding its Neumann series are obtained nonetheless (see Theorem 3.4.13).

In Section 3.5, we obtain some results regarding the spectrum and dynamics of the composition operators acting on $\mathcal{S}(\mathbb{R}^N)$, when $\psi : \mathbb{R}^N \to \mathbb{R}^N$ is a linear function.

In Chapter 5, we study the topologizability (and $m-$topologizability) of composition operators C_ψ acting on Gelfand-Shilov classes $\mathcal{S}_{(\omega)}(\mathbb{R})$.

In Chapter 6, we deal with the composition operators acting on global classes that are invariant under Fourier transform $\mathcal{F}$ with a different approach using modulation spaces.

Many other results from [53, 54] hold in the Gelfand-Shilov setting, some of which are proven using rather different approaches. We also identify some differences in the behavior of composition operators on $\mathcal{S}(\mathbb{R})$ (see, for instance, Chapter 5). Additionally, several problems remain unsolved (see Chapter 7).

1.2 Basic preliminaries of functional analysis

Assuming basic knowledge of general topology, we collect important facts about Fréchet spaces, the natural generalization of Banach spaces, we will work on. We follow the notation of [83, Part IV].

A topological vector space X is a vector space endowed with a topology such that addition $+ : X \times X \to X$ and scalar multiplication $* : \mathbb{C} \times X \to X$ are (jointly) continuous.

A subset M of a vector space X is called absolutely convex, if $\lambda x + \mu y \in M$, for each $x, y \in M$ and $\lambda, \mu \in \mathbb{C}$ with $|\lambda| + |\mu| \leq 1$. A locally convex vector space X is a topological vector space for which each point has neighborhood basis consisting of absolutely convex sets.

A family $\mathcal{U}$ of $0-$neighborhoods is said to be a fundamental system of $0-$neighborhoods, if for each $0-$neighborhood U there exist $V \in \mathcal{U}$ and $\varepsilon > 0$ such that $\varepsilon V \subset U$. A family $(p_\alpha)_{\alpha \in A}$ of semi-norms is called a fundamental system of semi-norms, if the sets $U_\alpha = \{x \in X : p_\alpha(x) < 1\}$ constitute a fundamental system of $0-$neighborhoods of X. Every locally convex space X admits a fundamental system of semi-norms $(p_\alpha)_{\alpha \in A}$. From now on we will assume that for every $x \in X$ there is $\alpha \in A$ with $p_\alpha(x) \neq 0$, which means that X is Hausdorff.

Let X and Y be locally convex vector spaces with respective fundamental systems $(p_\alpha)_{\alpha \in A}$ and $(q_\beta)_{\beta \in B}$ of semi-norms. A linear mapping $T : X \to Y$ is continuous, if and only if, for each $\beta \in B$ there exist $\alpha \in A$ and $C > 0$ such that

$$q_\beta(Tx) \leq C\, p_\alpha(x),$$

for each $x \in X$.

A subset M of a locally convex vector space X is said to be bounded if

$$\sup_{x\in M} p_\alpha(x) < \infty,$$

for every $\alpha \in A$, where $(p_\alpha)_{\alpha\in A}$ is a fundamental system of semi-norms for X.

Definition 1.2.1. *Given a locally convex space X, we say that X is a Fréchet space if the following conditions hold:*

1. *X is Hausdorff.*
2. *Its topology may be induced by a* ***countable*** *family of semi-norms $(\|\cdot\|_k)_{k\in\mathbb{N}_0}$.*
3. *It is complete.*

Remark 1.2.2. *The following statements hold:*

1. *If X is a Fréchet space, then X is metrizable. We can use as a metric the one given by*

$$d(x,y) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{\|x-y\|_k}{1+\|x-y\|_k}, \ \forall x, y \in X.$$

2. *A sequence $x_n \to x$ as $n \to \infty$ if and only if $\|x_n - x\|_k \to 0$ as $n \to \infty$, for each $k \in \mathbb{N}$.*

If each bounded set in a Fréchet space X is relatively compact in X then we say that X is Montel. There are many infinite-dimensional Fréchet spaces that are Montel. For instance, the Schwartz space $\mathcal{S}(\mathbb{R})$ and the space of holomorphic functions $H(U)$ on an open set $U \subset \mathbb{C}$ are Montel.

Let us recall some classical results of functional analysis that we will make use of:

Theorem 1.2.3 (Open mapping theorem, Closed graph theorem)**.** *Let E, F be Fréchet spaces and $T : E \to F$ a linear map. The following implications follow:*

1. *If T is surjective and continuous, then T is open.*

2. If the graph of T is closed in $E \times F$ then T is continuous.

Let E, F be locally convex spaces and $(p_\alpha)_\alpha$, $(q_\beta)_\beta$ be fundamental systems of seminorms in E and F respectively. Denote $L(E, F)$ (we will abbreviate with $L(E)$ in the case where $E = F$) the set of all linear continuous maps from E to F. A set $H \subset L(E, F)$ is equicontinuous if for every $\beta \in \mathbb{N}$ there exist $\alpha \in \mathbb{N}$ and $C > 0$ such that the following inequality holds:

$$q_\beta(T(x)) \leq C\, p_\alpha(x),$$

for every $x \in E$ and for every $T \in H$.

Let us recall that the Banach-Steinhaus theorem holds also for Fréchet spaces:

Theorem 1.2.4. *Let E be a Frechet space. $H \subset L(E, F)$ is equicontinuous if and only if for every $x \in E$, $H(x) := \{T(x) : T \in H\}$ is a bounded set of F.*

We are also going to use the following powerful result (see for instance [83, Corollary 26.22]):

Theorem 1.2.5. *Let E, F be Fréchet spaces. If $T \in L(E, F)$ is surjective then, for each compact subset K of F there exists a compact subset L of E such that $T(L) = K$.*

Let $T : X \to X$ be a continuous linear operator on a Fréchet space X. The resolvent set $\rho(T)$ of T consists of all $\lambda \in \mathbb{C}$ such that $R(\lambda, T) := (\lambda I - T)^{-1}$ is a continuous linear operator, that is $\lambda I - T : X \to X$ is bijective and has a continuous inverse. Here I stands for the identity operator on X. The set $\sigma(T) := \mathbb{C} \setminus \rho(T)$ is called the spectrum of T. The point spectrum $\sigma_p(T)$ of T consists of all $\lambda \in \mathbb{C}$ such that $\lambda I - T$ is not injective. Unlike in Banach spaces, $\sigma(T)$ may be unbounded.

Let X be a Fréchet space and $T : X \to X$ a linear continuous map. We say that the operator $T : X \to X$ is mean ergodic if the limits

$$Px = \lim_n \frac{1}{n} \sum_{k=1}^{n} T^k(x),$$

exist, for every $x \in X$. If furthermore the convergence is uniform on bounded sets in X we say that operator $T : X \to X$ is uniformly mean ergodic.

We say that the operator $T : X \to X$ is power bounded if the family of operators $\{T^m : X \to X : m \in \mathbb{N}\}$ is equicontinuous. By Theorem 1.2.4, we have that T is power bounded if and only if the sets $\{T^m x : m \in \mathbb{N}\}$ are bounded in X, for every $x \in X$.

We say that the operator $T : X \to X$ is hypercyclic if there exists a vector $x \in X$ whose orbit $O(T, x) := \{T^n(x) : \ n \in \mathbb{N}\}$ is dense in X for some $x \in X$. Such a point $x \in X$ is called hypercyclic vector for T. The operator is supercyclic if there exists $x \in X$ such that the projective orbit $\mathbb{K}O(T, x) = \{\lambda T^n(x) : \ \lambda \in \mathbb{K}, \ n \in \mathbb{N}\}$ is dense in X.

On the topic of hypercyclicy and other dynamical properties of linear operators, see the books [16, 64] and the articles [4, 5, 3, 60, 61] and the references therein.

1.3 Faà Di Bruno's formula

Since one of our most important tools is Faà Di Bruno's formula, let us recall it (see for instance [78, 1.3.1]):

$$(f \circ g)^{(j)}(x) = \sum \frac{j!}{k_1! \dots k_j!} f^{(k)}(g(x)) \left(\frac{g'(x)}{1!}\right)^{k_1} \dots \left(\frac{g^{(j)}(x)}{j!}\right)^{k_j}, \qquad (1.4)$$

for each $j \in \mathbb{N}$ and $x \in \mathbb{R}$, where the sum is extended over all $(k_1, \dots, k_j) \in \mathbb{N}_0^j$ such that $k_1 + 2k_2 + \dots + jk_j = j$ and we denote $k := k_1 + \dots + k_j$.

We will use the following combinatorial identity extensively, so we include a proof.

Lemma 1.3.1. *Let* $I_j := \left\{ \boldsymbol{k} = (k_1, k_2, \dots, k_j) \in \mathbb{N}_0^j : \sum_{\ell=1}^{j} \ell k_\ell = j \right\}, j \in \mathbb{N}$. *Then,*

$$\sum_{\boldsymbol{k} \in I_j} \frac{(k_1 + ... + k_j)!}{k_1! \dots k_j!} = 2^{j-1}. \qquad (1.5)$$

Proof. Let $g(x) = \frac{x}{1-x}$, for all $x \in \mathbb{R}$. An easy computation gives

$$g^{(\ell)}(x) = \frac{\ell!}{(1-x)^{\ell+1}}$$
$$(g \circ g)(x) = \frac{x}{1-2x} = \frac{1}{2} g(2x)$$

for all $\ell \geq 1$, $x \in \mathbb{R}$. On the one hand, it holds that

$$(g \circ g)^{(j)}(0) = 2^{j-1}\, g^{(j)}(0) = j!\, 2^{j-1}, \tag{1.6}$$

for each $j \in \mathbb{N}$. On the other hand, by applying Faà Di Bruno's formula (1.4) with $f = g$, we obtain that

$$(g \circ g)^{(j)}(0) = j! \sum_{(k_1,\ldots,k_j) \in\ I_j} \frac{k!}{k_1! \ldots k_j!}, \tag{1.7}$$

for each $j \in \mathbb{N}$, where $k = k_1 + \ldots + k_j$. Combining (1.6) and (1.7), we conclude the proof. □

1.4 Ultra-differentiable functions.

The existence of smooth functions that are not real-analytic is well-known. Less known is the fact that there are smooth functions that are nowhere real-analytic (see [84]). One could ask how 'large' the gap between the space of smooth functions and the space of real-analytic ones really is. The following theorem due originally to Borel will show that the gap is very large.

Theorem 1.4.1 (Borel's theorem)**.** *Any formal series* $\sum_{j=0}^{\infty} a_j\, x^j$ *is the Taylor series of a smooth function defined in an open neighborhood of the origin. In other words, the Borel map* $B : C^\infty(\mathbb{R}) \to \mathbb{R}^{\mathbb{N}}$*, defined by* $B(f) = (f^{(j)}(0))_j$*, is surjective.*

The proof of the result can be found in [80, Section 15.4]. An alternative approach can be found in [83, Theorem 26.29].

So the space of smooth functions is much larger than the space of real-analytic functions. Hence we might attempt to consider families of intermediate classes of functions with some nice properties. This is what Maurice Gevrey does in [57]. To do so, one observes that if f is a real-analytic function then, by Cauchy's integral formula, one has that for every compact subset K there exists $C = C_{K,f} > 0$ satisfying that

$$\sup_{x \in K} |f^{(j)}(x)| \leq C^{j+1}\, j!$$

for all $j \in \mathbb{N}_0$. If we replace the 1 in the exponent of the $j!$ on the right-hand side of the estimate above by $s > 1$, we obtain a larger class of smooth functions. In this way, he considers the following family of functions:

Definition 1.4.2. *The Gevrey class (of index $s \geq 1$) $G^s(\mathbb{R})$ is defined as the set of functions $f \in C^\infty(\mathbb{R})$ such that for every compact subset K there exists a $C = C_{K,f} > 0$ satisfying that*

$$\sup_{x \in K} |f^{(j)}(x)| \leq C^{j+1}\,(j!)^s$$

for all $j \in \mathbb{N}_0$.

Some basic well-known facts (see for instance [50, Theorem 2.3., Theorem 2.5., Theorem 2.8., Theorem 2.9., Theorem 2.11.]) about the Gevrey class $G^s(\mathbb{R})$:

1. For $s \geq 1$, $G^s(\mathbb{R})$ is an algebra.
2. For $s \geq 1$, $G^s(\mathbb{R})$ is both closed under differentiation and under composition.
3. The Inverse Function theorem holds in $G^s(\mathbb{R})$ for every $s \geq 1$.

It holds that $G^1(\mathbb{R}) = \mathcal{A}(\mathbb{R})$. It is easy to show that

$$G^s(\mathbb{R}) \subset G^{s+h}(\mathbb{R}),$$

for all $s \geq 1, h > 0$. It is not as easy to see that the inclusions are strictly increasing (see [40]).

In fact, the following inclusions are strict: $\bigcup_{s\geq 1} G^s(\mathbb{R}) \subsetneqq C^\infty(\mathbb{R})$ (this is an easy consequence of Borel's theorem (Theorem 1.4.1)) and also

$$\mathcal{A}(\mathbb{R}) \subsetneqq \bigcap_{s>1} G^s(\mathbb{R}).$$

This last result is harder to obtain, see for instance [21] or [25]. See [81] for further reading on this particular topic. These results motivate the research for further classes of functions which are contained in the gap between $C^\infty(\mathbb{R})$ and $\mathcal{A}(\mathbb{R})$.

At the end of the nineteenth century, Borel produced the first non-trivial examples of sets A of infinitely differentiable functions on $\mathbb{R}$, containing nowhere analytic functions, and such that any element f in A satisfies the implication

$$(*)\ f^{(j)}(0) = 0, \forall j \in \mathbb{N}_0 \Longrightarrow f \equiv 0.$$

Condition $(*)$ means that no element of the set $A \setminus \{0\}$ has flat points, the very basic requirement to construct non-trivial smooth functions with compact support.

After Borel's results, Hadamard was led to work in the same field of research due to considerations of PDE theory. Indeed, the work of Holmgren on the heat equation had already revealed that the solutions of certain partial differential equations are natural elements of classes of functions between $\mathcal{A}(\mathbb{R})$ and $C^\infty(\mathbb{R})$, defined by bounds on their successive derivatives. More explicitly, the heat operator L in $\mathbb{R}^n$, given by

$$L = \frac{\partial}{\partial x_n} - \sum_{j=1}^{n-1} \frac{\partial^2}{\partial x_j^2},$$

has as a fundamental solution the following function (see [58, 91]):

$$E(x_1, ..., x_n) = \begin{cases} (4\pi x_n)^{\frac{1-n}{2}} \exp\left(\frac{-(x_1^2+...+x_{n-1}^2)}{4x_n}\right) & \text{if } x_n > 0 \\ 0, & \text{if } x_n \leq 0 \end{cases}$$

It can be shown that E is not analytic in points of the form $(x_1, ..., x_{n-1}, 0)$, however $E \in C^\infty(\mathbb{R}^n \setminus \{0\})$. In fact, it can be proved that for every compact set $K \subset \mathbb{R}^n \setminus \{0\}$ we have the following estimate:

$$|(\partial^\alpha E)(x)| \leq C^{|\alpha|+1} (\alpha!)^2$$

for all $x \in K$ and $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$, where $C > 0$ only depends on K, $|\alpha| = \alpha_1 + ... + \alpha_n$, $\alpha! = \alpha_1! \, \alpha_2! \ldots \alpha_n!$ and $\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial^{\alpha_n} x_n}$. So $E \in G^2(\mathbb{R}^n)$.

In fact, there is the following more general result:

Proposition 1.4.3 ([40, Lemma 1])**.** *For $s > 1$, we define the function*

$$f_s(t) = \begin{cases} \exp(-t^{\frac{-1}{s-1}}) & \text{whenever } t > 0, \\ 0 & \text{otherwise.} \end{cases}$$

It holds that $f_s \in G^s(\mathbb{R})$. In particular, $f_{\frac{3}{2}}(t) = \exp(-t^{-2})$, for all $t > 0$, belongs to $G^{\frac{3}{2}}(\mathbb{R}) \subset G^2(\mathbb{R})$ and $f_2(t) = \exp(-\frac{1}{t})$, for all $t > 0$, belongs to $G^2(\mathbb{R})$.

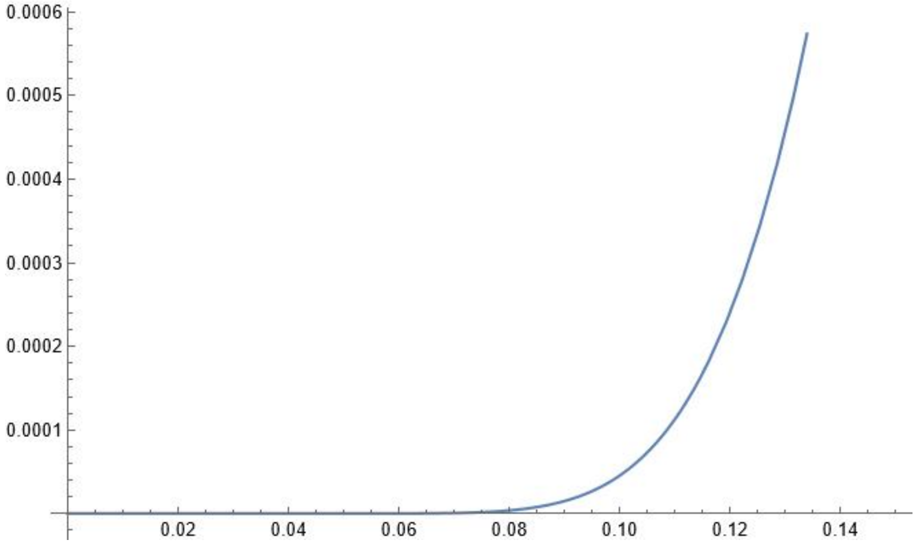

Figure 1.1: Plot of $f_{\frac{3}{2}}$ on $[0, \frac{3}{20}]$.

The functions $f_2, f_{\frac{3}{2}}$ above are usually the textbook examples of smooth functions which are not real-analytic at the origin and also the basic tools that are used to construct elementary bump functions (see for instance [2, Chapter 6, Exercise 6.6.6.]).

The proof of Proposition 1.4.3 relies on techniques of complex analysis. We are going to provide an independent proof for a particular case, which makes use of Faà Di Bruno's formula (1.4), which we will employ extensively afterwards.

Let us show that $h(x) = \exp(-t^{-2})$, for $x \neq 0$, and $h(0) = 0$, is not only smooth but belongs to $G^{\frac{3}{2}}(\mathbb{R})$. First, we need the following inequality:

Lemma 1.4.4. *For all $k, j \in \mathbb{N}$, it holds that*

$$\left(1 + \frac{j}{k}\right)^k \leq e^j.$$

Proof. From the obvious inequality

$$\log(1 + x) = \int_0^x \frac{1}{t+1} dt \leq x,$$

for all $x > 0$, we deduce that

$$(1 + x) \leq e^x,$$

for all $x > 0$ and hence, by setting $x = \frac{j}{k} > 0$, we obtain that

$$\left(1 + \frac{j}{k}\right) \leq e^{\frac{j}{k}},$$

and we easily conclude. □

We are ready to show that $h \in G^{\frac{3}{2}}(\mathbb{R})$.

Proof. Notice that $h = f \circ g$ in $\mathbb{R} \setminus \{0\}$ where $f(x) = \exp(-x)$ and $g(x) = \frac{1}{x^2}$ for all $x \neq 0$. It is known that $h^{(j)}(0) = 0$ for all $j \in \mathbb{N}$. We clearly have that

$$\begin{aligned} &f^{(k)}(x) = (-1)^k \exp(-x) \\ &g^{(\ell)}(x) = (-1)^\ell \, (\ell + 1)! \, x^{-(\ell+2)}, \end{aligned}$$

for all $k, \ell \in \mathbb{N}$ and $x \in \mathbb{R} \setminus \{0\}$. By Faà Di Bruno's formula (1.4), we get the following inequality:

$$|h^{(j)}(x)| \leq \sum \frac{j!}{k_1! \dots k_j!} \exp\left(-\frac{1}{x^2}\right) \frac{1}{|x|^{2k+j}} \prod_{\ell=1}^{j} (\ell + 1)^{k_\ell}, \tag{1.8}$$

for each $x \neq 0$, $j \in \mathbb{N}$, where the sum is extended over all $(k_1, \dots, k_j) \in \mathbb{N}_0^j$ such that $k_1 + 2k_2 + \dots + jk_j = j$ and we denote $k := k_1 + \dots + k_j$.

By the AM-GM inequality ([67, Theorem 1.2.1]) and Lemma 1.4.4, we have that

$$\begin{aligned} \prod_{\ell=1}^{j} (\ell + 1)^{k_\ell} &\leq \left(\frac{2k_1 + 3k_2 + ... + (j + 1)k_j}{k}\right)^k \\ &= \left(1 + \frac{j}{k}\right)^k \leq e^j \end{aligned} \tag{1.9}$$

for all $(k_1, ..., k_j) \in \mathbb{N}_0^j$ such that $k_1 + ... + jk_j = j$ and $j \in \mathbb{N}$, where $k = k_1 + ... + k_j$. Using the obvious inequality

$$e^y = \sum_{\alpha \geq 0} \frac{y^\alpha}{\alpha!} \geq \frac{y^\ell}{\ell!},$$

for all $y \geq 0$, $\ell \in \mathbb{N}_0$, we obtain the following inequalities:

$$\begin{cases} \sqrt{e^{\frac{1}{x^2}}} \geq \frac{1}{x^{2k}} \frac{1}{\sqrt{(2k)!}} \\ \sqrt{e^{\frac{1}{x^2}}} \geq \frac{1}{|x|^j} \frac{1}{\sqrt{j!}}, \end{cases} \tag{1.10}$$

for all $x \neq 0$, $j, k \in \mathbb{N}$. Combining (1.8), (1.9) and (1.10), and using the combinatorial identities Lemma 1.3.1 and $\frac{(2k)!}{k!^2} = \binom{2k}{k} \leq 4^k$, for all $k \in \mathbb{N}$, we finally deduce that

$$\begin{aligned} |h^{(j)}(x)| &\leq (j!)^{\frac{3}{2}} e^j \sum \frac{k!}{k_1!...k_j!} \sqrt{\frac{(2k)!}{k!^2}} \\ &\leq \frac{(4e)^j}{2} (j!)^{\frac{3}{2}}, \end{aligned}$$

where the sum is extended over all $(k_1, \ldots, k_j) \in \mathbb{N}_0^j$ such that $k_1 + 2k_2 + \ldots + jk_j = j$ and we denote $k = k_1 + \ldots + k_j$, for each $j \in \mathbb{N}$ and $x \neq 0$. So, in particular, $h \in G^{\frac{3}{2}}(\mathbb{R})$. □

Notice that the estimations above hold globally.

Hadamard was set to investigate whether the implication $(*)$ could also be characterized in terms of a growth condition on derivatives. It turns out that it can be done (see [101, Theorem 2]). For a more comprehensive study of the use of Gevrey classes in PDE theory, we refer to [39, 70, 91].

In [36, 37, 38, 49] the work of Gevrey is extended; much broader classes of ultra-differentiable functions by means of weight sequences are introduced and studied. A weight sequence in the sense of [76] is defined as follows.

Definition 1.4.5. *A sequence $(M_p)_{p\in\mathbb{N}_0}$ is a weight sequence if it satisfies*

(M0) There exists $c > 0$ such that $\big(c(p+1)\big)^p \leq M_p,\ p \in \mathbb{N}_0$.

(M1) $M_p^2 \leq M_{p-1}M_{p+1},\ p \in \mathbb{N}$ and $M_0 = 1$.

(M2) There are $A, H > 0$ such that $M_p \leq AH^p \min_{0\leq q\leq p} M_q M_{p-q},\ p \in \mathbb{N}_0$.

(γ_1) $\sup_p \dfrac{m_p}{p} \sum_{j\geq p} \dfrac{1}{m_j} < \infty$, where $m_p = \dfrac{M_p}{M_{p-1}}$.

An example of a weight sequence is $M_p = p!^s$, with $s > 1$.

We define the Carleman classes of both Roumieu and Beurling type in $\mathbb{R}^N$ (see [76]):

Definition 1.4.6. *For a weight sequence $(M_p)_{p\in\mathbb{N}_0}$ and an open set G in $\mathbb{R}^N$. The Carleman class $\mathcal{E}_{\{M_p\}}(G)$ of Roumieu type on G is defined as*

$$\mathcal{E}_{\{M_p\}}(G) := \{f \in C^\infty(G) : \textit{ For each } K \subset G \textit{ compact there is } h > 0 : \\ \sup_{x\in K,\alpha\in\mathbb{N}_0^N} \frac{|f^{(\alpha)}(x)|}{h^{|\alpha|}\, M_{|\alpha|}} < \infty\} \tag{1.11}$$

and the Carleman class $\mathcal{E}_{(M_p)}(G)$ of Beurling type on G is defined as

$$\mathcal{E}_{\{M_p\}}(G) := \{f \in C^\infty(G) : \textit{ For each } K \subset G \textit{ and each } h > 0 : \\ \sup_{x\in K,\alpha\in\mathbb{N}_0^N} \frac{|f^{(\alpha)}(x)|}{h^{|\alpha|}\, M_{|\alpha|}} < \infty\} \tag{1.12}$$

Notice that $\mathcal{E}_{\{M_p\}}(\mathbb{R}) = G^s(\mathbb{R})$ with $M_p = p!^s$, for all $p \geq 1$, and also that $\mathcal{E}_{(M_p)}(G) \subset \mathcal{E}_{\{M_p\}}(\mathbb{R})$.

Much later on, Beurling in the 60s (see [19]) pointed out that one can also use weight functions ω to measure the smoothness of functions with compact support by the decay properties of their Fourier transform. Now we define the corresponding ultra-differentiable classes using weight functions in the spirit of Beurling (see [21] and [33] for detailed expositions):

Definition 1.4.7. *A continuous increasing function $\omega : [0,\infty[\longrightarrow [0,\infty[$ is called a weight if it satisfies:*

(α) *there exists $K \geq 1$ with $\omega(2t) \leq K(\omega(t) + 1)$ for all $t \geq 0$,*

(β) $\displaystyle\int_0^\infty \frac{\omega(t)}{1+t^2}\, dt < \infty$,

(γ) $\log(1 + t^2) = o(\omega(t))$ *as t tends to ∞,*

(δ) $\varphi_\omega : t \to \omega(e^t)$ *is convex.*

ω is extended to $\mathbb{R}$ as $\omega(x) = \omega(|x|)$.

We say that two weights ω and σ are equivalent if there exist $A, B > 0$ such that $\omega(t) \leq A\,(1 + \sigma(t))$ and $\sigma(t) \leq B\,(1 + \omega(t))$ for all $t \geq 0$.

In the main results of this work, the weights are assumed to be subadditive i.e. that it holds

$$\omega(s + t) \leq \omega(s) + \omega(t),\ \forall s, t \geq 0,$$

or equivalent to a subadditive one. The most notable example of sub-additive weight is $\omega(t) = |t|^{\frac{1}{s}}$, with $s > 1$. Notice that sub-additivity is a more restrictive condition than (α).

Definition 1.4.8. *Let K be a compact subset of $\mathbb{R}^N$, the set $\mathcal{D}_{(\omega)}(K)$ consists of the integrable functions f with compact support contained in K such that*

$$\|f\|_\lambda := \int |(\mathcal{F}f)(t)|\; e^{\lambda\omega(t)}\, dt < \infty,$$

for every $\lambda > 0$.

It holds that $\mathcal{D}_\omega(K)$ is a Fréchet space when it is endowed with the family of semi-norms $(\| \bullet \|_m)_{m\in\mathbb{N}}$.

Definition 1.4.9. *Let G be an open subset of $\mathbb{R}^N$ and $(K_\nu)_\nu$ an increasing family of compact sets contained in G whose union is G. We define $\mathcal{D}_{(\omega)}(G)$ as the inductive limit (see [83, Pag. 276] for a detailed exposition of inductive and projective limits) of the Fréchet spaces $\mathcal{D}_{(\omega)}(K_\nu)$.*

Definition 1.4.10. *For a weight function ω and an open set $G \subset \mathbb{R}$ we define the ω-ultradifferentiable functions $\mathcal{E}_{(\omega)}(G)$ of Beurling type as those complex-valued functions f in G such that $f\varphi \in \mathcal{D}_{(\omega)}(G)$ whenever $\varphi \in \mathcal{D}_{(\omega)}(G)$. The topology of $\mathcal{E}_{(\omega)}(G)$ is given by the family of semi-norms $\|f\|_{\lambda,\varphi} := \|f\varphi\|_\lambda$, for each $\varphi \in \mathcal{D}_{(\omega)}(G), \lambda > 0$.*

The technique was later modified by Braun, Meise, and Taylor (see [33]), who showed that these classes can also be defined by the decay behaviour of their derivatives, if one uses the Young conjugate (sometimes called Legendre or Fenchel transform, see [67, Definition 2.2.3]) of the function $t \to \varphi_\omega(t) = \omega(e^t)$. The *Young conjugate* $\varphi^*_\omega : [0, \infty[\longrightarrow \mathbb{R}$ of φ_ω is defined by

$$\varphi^*_\omega(s) := \sup\{st - \varphi_\omega(t) :\ t \geq 0\},\ s \geq 0.$$

It is well-known (see, for instance, [67] or [103] for a modern treatment) in convex analysis that if $\varphi : [0,+\infty[\to [0,+\infty[$ is convex, the Young conjugate $\varphi^*_\omega : [0,+\infty[\to [0,+\infty[$ is also convex and verifies that $(\varphi^*)^* = \varphi$. For the convenience of the reader, we are going to write some important properties that will be used very often from now on:

Proposition 1.4.11 ([24, Lemma 4.6.]). *Let $\omega : [0,+\infty[\to [0,+\infty[$ be a continuous increasing function such that $\varphi := \omega \circ \exp$ is convex. Then the following properties hold:*

1. *$\frac{\varphi^*(s)}{s}$ is increasing.*
2. *$\varphi^*(s) + \varphi^*(s) \leq \varphi^*(s+t)$ for all $s,t \geq 0$.*
3. *If there exist $A \geq 0$, $B \geq 1$ such that $\omega(et) \leq A + B\omega(t)$ for all $t \geq 0$, then for all $\rho \geq 1, \lambda > 0$ and $j \in \mathbb{N}_0$:*
$$\rho^j \exp\left(\lambda\varphi^*(\frac{j}{\lambda})\right) \leq \Lambda_{\rho,\lambda} \exp\left(\lambda'\varphi^*(\frac{j}{\lambda'})\right),$$
for all $0 < \lambda' \leq \frac{\lambda}{B^{E[\log\rho+1]}}$, where $\Lambda_{\rho,\lambda} = \exp\left(\lambda\frac{A}{B}E[\log\rho + 1]\right)$ and $E[x]$ is the integer part of x.
4. *Assume that there exist $A \geq 0$, $B \geq 1$ such that $\omega(et) \leq A + B\omega(t)$ for all $t \geq 0$ and that $\lim_{t\to\infty}\frac{\omega(t)}{t} = 0$. Then, for all $D, \lambda > 0$ there is $C_{D,\lambda} > 0$ such that for all $n \in \mathbb{N}_0$:*
$$D^n n! \leq C_{D,\lambda} \exp\left(\lambda\varphi^*(\frac{n}{\lambda})\right).$$

Theorem 1.4.12. *For a weight function ω and an open set $G \subset \mathbb{R}^N$, the ω-ultradifferentiable functions $\mathcal{E}_{\{\omega\}}(G)$ of Roumieu type can be described as*

$$\mathcal{E}_{\{\omega\}}(G) = \{f \in C^\infty(G) : \textit{ for each compact } K \subset G \textit{ there is } m \in \mathbb{N} : \sup_{x\in K,\alpha\in\mathbb{N}_0^N} |f^{(\alpha)}(x)| \exp\left(-\frac{1}{m}\varphi^*_\omega(m|\alpha|)\right) < \infty\}$$

and the ω-ultradifferentiable functions $\mathcal{E}_{(\omega)}(G)$ of Beurling type as

$$\mathcal{E}_{(\omega)}(G) = \{f \in C^\infty(G) : \textit{ for each compact } K \subset G \textit{ and for each } m \in \mathbb{N} : p_{K,m}(f) < \infty\}$$

where $p_{K,m}(f) = \sup_{x\in K,\alpha\in\mathbb{N}_0^N} |f^{(\alpha)}(x)| \exp\left(-m\varphi^*_\omega(\frac{|\alpha|}{m})\right)$.

The topology of $\mathcal{E}_{\{\omega\}}(G)$ is given by first taking the inductive limit over all $m \in \mathbb{N}$ for each compact $K \subset G$ and then taking the projective limit of these (see [83, p. 276]), while $\mathcal{E}_{(\omega)}(G)$ carries the straightforward locally convex topology given by the semi-norms $P_{K,m}$, where K is a compact subset of G and $m \in \mathbb{N}$. $\mathcal{E}_{(\omega)}(G)$ is a Fréchet Montel space, whereas $\mathcal{E}_{\{\omega\}}(G)$ has a rather cumbersome topology but it is worth mentioning that $\mathcal{E}_{\{\omega\}}(G)$ is a countable projective limit of $(DFN)-$spaces, which is in turn ultrabornological (i.e. every linear bounded operator from $\mathcal{E}_{\{\omega\}}(G)$ to any other topological vector space is continuous), reflexive and complete ([29]).

Obviously, the following relation hold:

$$\mathcal{E}_{(\omega)}(G) \subset \mathcal{E}_{\{\omega\}}(G),$$

with continuous inclusion, for each weight ω. Moreover, if $\frac{\omega(t)}{\sigma(t)} \to 0$, as $t \to \infty$, then,

$$\mathcal{E}_{\{\omega\}}(G) \subset \mathcal{E}_{(\sigma)}(G),$$

and the inclusion is continuous.

The classical Gevrey class $G^s(\mathbb{R})$, $s > 1$, corresponds to the $\omega-$ultradifferentiable class $\mathcal{E}_{\{\omega\}}(\mathbb{R})$ of Roumieu type when $\omega(t) = |t|^{\frac{1}{s}}$.

For its use in later chapters we now calculate the Young's conjugate of φ_ω for the most important weights we consider in the thesis.

Lemma 1.4.13. *1. If $\omega(t) = |t|^{\frac{1}{s}}$, with $s > 1$, then*

$$\varphi^*_\omega(x) = xs \log\left(\frac{xs}{e}\right), \ x \geq 0. \tag{1.13}$$

2. If $\omega(t) = \max\{0, \log t\}^p$, with $p > 1$, then it holds that

$$\varphi^*_\omega(s) = \left(\frac{p-1}{p^{\frac{p}{p-1}}}\right) s^{\frac{p}{p-1}}, \ \forall s \geq 0.$$

Proof. (1). Fix $x \geq 0$. Let us calculate the maximum of $g(y) := xy - \varphi_\omega(y)$ on $[0, \infty[$. Notice that $\lim_{y\to 0} g(y) = -1$ and $\lim_{y\to\infty} g(y) = -\infty$. We also have that

$$g'(y) = x - \exp(\frac{y}{s})\frac{1}{s} = 0 \Leftrightarrow y = s\log(xs),$$
$$g''(y) = -\exp(\frac{y}{s})\frac{1}{s^2} < 0,$$

for all $y \geq 0$. So, we obtain that

$$\varphi_\omega^*(x) = g(s\log(xs)) = xs\log(\frac{xs}{e}).$$

(2). Clearly, $\varphi_\omega^*(s) = \sup_{t>0}\{st - t^p\}$ for all $s \geq 0$. Consider $h_s(t) = st - t^p$ for all $t \geq 0$. Obviously, $\varphi_\omega^*(s) = \sup\{h_s(t) : t > 0\}$ for all $s \geq 0$. Note that $\varphi_\omega^*(s) \geq h_s(0) = 0$ for all $s \geq 0$ and $h_s(t) = t(s - t^{p-1}) \to -\infty$ as $t \to \infty$ because $p - 1 > 0$. As $s \geq 0$, $t \geq 0$, $p - 1 > 0$ we also have that

$$h_s^{'}(t) = s - pt^{p-1} = 0 \textit{ if and only if, } t = (\frac{s}{p})^{\frac{1}{p-1}}$$

Finally, we have that

$$\begin{aligned}\varphi_\omega^*(s) = h_s(t_s) &= s\,\frac{s^{\frac{1}{p-1}}}{p^{\frac{1}{p-1}}} - \frac{s^{\frac{p}{p-1}}}{p^{\frac{p}{p-1}}} \\ &= \left(\frac{p}{pp^{\frac{1}{p-1}}} - \frac{1}{p^{\frac{p}{p-1}}}\right) s^{\frac{p}{p-1}} \\ &= \left(\frac{p-1}{p^{\frac{p}{p-1}}}\right) s^{\frac{p}{p-1}}\end{aligned}$$

for all $s \geq 0$, as we wanted. □

The following will be frequently used.

Lemma 1.4.14. *Let ω be a weight function and $0 < a \leq 1$. If $\sigma(x) = \omega(x^a)$, for all $x \in \mathbb{R}$, then, $\varphi_\sigma^*(x) = \varphi_\omega^*(\frac{x}{a})$, for all $x \in \mathbb{R}$, and σ is a weight function. Moreover, if ω is sub-additive, so is σ.*

Proof. Obviously, it holds for every $y \geq 0$ that

$$\varphi_\sigma(y) = \sigma(\exp(y)) = \omega(\exp(ay)) = \varphi_\omega(ay).$$

From which it follows for each $x \in \mathbb{R}$ that

$$\begin{aligned}\varphi_\sigma^*(x) &= \sup_{y>0}\{xy - \varphi_\sigma(y)\} = \sup_{y>0}\{\frac{x}{a}(ay) - \varphi_\omega(ay)\} \\ &= \sup_{q>0}\{\frac{x}{a}q - \varphi_\omega(q)\} = \varphi_\omega^*(\frac{x}{a}).\end{aligned}$$

Now let us check that σ is indeed a weight. Clearly, σ is continuous and increasing. Also $\varphi_\sigma = \sigma \circ \exp$ is convex. Moreover, σ verifies that

$$\int_1^\infty \frac{\sigma(t)}{1+t^2}dt = \int_1^\infty \frac{\omega(t^a)}{1+t^2}dt \leq \int_1^\infty \frac{\omega(t)}{1+t^2}dt < \infty$$

and

$$\begin{aligned}\lim_{t\to\infty} \frac{\log(1+t)}{\sigma(t)} &= \lim_{t\to\infty} \frac{\log(1+t^a)}{\omega(t^a)} \frac{\log(1+t)}{\log(1+t^a)} \\ &= \frac{1}{a} \lim_{s\to\infty} \frac{\log(1+s)}{\omega(s)} = 0.\end{aligned}$$

Finally, if $n \in \mathbb{N}$ then it holds that

$$\sigma(2t) \leq \omega(2t^a) \leq K\,(1+\omega(t^a)) = K\,(1+\sigma(t)),$$

for all $t \geq 0$.

If we further assume that ω is sub-additive then, σ also is sub-additive. Indeed, since $0 < a < 1$, we have that

$$(s+t)^a \leq s^a + t^a,$$

for all $s, t \geq 0$. Using that ω is increasing and sub-additive, we deduce that

$$\begin{aligned}\sigma(s+t) = \omega((s+t)^a) &\leq \omega(s^a+t^a) \\ &\leq \omega(s^a) + \omega(t^a) = \sigma(s) + \sigma(t),\end{aligned}$$

for all $s, t \geq 0$. □

Braun, Meise and Taylor also showed in [33] that, under rather strong conditions, Theorem 1.4.12 and Definition 1.4.6 lead to the same class. But it remained an open question as to whether both approaches were identical under milder assumptions or not. In [29] this question was finally settled. It turns out that, in general, there are classes defined by mean of weight functions ω which cannot be defined by mean of a weight sequence $(M_p)_{p\in\mathbb{N}_0}$. We will summarize the relevant results for our purposes in Proposition 1.4.15 down below.

It is worth pointing out that there are more recent approaches using weight matrices that allow one to unify, under some mild assumptions, the previous approaches (see for instance [92] and the references therein).

The precise condition that makes both approaches equivalent is the following one:

Proposition 1.4.15 ([29, 14 Theorem, 16 Corollary, 20 Example])**.** *It holds:*

1. *For each weight sequence $(M_p)_p$, the following assertions are equivalent:*
    (a) *There exists a weight function ω such that for each $n \in \mathbb{N}$ and each open set G in $\mathbb{R}^n$ the spaces $\mathcal{E}_{\{\omega\}}(\mathbb{R}^n)$ ($\mathcal{E}_{(\omega)}(\mathbb{R}^n)$) and $\mathcal{E}_{\{M_p\}}(\mathbb{R}^n)$ (resp. $\mathcal{E}_{(M_p)}(\mathbb{R}^n)$) are equal as vector spaces and/or locally convex spaces.*
    (b) *There exists $Q \in \mathbb{N}$ such that $\liminf_{j\to\infty} \frac{m_{Qj}}{m_j} > 1$ and 1.(a) holds with $\omega = M$, where M is a weight function given by*

    $$M(t) = \sup_{p\in\mathbb{N}} \log\left(\frac{|t|^p}{M_p}\right)$$

    for all $t > 0$ and $M(0) := 0$.

2. *For each weight function ω the following assertions are equivalent:*
    (a) *There exists a weight sequence $(M_p)_{p\in\mathbb{N}_0}$ such that for each $n \in \mathbb{N}$ and each open set G in $\mathbb{R}^n$ the spaces $\mathcal{E}_{\{\omega\}}(\mathbb{R}^n)$ ($\mathcal{E}_{(\omega)}(\mathbb{R}^n)$) and $\mathcal{E}_{\{M_p\}}(\mathbb{R}^n)$ (resp. $\mathcal{E}_{(M_p)}(\mathbb{R}^n)$) are equal as vector spaces and/or locally convex spaces.*
    (b) *There exists $H \geq 1$ such that*

    $$2\omega(t) \leq \omega(Ht) + H \tag{1.14}$$

    for all $t \geq 0$, and the sequence $(M_p)_p$ given by $M_p = \varphi_\omega(p)$, for all $p \in \mathbb{N}$, is a weight sequence for which 2.(a) holds.

3. *For each $p > 1$, the function $\omega(t) := \max\{0, \log(t)\}^p$ is a weight function which satisfies that*

    $$\mathcal{E}_{(\omega)}(\mathbb{R}) \neq \mathcal{E}_{(M_p)}(\mathbb{R})$$

    for each weight sequence $(M_p)_p$.

Moreover, the weight $\omega(t) := \max\{0, \log(t)\}^p$ is equivalent to a subadditive weight.

When we write $\mathcal{E}_*(G)$ we mean that the result holds for both $\mathcal{E}_{\{\omega\}}(G)$ and $\mathcal{E}_{(\omega)}(G)$.

In regard to the composition operators acting on $\mathcal{E}_*(G)$, we have the following well-known fact (see [33]):

Proposition 1.4.16. *Let $G_1, G_2 \subset \mathbb{R}^N$ be open sets, let $g : G_1 \to G_2$ be a real-analytic function and $f \in \mathcal{E}_*(G_2)$. Then $f \circ g \in \mathcal{E}_*(G_1)$. In other words, the composition operator $C_g : \mathcal{E}_*(G_2) \to \mathcal{E}_*(G_1)$ is continuous. In particular, $\mathcal{E}_*(G)$ contains all real-analytic functions for all open set $G \subset \mathbb{R}^N$.*

Composition operators on ω-ultradifferentiable function spaces have been widely studied. In particular, the problem of characterizing inverse closed (or equivalently, holomorphically closed) Denjoy–Carleman classes has been addressed — see [52, Theorem 3.8] and references therein.

We recall that the Schwartz class $\mathcal{S}(\mathbb{R})$ consists of those smooth functions $f : \mathbb{R} \to \mathbb{C}$ with the property that

$$p_n(f) := \sup_{x\in\mathbb{R}} \sup_{1\leq j\leq n} (1+x^2)^n |f^{(j)}(x)| < \infty$$

for each $n \in \mathbb{R}$. It turns out that $\mathcal{S}(\mathbb{R})$ is a Fréchet Montel space when it is endowed with the natural locally convex topology generated by the sequence of seminorms $(p_n)_{n\in\mathbb{N}}$. The most important reason as to why $\mathcal{S}(\mathbb{R})$ was introduced by Laurent Schwartz is the fact that $\mathcal{F} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is an isomorphism onto, where $\mathcal{F}$ is the usual Fourier transform. $\mathcal{S}(\mathbb{R})$ is one of the pillars of harmonic analysis and the theory of distributions, see for instance [44, 65, 66, 97, 105]).

The first characterization for φ being a so-called symbol (i.e. giving a continuous composition operator $C_\varphi : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ was given in [55]:

Theorem 1.4.17 ([55, Theorem 2.3.]). *A function $\varphi \in C^\infty(\mathbb{R})$ is a symbol for $\mathcal{S}(\mathbb{R})$ if and only if the following conditions are satisfied:*

(i) For all $j \in \mathbb{N}_0$ there exist $C, p > 0$ such that

$$|\varphi^{(j)}(x)| \leq C(1 + |\varphi(x)|^2)^p$$

for every $x \in \mathbb{R}$.

(ii) There exists $k > 0$ such that $|\varphi(x)| \geq |x|^{1/k}$ for all $|x| \geq k$.

By [55, Remark 2.4.], Theorem 1.4.17 also holds for $\mathcal{S}(\mathbb{R}^N)$, where $N \geq 1$.

Notice that by condition *(ii)*, it holds that every symbol φ for $\mathcal{S}(\mathbb{R})$ goes to infinity as $|x|$ goes to infinity.

In the followup paper [53], the dynamics of the composition operators C_φ acting on $\mathcal{S}(\mathbb{R})$ is studied.

In a recent paper [13], the dynamics of the (weighted) composition operators $C_{\psi,\varphi}$ acting on $\mathcal{S}(\mathbb{R}^N)$ is examined.

1.5 Global spaces of ultradifferentiable functions.

The Gevrey classes are made of functions whose derivatives verify certain local estimations, whereas the Schwartz class is made of functions whose derivatives asymptotically decrease fast 'enough'. Combining both the Gevrey classes and the Schwartz class, we define the following well-known family of smooth functions (originally introduced in [56], see [85] and the references therein for further information):

Definition 1.5.1. *The Gelfand-Shilov space* $\Sigma_d(\mathbb{R})$*, with* $d > 1$*, consists of those functions* $f \in C^\infty(\mathbb{R})$ *such that, for each* $h > 0$ *:*

$$\sup_{x\in\mathbb{R}} \sup_{j,\ell\in\mathbb{N}_0} \frac{|x^\ell f^{(j)}(x)|}{h^{j+\ell}[(j+\ell)!]^d} < +\infty.$$

We can define more general families of functions, by changing the sequence $([(j+\ell)!]^d)_{j,\ell}$ above for a weight sequence $(M_{j+\ell})_{j,\ell}$ in the sense of Carleman (see Definition 1.4.5):

Definition 1.5.2. *The space* $\mathcal{S}_{(M_p)}(\mathbb{R})$ *associated to the weight sequence* $(M_p)_{p\in\mathbb{N}_0}$ *consists of those functions* $f \in C^\infty(\mathbb{R})$ *such that, for each* $h > 0$ *:*

$$\sup_{x\in\mathbb{R}} \sup_{j,\ell\in\mathbb{N}_0} \frac{|x^\ell f^{(j)}(x)|}{h^{j+\ell} M_{j+\ell}} < +\infty.$$

With little adjustments from $\mathcal{E}_{(M_p)}(\mathbb{R})$ and $\mathcal{E}_{(\omega)}(\mathbb{R})$, we can define the following global class of smooth functions using weight functions in the sense of Braun-Meise-Taylor (see Definition 1.4.7) rather than weight sequence:

Definition 1.5.3. *Let ω be a weight function. The Gelfand-Shilov space of Beurling type $\mathcal{S}_{(\omega)}(\mathbb{R})$ consists of those functions $f \in L^1(\mathbb{R})$ with the property that $f, \widehat{f} \in C^\infty(\mathbb{R})$ and*

$$q_{\lambda,j}(f) := \max\big\{ \sup_{x\in\mathbb{R}} |f^{(j)}(x)|\, e^{\lambda\omega(x)},\ \sup_{\xi\in\mathbb{R}} |\widehat{f}^{(j)}(\xi)|\, e^{\lambda\omega(\xi)}\big\} < +\infty$$

for every $\lambda > 0, j \in \mathbb{N}_0$.

$\big(\mathcal{S}_{(\omega)}(\mathbb{R}), (q_{\lambda,j})_{\lambda>0, j\in\mathbb{N}_0}\big)$ is a Fréchet space which can be endowed with different equivalent systems of seminorms (see for instance [14, 22]). In particular we shall use the families of seminorms

$$p_\lambda(f) := \sup_{j,k\in\mathbb{N}_0} \sup_{x\in\mathbb{R}} |x^k f^{(j)}(x)| e^{-\lambda\varphi^*_\omega(\frac{j+k}{\lambda})}, \quad \lambda > 0$$

or

$$\pi_{\lambda,\mu}(f) := \sup_{j\in\mathbb{N}_0} \sup_{x\in\mathbb{R}} |f^{(j)}(x)| e^{-\lambda\varphi^*_\omega(\frac{j}{\lambda}) + \mu\omega(x)}, \quad \lambda > 0, \mu > 0.$$

Let $d > 1$ be given. The Gelfand-Shilov space $\Sigma_d(\mathbb{R})$ is

$$\Sigma_d(\mathbb{R}) = \mathcal{S}_{(M_p)}(\mathbb{R}) = \mathcal{S}_{(\omega)}(\mathbb{R}),$$

where

$$M_p = p!^d, \quad \omega(t) = t^{\frac{1}{d}}.$$

It holds that the Gaussian function $h(x) = \exp(-x^2)$, for $x \in \mathbb{R}$, belongs to $\mathcal{S}_{(\omega)}(\mathbb{R})$, for every weight ω. A way of showing this is by using Proposition 1.4.11, Stirling's formula and the following estimate from [56, p. 220]: there are $A, B, C > 0$ such that

$$|x|^k\, |h^{(q)}(x)| \le C\, A^k\, B^q\, k^{\frac{k}{2}}\, q^{\frac{q}{2}}$$

for all $x \in \mathbb{R}$, $k \in \mathbb{N}$, $q \in \mathbb{N}$.

Also, $\mathcal{S}_{(\omega)}(\mathbb{R})$ is an algebra and Montel. In fact, $\mathcal{S}_{(\omega)}(\mathbb{R})$ is nuclear (see [23]). It is well-known, even since [21], that the Fourier transform $\mathcal{F} : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\omega)}(\mathbb{R})$ is an isomorphism onto, as it was the case for $\mathcal{S}(\mathbb{R})$, which happens to be a particular case of the global $\omega-$ultradifferentiable classes $\mathcal{S}_{(\omega)}(\mathbb{R})$. Indeed,

Remark 1.5.4. *If $\omega(t) = \max\{0, \log(t)\}$ then the corresponding $\mathcal{S}_{(\omega)}(\mathbb{R}) = \mathcal{S}(\mathbb{R})$ as Fréchet spaces. However, ω does not verify Definition 1.4.7(γ).*

In the last decades, the classes of ultradifferentiable functions and their duals have been intensively studied for many reasons and have become the right setting to study many different problems in analysis in a very general way (partial differential equations, Paley-Wiener theorem-like results, Whitney jets, Borel theorem-like results, etc). Just to mention some of the literature on these classes: [44, 5.2.1], and also [7, 8, 9, 14, 22, 23, 24, 35, 43, 48, 56, 76, 87, 93]. In the next section we will discuss in detail more properties of $\mathcal{S}_{(\omega)}(\mathbb{R})$ on which we rely.

1.6 Surjectivity of the Borel map in $\mathcal{S}_{(\omega)}(\mathbb{R})$

After Borel's classical work and results such as Theorem 1.4.1, many authors have further investigated conditions on the weight ω in the Beurling-Braun-Meise-Taylor setting (or on the weight sequence $(M_p)_p$) and on a sequence $(a_p)_{p\in\mathbb{N}_0}$ to ensure the existence of a function f in the class associated with ω (or with $(M_p)_p$ resp.) such that

$$f^{(p)}(0) = a_p,$$

for all $p \in \mathbb{N}_0$. See for instance [28, 30, 82, 90] and the references therein.

We define the following Köthe spaces (see [83, 27 Sequences spaces]):

$$\mathcal{E}_{(\omega)}(\{0\}) = \left\{ (x_j)_j \in \mathbb{C}^{\mathbb{N}_0} : \sup_j |x_j| \exp\left(-k\varphi^*_\omega(\frac{j}{k})\right) < \infty \ \forall k > 0 \right\},$$
$$\Lambda_{(M_p)} = \left\{ (x_p)_p \in \mathbb{C}^{\mathbb{N}_0} : \sup_p \frac{|x_p|}{h^p M_p} < \infty \ \forall h > 0 \right\}.$$

The Borel map B, defined by $Bf = \left(f^{(j)}(0)\right)_{j\in\mathbb{N}_0}$, for each $f \in C^\infty(\mathbb{R})$, verifies that

$$B : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{E}_{(\omega)}(\{0\})$$
$$B : \mathcal{E}_{(M_p)}(\mathbb{R}) \to \Lambda_{(M_p)}.$$

In regard to the Carlemann classes $\mathcal{E}_{(M_p)}(\mathbb{R})$ of Beurling type, Petzsche [90] proved the following result:

Theorem 1.6.1 ([90, 1.1. Proposition, 3.4. Theorem]). *Let $(M_p)_p$ be a weight sequence. The following conditions are equivalent:*

1. *$(M_p)_p$ satisfies* (γ_1) $\sup_p \dfrac{m_p}{p}\sum_{j\geq p}\dfrac{1}{m_j}<\infty$, *where* $m_p=\dfrac{M_p}{M_{p-1}}$.
2. *the Borel map* $B:\mathcal{E}_{(M_p)}(\mathbb{R})\to\Lambda_{(M_p)}, f\mapsto \left(f^{(j)}(0)\right)_{j\in\mathbb{N}_0}$, *is surjective.*

For further reading on Carleman's classes $\mathcal{E}_*(\mathbb{R})$ see for instance [90, 99, 101, 100, 102, 106] and the references therein.

Later on, Meise and Taylor extended the result to the $\omega-$ultradifferentiable setting $\mathcal{E}_{(\omega)}(\mathbb{R})$ of Beurling type (see [82]). In a subsequent series of papers by Bonet, Meise and Taylor ([28, 30, 31]), the study of the range of the Borel map B is extended to the quasi-analytic setting and of Roumieu type for the non-quasi-analytic.

We will make use of a version of Theorem 1.6.1 for the ultra-differentiable setting $\mathcal{S}_{(\omega)}(\mathbb{R})$ of Beurling type, with ω being a non-quasianalytic weight in the sense of Braun-Meise-Taylor (Definition 1.4.7). Since there are bump functions in $\mathcal{E}_{(\omega)}(\mathbb{R})$ (i.e. $\mathcal{D}_{(\omega)}(\mathbb{R})\neq\emptyset$) and $\mathcal{E}_{(\omega)}(\mathbb{R})\cap\mathcal{D}_{(\omega)}(\mathbb{R})=\mathcal{S}_{(\omega)}(\mathbb{R})\cap\mathcal{D}_{(\omega)}(\mathbb{R})$, surjectivity of the Borel map B on $\mathcal{E}_{(\omega)}(\mathbb{R})$ suffices. Indeed, Meise and Taylor proved the following.

Theorem 1.6.2 ([82])**.** *The following conditions are equivalent:*

1. *The Borel map* $B:\mathcal{E}_{(\omega)}(\mathbb{R})\to\mathcal{E}_{(\omega)}(\{0\}), f\mapsto\left(f^{(j)}(0)\right)_{j\in\mathbb{N}_0}$ *is surjective.*
2. *There exist* $K>1$ *and* $t_0>0$ *such that*
$$\omega(Kt)\leq\frac{K}{2}\omega(t),$$
for all $t\geq t_0$.
3. *There exists* $C>0$ *such that*
$$(\varepsilon)\qquad \int_1^\infty\frac{\omega(yt)}{t^2}dt\leq C\,(1+\omega(y)),\tag{1.15}$$
for all $y>0$.

A weight ω is said to be a strong weight if it satisfies the condition (ε), and therefore any of the equivalent conditions of the previous theorem.

Theorem 1.6.3 ([82, 1.4. Corollary])**.** *Let ω be a strong weight. Then, there are $0 < \alpha < 1$ and $M > 0$ such that*

$$\frac{\omega(t)}{t^\alpha} \le M,$$

for all $t > 0$ large enough. In particular, $\mathcal{E}_{(\omega)}(\mathbb{R}) \subset \mathcal{E}_{(\omega_\alpha)}(\mathbb{R})$, where $\omega_\alpha(t) = t^\alpha$, for $t \ge 0$.

Condition (β) is equivalent to the existence of non-trivial functions with compact support on $\mathcal{S}_{(\omega)}(\mathbb{R})$, hence by [82], condition (ε) is equivalent to the surjectivity of the Borel map

$$B : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{E}_{(\omega)}(\{0\}), f \mapsto \left(f^{(j)}(0)\right)_{j\in\mathbb{N}_0},$$

where

$$\mathcal{E}_{(\omega)}(\{0\}) = \left\{(x_j)_j \in \mathbb{C}^{\mathbb{N}_0} : \sup_j |x_j| \exp(-k\varphi^*_\omega(\frac{j}{k})) < \infty \ \forall k > 0\right\}.$$

Now we are ready to make the connection between $\mathcal{S}_{(M_p)}(\mathbb{R})$ and $\mathcal{S}_{(\omega)}(\mathbb{R})$.

Condition (γ_1) implies (M3)': $\sum_p \frac{M_{p-1}}{M_p} < \infty$, which is equivalent to the fact that $\mathcal{S}_{(M_p)}(\mathbb{R})$ contains non trivial compactly supported functions. Moreover, from [90, Proposition 1.1], (γ_1) is equivalent to

$$\exists Q \in \mathbb{N} : \ \lim\inf_{j\to\infty} \frac{m_{Qj}}{m_j} > 1.$$

By ([29, Theorem 14]), $M(t) := \sup_p \log \frac{|t|^p}{M_p}, \ t \ne 0, M(0) = 1$, is a weight function and the following assertions hold:

For each $0 < h < 1$ there exist $k \in \mathbb{N}$ and $C > 0$ such that

$$\exp\left(k\varphi^*_M(p/k)\right) \le Ch^p M_p, \ p \in \mathbb{N}_0.$$

For each $k \in \mathbb{N}$ there exist $0 < h < 1$ and $D > 0$ such that

$$h^p M_p \le D \exp\left(k\varphi^*_M(p/k)\right), \ p \in \mathbb{N}_0.$$

Consequently,

$$\mathcal{S}_{(M_p)}(\mathbb{R}) = \mathcal{S}_{(\omega)}(\mathbb{R})$$

for $\omega = M$. From the surjectivity of the Borel map we conclude that $\omega = M$ is a strong weight. Finally, by [76, Proposition 3.6] there is $H \ge 1$ such that

$$2\omega(t) \le \omega(Ht) + H, \ t \ge 0. \tag{1.16}$$

Conversely, [29, Corollary 16] and [23, Section 3] imply that for any weight function ω satisfying condition (1.16) there exists a weight sequence $(M_p)_{p\in\mathbb{N}_0}$ such that $\mathcal{S}_{(\omega)}(\mathbb{R}) = \mathcal{S}_{(M_p)}(\mathbb{R})$.

1.7 Modulation spaces

Time-frequency analysis is a part of harmonic analysis that treats simultaneously and simmetrically time and frequency. Given a signal f, that is a square integrable function in $\mathbb{R}^d$, its Fourier transform $\hat{f}$ represents the distribution of frequencies in the signal, but it does not give information about when these frequencies occur. To overcome this difficulty, following an idea due to Gabor, we can consider a cut-off, or window function φ localized around the origin. Then, to analyze the frequencies near a fixed x, we evaluate the Fourier transform of $\overline{\varphi(t-x)}f(t)$, thus obtaining a function both of time and frequency

$$V_\varphi f(x,\omega) = \int_{\mathbb{R}^d} f(t)\overline{\varphi(t-x)}e^{-2\pi it\omega}\,dt,$$

which is called the short time Fourier transform (STFT) of f with respect to the window φ.

The signal f can be reconstructed from its STFT by the formula

$$f(t) = \int_{\mathbb{R}^{2d}} V_\varphi f(x,\omega)\varphi(t-x)e^{2\pi it\omega}\,dx\,d\omega$$

provided that $\|\varphi\|_{L^2} = 1$. Moreover we have the ortogonality relations

$$\langle V_{\varphi_1}f_1, V_{\varphi_2}f_2\rangle = \langle f_1, f_2\rangle\overline{\langle g_1, g_2\rangle}, \tag{1.17}$$

where the brackets stand for the inner product in L^2.

We will use brackets $\langle f, g\rangle$ to denote the extension to $\mathcal{S}'(\mathbb{R}^d)\times\mathcal{S}(\mathbb{R}^d)$ (or to $\mathcal{S}'_{(\omega)}(\mathbb{R}^d)\times\mathcal{S}_{(\omega)}(\mathbb{R}^d)$) of the inner product

$$\langle f, g\rangle = \int_{\mathbb{R}^d} f(t)\overline{g(t)}dt.$$

The modulation and translation operators are defined by $M_\omega f(t) = e^{2\pi\imath\omega t}f(t)$ and $T_x f(t) = f(t-x)$. For a non-zero $\varphi\in\mathcal{S}(\mathbb{R}^d)$ (resp. $\varphi\in\mathcal{S}_{(\omega)}(\mathbb{R}^d)$) and a tempered distribution $f\in\mathcal{S}'(\mathbb{R}^d)$ (resp. $f\in\mathcal{S}'_{(\omega)}(\mathbb{R}^d)$), the STFT of f with window φ is given by the continuous function

$$V_\varphi f(x,\omega) = \langle f, M_\omega T_x\varphi\rangle\,.$$

Modulation space norms are measures of the time frequency concentration of a function or a (ultra-)distribution. The modulation spaces are defined as follows: Given a non-zero $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $1 \leq p,\ q \leq \infty$, the space $M^{p,q}(\mathbb{R}^N)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $V_\varphi f$ belongs to the mixed Lebesgue space

$$L^{p,q}(\mathbb{R}^{2d}) := \{g : \mathbb{R}^{2d} \to \mathbb{C} \text{ measurable} : \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |g(x,y)|^p \, dx \right)^{q/p} dy < \infty\}.$$

$M^{p,q}(\mathbb{R}^N)$ is endowed with the norm

$$\|f\|_{M^{p,q}} := \|V_\varphi f\|_{L^{p,q}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_\varphi f(x,\omega)|^p \, dx \right)^{q/p} d\omega \right)^{1/q},$$

with obvious changes if p or q are ∞. When $p = q$, we simply write M^p.

$M^{p,q}(\mathbb{R}^d)$ is a Banach space, its definition is independent of the choice of φ and different windows give equivalent norms. Moreover, M^2 is exactly L^2. In the case $1 \leq p, q < \infty$, the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ is dense in $M^{p,q}(\mathbb{R}^d)$ and its dual can be identified with $M^{p',q'}(\mathbb{R}^d)$ where $\dfrac{1}{p} + \dfrac{1}{p'} = \dfrac{1}{q} + \dfrac{1}{q'} = 1$.

To consider more general spaces of functions or distributions as, for instance, weighted L^2 spaces and Sobolev spaces we consider weighted modulation spaces.

Let m and v be positive functions on $\mathbb{R}^N$, v is said to be submultiplicative if

$$v(x_1 + x_2) \leq v(x_1)v(x_2), \quad x_1, x_2 \in \mathbb{R}^N.$$

If v is submultiplicative, m is called v-moderate if for some constant $C > 0$,

$$m(x_1 + x_2) \leq Cm(x_1)v(x_2), \quad x_1, x_2 \in \mathbb{R}^N.$$

Important examples of submultiplicative weights are the functions

$$v_s(z) = (1 + |z|)^s \ \ (s \geq 0),\ v(z) = e^{a|z|},\ v(z) = e^{a\omega(z)} \ \ (a > 0),$$

when ω is a subadditive weight function. Here $|\cdot|$ denotes the euclidean nom in $\mathbb{R}^N$. The weights

$$m(z) = (1 + |z|)^s \ \ (s \in \mathbb{R})$$

are $v_{|s|}-$moderate and $m(z) = e^{a\omega(z)}$, $a \in \mathbb{R}$, is $e^{|a|\omega(z)}$-moderate.

Given a submultiplicative weight v, $\mathcal{M}_v(\mathbb{R}^N)$ stands for the set of all v-moderate positive functions λ on $\mathbb{R}^N$.

Let us denote by $\mathcal{P}(\mathbb{R}^{2d})$ the set of all continuous and positive functions m on $\mathbb{R}^{2d}$ such that m is polynomial moderate, that is, there are C_m and s_m such that

$$m(x_1 + x_2) \leq C_m m(x_1)(1 + |x_2|)^{s_m}, \quad x_1, x_2 \in \mathbb{R}^{2d}.$$

Given $m \in \mathcal{P}(\mathbb{R}^{2d}) \setminus \{0\}$, $1 \leq p, q \leq \infty$ and $\varphi \in \mathcal{S}(\mathbb{R}^N) \setminus \{0\}$, we denote by $M^{p,q}_m(\mathbb{R}^N)$ the space

$$M^{p,q}_m(\mathbb{R}^d) := \{f \in \mathcal{S}'(\mathbb{R}^d) : \, m\, V_\varphi f \in L^{p,q}(\mathbb{R}^{2d})\}$$

endowed with the norm

$$|\|f\|_{M^{p,q}_m} = \|m\, V_\varphi f\|_{L^{p,q}}.$$

Focusing our attention in the Hilbert cases, we mention that $M^2(\mathbb{R}^d)$ is exactly $L^2(\mathbb{R}^d)$. In the weighted case, if m is only time dependent, that is $m(x, \omega) = \lambda(x)$, then

$$M^2_m(\mathbb{R}^d) = L^2_m(\mathbb{R}^d) := \{f : \ \lambda\, f \in L^2(\mathbb{R}^d)\}$$

whereas for $m(x, \omega) = \lambda(\omega)$, the modulation space is

$$\{f \in \mathcal{S}'(\mathbb{R}^d) : \, \lambda \hat{f} \in L^2(\mathbb{R}^d)\}.$$

In particular, the Sobolev spaces $H^s(\mathbb{R}^d)$ may be obtained in this way.

Moreover,

$$\mathcal{S}(\mathbb{R}^d) = \bigcap_{s>0} M^\infty_{v_s}(\mathbb{R}^d).$$

Finally, to work with ultradistributions in $\mathcal{S}'_{(\omega)}(\mathbb{R}^d)$ we fix a non-zero function $\varphi \in \mathcal{S}_{(\omega)}(\mathbb{R}^d)$. Given a weight function ω, $a > 0$, $1 \leq p, q \leq \infty$ and a function $m : \mathbb{R}^{2N} \to [0, \infty[$ which is $e^{a\omega}$-moderate, the modulation space $M^{p,q}_m(\mathbb{R}^d)$ consists of all $f \in \mathcal{S}'_{(\omega)}(\mathbb{R}^d)$ such that $mV_\varphi f \in L^{p,q}(\mathbb{R}^{2d})$ with the norm

$$\|f\|_{M^{p,q}_m} = \|m\, V_\varphi f\|_{L^{p,q}}.$$

We will refer to them as modulation spaces of ultradistributions or ultramodulation spaces.

Again $M^{p,q}_m(\mathbb{R}^d)$ is a Banach space and its definition is independent on the choice of the window φ. For $p = q$ we simply write $M^p_m(\mathbb{R}^d)$. For $p = 2$, M^2_m is a Hilbert space.

The dual of $M^{p,q}_m(\mathbb{R}^d)$ for $1 \leq p, q < \infty$ can be identified with the modulation space $M^{p',q'}_{\frac{1}{m}}(\mathbb{R}^d)$ where $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$.

The ortogonality relations (1.17) extend to de dual pars $\left(M^{p,q}_m(\mathbb{R}^d), M^{p',q'}_{\frac{1}{m}}(\mathbb{R}^d)\right)$.

Given a subadditive weight ω, let us write $\nu_s(z) = e^{s\omega(z)}$, then

$$\bigcap_{s>0} M^{\infty}_{\nu_s}(\mathbb{R}) = S_\omega(\mathbb{R}).$$

We refer to [44, 62] for the necessary background on modulation spaces.

Chapter 2

Continuity of composition operators on Gelfand-Shilov classes

2.1 Introduction

The aim of this chapter is to start the investigation of composition operators on global classes of ultradifferentiable functions invariant under Fourier transform. We first prove that, as it happens with the Schwartz class, composition operators in this setting are never compact, at least under the assumption that ω is strongly non-quasianalytic. From this point, the behaviour of composition operators on Gelfand-Shilov spaces and on the Schwartz class is completely different, as follows from the results in section 3. Then, in Section 4, we give some sufficient conditions on a smooth function ψ to guarantee that the corresponding composition operator is well defined on a given Gelfand-Shilov class.

In regard to the Gelfand Shilov space $\Sigma_d(\mathbb{R})$, $d > 1$, a necessary condition for the composition operator C_ψ to leave the class invariant is the boundedness of ψ'. We will also see that it is possible to find the optimal index d' for which $C_\psi(\Sigma_d(\mathbb{R})) \subset \Sigma_{d'}(\mathbb{R})$ holds for any non-constant polynomial ψ (see Corollary 2.3.9 and Theorem 2.4.7).

This chapter consists mostly of the article:

-Ariza, H., Fernández, C., Galbis, A.: *Composition operators on Gelfand-Shilov classes* J. Math. Anal. Appl. **531**, 127869 (2024).

2.2 Compactness

We recall that a linear operator $T : E \to F$ between two Fréchet spaces is said to be compact if there is a 0-neighbourhood U in E such that $T(U)$ is a relatively compact subset of F. In [55] it is proved that composition operators in the Schwartz class are never compact. The same is true for Gelfand-Shilov classes, although the proof is very different from that of [55].

Lemma 2.2.1. *Let $h, \psi \in C^\infty(\mathbb{R})$, $n \in \mathbb{N}$ and $x_0 \in \mathbb{R}$ such that $h^{(k)}(\psi(x_0)) = 0$ whenever $k \neq n$. Then*

$$(h \circ \psi)^{(n)}(x_0) = h^{(n)}(\psi(x_0)) \left(\psi'(x_0)\right)^n .$$

Proof. According to Faà di Bruno's formula (1.4) we have

$$(h \circ \psi)^{(n)}(x_0) = \sum \frac{n!}{k_1! \dots k_n!} h^{(k)}(\psi(x_0)) \left(\frac{\psi'(x_0)}{1!}\right)^{k_1} \dots \left(\frac{\psi^{(n)}(x_0)}{n!}\right)^{k_n} \tag{2.1}$$

where the sum is extended over all $(k_1, \dots, k_n)$ such that $k_1 + 2k_2 + \dots + nk_n = n$ and we denote $k := k_1 + \dots + k_n$. The only term in the summation that does not vanish is obtained when $k = n$, therefore, $k_1 = n,\ k_2 = \dots = k_n = 0$. From here the conclusion follows. □

Theorem 2.2.2. *Let ω be a strong weight and let us assume that $\psi \in C^\infty(\mathbb{R})$ satisfies $C_\psi(\mathcal{S}_{(\omega)}(\mathbb{R})) \subset \mathcal{S}_{(\omega)}(\mathbb{R})$. Then $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\omega)}(\mathbb{R})$ is not a compact operator.*

Proof. As in [55, Lemma 2.2], we have $\lim_{|x|\to\infty} \psi(x) = \infty$. Without loss of generality we can assume there is $x_0 \in \mathbb{R}$ such that $|\psi'(x_0)| > 1$ (on the contrary we consider $\sigma(x) = \psi(ax)$ for appropriate $a > 0$ and observe that the compactness of C_ψ is equivalent to that of C_σ). We now assume that C_ψ is compact and we fix a 0-neighbourhood U in $\mathcal{S}_{(\omega)}(\mathbb{R})$ with the property that $C_\psi(U)$ is relatively compact (hence bounded) in $\mathcal{S}_{(\omega)}(\mathbb{R})$. The map

$$B_{x_0} : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{E}_{(\omega)}(\{0\}), f \mapsto \left(f^{(j)}(\psi(x_0))\right)_{j\in\mathbb{N}_0},$$

being the composition of a translation with the Borel map, is continuous, linear and surjective. Hence it is open and $B_{x_0}(U)$ is a 0-neighbourhood in

$\mathcal{E}_{(\omega)}(\{0\})$. Consequently there are $p \in \mathbb{N}$ and $\varepsilon > 0$ such that the conditions $(a_j)_j \in \mathcal{E}_{(\omega)}(\{0\})$ and

$$\sup_j |a_j| \exp\left(-p\varphi^*_\omega(\frac{j}{p})\right) \leq \varepsilon$$

imply $(a_j)_j \in B_{x_0}(U)$. For every $n \in \mathbb{N}$ we consider $b_n = (b_n(j))_j \in \mathcal{E}_{(\omega)}(\{0\})$ defined by $b_n(n) = \varepsilon \exp(p\varphi^*_\omega(\frac{n}{p}))$ while $b_n(j) = 0$ for $j \neq n$, and we take $f_n \in U$ such that $B_{x_0}(f_n) = b_n$. In particular $f_n^{(k)}(\psi(x_0)) = 0$ whenever $k \neq n$. Since $C_\psi(U)$ is bounded in $\mathcal{S}_{(\omega)}(\mathbb{R})$ then $(f_n \circ \psi)_n$ is a bounded sequence in $\mathcal{S}_{(\omega)}(\mathbb{R})$, from where it follows

$$\sup_{n\in\mathbb{N}_0} \sup_{m\in\mathbb{N}_0} \left|(f_n \circ \psi)^{(m)}(x_0)\right| \exp\left(-p\varphi^*_\omega(\frac{m}{p})\right) < \infty.$$

From Lemma 2.2.1 we conclude

$$\begin{aligned}
\varepsilon \sup_{n\in\mathbb{N}} |\psi'(x_0)|^n &= \sup_{n\in\mathbb{N}} \left|f_n^{(n)}(\psi(x_0))\right| \cdot |\psi'(x_0)|^n \exp\left(-p\varphi^*_\omega(\frac{n}{p})\right) \\
&= \sup_{n\in\mathbb{N}} \left|(f_n \circ \psi)^{(n)}(x_0)\right| \exp\left(-p\varphi^*_\omega(\frac{n}{p})\right) < \infty,
\end{aligned}$$

which is a contradiction. □

Compare this result with what happens in the holomorphic case $H(\mathbb{D})$:

Theorem 2.2.3 ([10, Theorem 2.5.]). *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be holomorphic. The following assertions are equivalent:*

1. *$C_\varphi : H(\mathbb{D}) \to H(\mathbb{D})$ is a compact operator.*
2. $\sup_{z\in\mathbb{D}} |\varphi(z)| < 1$.

2.3 Negative results and necessary conditions

In this section we will prove that, unlike in the Schwartz class, the composition operators with polynomials do not in general leave the Gelfand-Shilov classes $\mathcal{S}_{(\omega)}(\mathbb{R})$ invariant. The construction of counterexamples depends on the surjectivity of the Borel map $B : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{E}_{(\omega)}(\{0\})$.

We will need some preliminaries.

Lemma 2.3.1. *Let* ω *be a weight and* $(I_m)_{m\in\mathbb{N}}$ *be a partition of* $\mathbb{N}_0$ *where* I_m *is a finite set for all* m*. We put* $a_j := \exp(m\varphi^*_\omega(\frac{j}{m}))$ *for all* $j \in I_m$*. Then* $(a_j)_{j\in\mathbb{N}_0} \in \mathcal{E}_{(\omega)}(\{0\})$*.*

Proof. We fix $k \in \mathbb{N}$. Since $m\varphi^*_\omega\left(\frac{j}{m}\right) \le k\varphi^*_\omega\left(\frac{j}{k}\right)$ for every $j \in \mathbb{N}_0$ and $m \ge k$, then we have

$$\sup_{j\in\mathbb{N}_0} |a_j| \exp(-k\varphi^*_\omega(\frac{j}{k})) = \max\{\sup_{1\le m\le k}\sup_{j\in I_m} \exp(m\varphi^*_\omega(\frac{j}{m}) - k\varphi^*_\omega(\frac{j}{k})), 1\} < +\infty.$$

$\square$

For every $x \in \mathbb{R}$ we consider the translation operator

$$T_x : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\omega)}(\mathbb{R}),\ (T_x f)(t) := f(t-x).$$

Lemma 2.3.2. *Let* $(x_j)_{j\in\mathbb{N}}$ *and* $(\lambda_j)_{j\in\mathbb{N}}$ *be two sequences of positive real numbers such that* $\lim_j \lambda_j = +\infty$*. We consider*

$$U_j : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\omega)}(\mathbb{R}),\ U_j f = \exp\left(-\lambda_j \omega(x_j)\right) T_{x_j}.$$

Then $\{U_j :\ j \in \mathbb{N}\}$ *is equicontinuous.*

Proof. We recall that $(\pi_{n,n})_{n\in\mathbb{N}}$ is a fundamental system of seminorms. From Definition 1.4.5(α) there exists $k \in \mathbb{N}$ such that

$$\omega(x) \le k\left(1 + \omega(x_j) + \omega(x - x_j)\right) \quad \forall x \in \mathbb{R}, j \in \mathbb{N},$$

(see for instance [24, Remark 2.2]). Let us fix $n \in \mathbb{N}$ and take $m = kn$. Then, for every $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$, we have

$$\begin{aligned}\pi_{n,n}(U_j f) &= \exp\left(-\lambda_j\omega(x_j)\right) \sup_{\ell\in\mathbb{N}_0}\sup_{x\in\mathbb{R}} |f^{(\ell)}(x - x_j)| e^{-n\varphi^*_\omega(\frac{\ell}{n}) + n\omega(x)} \\ &\le \exp\left(-\lambda_j\omega(x_j)\right) \pi_{m,m}(f) \exp\left(m + m\omega(x_j)\right) \\ &\le C_n\, \pi_{m,m}(f)\end{aligned}$$

for all $j \in \mathbb{N}$, where

$$C_n := \sup_{j\in\mathbb{N}} \exp\left((-\lambda_j + m)\omega(x_j) + m\right) < +\infty.$$

$\square$

The next result can be found in [33]. We include a proof for completeness.

Lemma 2.3.3. *Let* $L \in \mathbb{N}$ *satisfy* $\omega(et) \leq L(1 + \omega(t))$ *for all* $t \geq 0$. *For* $\lambda > 0, N \in \mathbb{N}$ *we take* $\mu = L^N \lambda$. *Then*

$$\mu\varphi^*_\omega(\frac{j}{\mu}) + Nj \leq \lambda\varphi^*_\omega(\frac{j}{\lambda}) + \lambda \sum_{k=1}^{N} L^k,\ j \in \mathbb{N}_0.$$

Proof. Take $N = 1$. Then

$$\begin{aligned} \mu\varphi^*_\omega(\frac{j}{\mu}) + j \quad &= \sup_{s\geq 0}\left(j(s+1) - \mu\omega(e^s)\right) \leq \sup_{s\geq 0}\big(js - \mu\omega(e^{s-1})\big) \\ &\leq \lambda L + \sup_{s\geq 0}\left(js - \lambda\omega(e^s)\right) = \lambda L + \lambda\varphi^*_\omega(\frac{j}{\lambda}). \end{aligned}$$

Now proceed by induction on N. □

It easily follows that, for every $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$ and $m \in \mathbb{N}$,

$$\sup_{j,k\in\mathbb{N}_0} \sup_{x\in\mathbb{R}} (1 + |x|)^k |f^{(j)}(x)| e^{-\lambda\varphi^*(\frac{j+k}{\lambda})} < +\infty.$$

Since $\mathcal{E}_{(\omega)}(\{0\})$ is a quotient of $\mathcal{S}_{(\omega)}(\mathbb{R})$, the next lemma follows from [23, Theorem 3] under the extra hypothesis that ω satisfies condition (1.16). See also [48].

Lemma 2.3.4. $\mathcal{E}_{(\omega)}(\{0\})$ *is a Fréchet nuclear space. In particular, every bounded set in* $\mathcal{E}_{(\omega)}(\{0\})$ *is relatively compact.*

Proof. We first observe that $\mathcal{E}_{(\omega)}(\{0\}) = \bigcap_{n=1}^{\infty} \ell^\infty(v_n)$ where

$$v_n(j) := \exp\left(-n\varphi^*_\omega(\frac{j}{n})\right).$$

As in Lemma 2.3.3, for every $m \in \mathbb{N}$ we take $\ell = Lm$. Then

$$\sum_{j=1}^{\infty} \frac{v_m(j)}{v_\ell(j)} \leq e^{mL} \sum_{j=1}^{\infty} e^{-j} < +\infty.$$

Now the conclusion follows from the Grothendieck-Pietsch criterion (see for instance [69, 21.6.2]). □

Our next result gives a condition on ψ' which implies that C_ψ does not leave invariant the Gelfand-Shilov class.

Theorem 2.3.5. *Let* $\psi \in C^\infty(\mathbb{R})$ *satisfy* $\lim_{x\to+\infty} \psi(x) = +\infty$ *and* $\psi'(x) \geq (\psi(x))^k$ *for some* $k > 0$ *and every* x *large enough. Let* ω *and* σ *be two strong weights such that* $\omega(t^{\frac{1}{k+1}}) = o(\sigma(t))$ *as* $t \to \infty$. *Then there exists* $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$ *such that* $f \circ \psi \notin \mathcal{S}_{(\sigma)}(\mathbb{R})$.

Proof. Proceeding by contradiction we assume that $f \circ \psi \in \mathcal{S}_{(\sigma)}(\mathbb{R})$ for every $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$. Let τ denote the compact-open topology on $\mathcal{S}_{(\sigma)}(\mathbb{R})$. Since $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \left(\mathcal{S}_{(\sigma)}(\mathbb{R}), \tau\right), \ f \mapsto f \circ \psi$, is continuous then its graph is closed in $\mathcal{S}_{(\omega)}(\mathbb{R}) \times \left(\mathcal{S}_{(\sigma)}(\mathbb{R}), \tau\right)$, hence in $\mathcal{S}_{(\omega)}(\mathbb{R}) \times \mathcal{S}_{(\sigma)}(\mathbb{R})$. By the closed graph theorem for Fréchet spaces,

$$C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\sigma)}(\mathbb{R})$$

is a continuous operator. We put $\tilde{\omega}(t) = \omega(t^{\frac{1}{k+1}})$ and take $L \in \mathbb{N}$ so that $\tilde{\omega}(et) \leq L(1 + \tilde{\omega}(t)), \ t \geq 0$. Since $\tilde{\omega} = o(\sigma)$ then for every $n \in \mathbb{N}$ there is $s_n \in \mathbb{N}$ such that

$$\varphi^*_\sigma(s) \leq 2Ln\varphi^*_{\tilde{\omega}}(\frac{s}{2Ln}) \ \ \forall s \geq s_n.$$

We select

$$j_n := \max\left(2^n, s_{n+1}\right), n \in \mathbb{N}_0, \quad I_m := \{j \in \mathbb{N} : \ j_{m-1} < j \leq j_m\}, m \in \mathbb{N}.$$

Now, for every $j \in I_m, \ m \in \mathbb{N}$, we put

$$\lambda_j := m, \quad a_j := \exp\left(\lambda_j \varphi^*_\omega(\frac{j}{\lambda_j})\right).$$

The set

$$\mathcal{C} := \{(a_j\delta_{j\ell})_{\ell\in\mathbb{N}_0} : \ j > j_0\}$$

is bounded, hence relatively compact, in $\mathcal{E}_{(\omega)}(\{0\})$. Since the Borel map $B : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{E}_{(\omega)}(\{0\})$ is surjective we can apply [83, 26.22] to find a relatively compact set $\mathcal{K} = \{g_j : \ j > j_0\} \subset \mathcal{S}_{(\omega)}(\mathbb{R})$ such that $\mathcal{C} = B(\mathcal{K})$. Then $\mathcal{K}$ is a bounded set and $g_j^{(\ell)}(0) = a_j$ for $\ell = j$ while $g_j^{(\ell)}(0) = 0$ for $\ell \neq j$. Now, for every $j > j_0$ we select $x_j > 0$ such that

$$kj \log x_j - \lambda_j \omega(x_j) = \lambda_j \varphi^*_\omega(\frac{kj}{\lambda_j}).$$

This selection is possible because the continuity of ω and Definition 1.4.7(γ) ensure that the supremum that appears in the definition of φ^*_ω is in fact a maximum. From the continuity of C_ψ and Lemma 2.3.2 we conclude that

$$\left\{\exp\left(-\lambda_j\omega(x_j)\right) C_\psi(T_{x_j}g_j) :\ j > j_0\right\}$$

is a bounded set in $\mathcal{S}_{(\sigma)}(\mathbb{R})$. In particular, for $h_j := C_\psi(T_{x_j}g_j)$, we have

$$\sup_{j>j_0}\sup_{y\in\mathbb{R}} \exp\left(-\lambda_j\omega(x_j)\right)\left|h_j^{(j)}(y)\right| e^{-\varphi^*_\sigma(j)} < +\infty. \tag{2.2}$$

Since $\lim\limits_{j\to+\infty} \dfrac{\lambda_j}{kj}\varphi^*_\omega(\dfrac{kj}{\lambda_j}) = +\infty$ and $\log x_j = \dfrac{\lambda_j}{kj}\varphi^*_\omega(\dfrac{kj}{\lambda_j}) + \dfrac{\lambda_j}{kj}\omega(x_j)$ then $\lim\limits_{j\to\infty} x_j = +\infty$ and (for j large enough, let say $j \geq J$) there is $y_j \in \mathbb{R}$ with $\psi(y_j) = x_j$. From (2.2) we obtain

$$\sup_{j>J} \exp\left(-\lambda_j\omega(x_j)\right)\left|h_j^{(j)}(y_j)\right| e^{-\varphi^*_\sigma(j)} < +\infty. \tag{2.3}$$

To finish we will prove that (2.3) does not hold. Indeed, by hypothesis and Lemma 2.2.1 we get for all j that

$$h_j^{(j)}(y_j) = g_j^{(j)}(0)\left(\psi'(y_j)\right)^j \geq a_j x_j^{kj} = \exp\left(\lambda_j\varphi^*_\omega(\frac{j}{\lambda_j}) + kj\log x_j\right).$$

Since φ^*_ω is a convex function and $\varphi^*_\omega((k+1)s) = \varphi^*_{\tilde\omega}(s)$ we get

$$\begin{aligned}\exp(-\lambda_j\omega(x_j))h_j^{(j)}(y_j) &\geq \exp\left(\lambda_j\varphi^*_\omega(\frac{j}{\lambda_j}) + \lambda_j\varphi^*_\omega(\frac{kj}{\lambda_j})\right)\\ &\geq \exp\left(2\lambda_j\varphi^*_\omega(\frac{(k+1)j}{2\lambda_j})\right) = \exp\left(2\lambda_j\varphi^*_{\tilde\omega}(\frac{j}{2\lambda_j})\right).\end{aligned}$$

From Lemma 2.3.3 we obtain

$$2\lambda_j\varphi^*_{\tilde\omega}(\frac{j}{2\lambda_j}) \geq 2L\lambda_j\varphi^*_{\tilde\omega}(\frac{j}{2L\lambda_j}) + j - 2\lambda_j L \geq \varphi^*_\sigma(j) + j - 2\lambda_j L.$$

The last inequality follows from the fact that for $j \in I_m$ we have $\lambda_j = m$ and $j > s_m$. Consequently

$$\exp\left(-\lambda_j\omega(x_j)\right)\left|h_j^{(j)}(y_j)\right| e^{-\varphi^*_\sigma(j)} \geq \exp(j - 2\lambda_j L),$$

which goes to infinity as $j \to \infty$. This is due to the fact that if $j \in I_m$ then $\log_2(j) > m - 1$ and so $\lambda_j < \log_2(j) + 1$ for all j. We have reached a contradiction with (2.3) and the proof is complete. □

We recall that if ω satisfies that there is $H \geq 1$ such that

$$2\omega(t) \leq \omega(Ht) + H, \ \forall t \geq 0, \tag{2.4}$$

then the corresponding $\mathcal{S}_{(\omega)}(\mathbb{R})$ can be described by means of a sequence weight $(M_p)_p$. Notice that power of logarithms are strong weights for which condition (2.4) does not hold.

Under some conditions on ω we have the following result:

Corollary 2.3.6. *Let $\psi \in C^\infty(\mathbb{R})$ satisfy $\lim_{x\to+\infty} \psi(x) = +\infty$ and $\psi'(x) \geq (\psi(x))^k$ for some $k > 0$ and every $x \in \mathbb{R}$ large enough. Let ω be a strong weight such that condition (2.4) is satisfied. Then there exists $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$ such that $f \circ \psi \notin \mathcal{S}_{(\omega)}(\mathbb{R})$.*

Proof. Take $s = k+1 > 1$. It suffices to show that $\omega(t^{\frac{1}{s}}) = o(\omega(t))$ as $t \to \infty$. In fact, for every $j \in \mathbb{N}$ we have

$$2^j\omega(t) \leq \omega(H^j t) + H\sum_{k=0}^{j-1} 2^k, \quad t \geq 0.$$

For every $t \geq 1$ there is $j \in \mathbb{N}_0$ with $H^j \leq t^{s-1} < H^{j+1}$. Hence

$$\omega(t^s) \geq \omega(tH^j) \geq 2^j\omega(t) - (2^j - 1)H,$$

from where it follows

$$\frac{\omega(t)}{\omega(t^s)} \leq 2^{-j} + \frac{H}{\omega(t^s)}.$$

The conclusion easily follows. ☐

Remark 2.3.7. The conclusion of Theorem 2.3.5 is still valid if ψ satisfies $\lim_{x\to+\infty} |\psi(x)| = +\infty$ and $|\psi'(x)| \geq c|\psi(x)|^k$ for some $c > 0$ and $x > 0$ large enough.

In fact, let us assume for instance that $\lim_{x\to+\infty} \psi(x) = -\infty$. Then $\psi'(x)$ is negative for x large enough. Take $a := c^{-1}$ and consider $\psi_a(x) := -\psi(ax)$. Then $\lim_{x\to+\infty} \psi_a(x) = +\infty$ and, for x large enough, $\psi_a'(x) = a|\psi'(ax)| \geq (\psi_a(x))^k$. Then there is $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$ such that $f \circ \psi_a \notin \mathcal{S}_{(\omega)}(\mathbb{R})$. Since $\mathcal{S}_{(\omega)}(\mathbb{R})$ is invariant under dilations and reflections, the conclusion follows. ☐

We recall that $\Sigma_d(\mathbb{R}) \subset \Sigma_{d'}(\mathbb{R})$ whenever $1 < d < d'$.

Corollary 2.3.8. *Let ψ be given as in Remark 2.3.7. Then for every $d \leq d' < (k+1)d$ there exists $f \in \Sigma_d(\mathbb{R})$ such that $f \circ \psi \notin \Sigma_{d'}(\mathbb{R})$.*

The following result quantifies the loss of regularity when we compose a function in $\Sigma_d(\mathbb{R})$ with a polynomial of degree greater than one.

Corollary 2.3.9. *Let ψ be a polynomial of degree $N > 1$. Then for every $d \leq d' < \frac{2N-1}{N}d$ there is $f \in \Sigma_d(\mathbb{R})$ such that $f \circ \psi \notin \Sigma_{d'}(\mathbb{R})$. In particular, for any polynomial ψ of degree greater than one and $d \leq d' < \frac{3}{2}d$ there is $f \in \Sigma_d(\mathbb{R})$ such that $f \circ \psi \notin \Sigma_{d'}(\mathbb{R})$.*

In fact, we can apply Corollary 2.3.8 with $k = \frac{N-1}{N}$. Observe that $k+1 \geq \frac{3}{2}$.

Remark 2.3.10. The *Gelfand-Shilov space* $\mathcal{S}_d(\mathbb{R})$ $(d > 1)$ of Roumieu type is the set of all smooth functions f such that

$$\sup\{\frac{|x^\ell f^{(j)}(x)|}{h^{\ell+j}(\ell!\,j!)^d} : x \in \mathbb{R}, \ell, j \in \mathbb{N}_0\}$$

is finite for some $h > 0$. Endowed with the natural inductive topology, it is a DFS space. It is immediate to see that $\mathcal{S}_d(\mathbb{R}) \subset \Sigma_{d'}(\mathbb{R})$ with continuous inclusion whenever $d < d'$. Hence, as a direct application of Corollary 2.3.9, the composition with polynomials of degree greater than one does not leave invariant the Gelfand-Shilov spaces of Roumieu type. In fact, if ψ is a polynomial of degree $N > 1$, and $d \leq d' < \frac{2N-1}{N}d$ there is $f \in \mathcal{S}_d(\mathbb{R})$ such that $f \circ \psi \notin \mathcal{S}_{d'}(\mathbb{R})$.

For weights of Gevrey type Corollary 2.3.6 can be improved.

Theorem 2.3.11. *Let $d > 1$ and $\psi \in C^\infty(\mathbb{R})$ be given such that $C_\psi(\Sigma_d) \subset \Sigma_d$. Then ψ' is bounded.*

Proof. We recall that $\Sigma_d = \mathcal{S}_{(\omega)}(\mathbb{R})$ for $\omega(t) = t^{\frac{1}{d}}$. An easy calculation gives

$$\varphi^*_\omega(x) = xd \log\left(\frac{xd}{e}\right). \tag{2.5}$$

For every $m \in \mathbb{N}$ and $j \in \mathbb{N}_0$ we put $x_{m,j} = \left(\frac{jd}{m}\right)^d$. Proceeding by contradiction, let us assume that ψ' is unbounded. Then we can find a sequence $(y_m)_m$

such that $\lim_{m\to\infty} |y_m| = +\infty$ and $|\psi'(y_m)| \geq 2^{md}$. Without loss of generality we can assume $\psi(y_m) > 0$. For every $m \in \mathbb{N}$ we put $x_m := \psi(y_m)$ and select $j(m) \in \mathbb{N}$ such that

$$x_{m,j(m)} \leq x_m < x_{m,j(m)+1},$$

or equivalently $j(m) \leq \frac{m}{d}\psi(y_m)^{\frac{1}{d}} < j(m)+1$, from where it follows $\lim_{m\to\infty} j(m) = +\infty$. Proceeding as in Theorem 2.3.5 we can find a bounded sequence $(g_m)_m$ in Σ_d such that

$$g_m^{(j(m))}(0) = \exp\left(m\varphi_\omega^*(\frac{j(m)}{m})\right)$$

while $g_m^{(\ell)}(0) = 0$ for any $\ell \neq j(m)$. From the continuity of C_ψ and Lemma 2.3.2 we conclude that

$$\{\exp\left(-m\omega(x_m)\right) C_\psi(T_{x_m} g_m) : \ m \in \mathbb{N}\}$$

is a bounded set in Σ_d. In particular, for $h_m := C_\psi(T_{x_m} g_m)$, we have

$$\sup_{m\in\mathbb{N}} \exp\left(-m\omega(x_m)\right) \left|h_m^{(j(m))}(y_m)\right| e^{-\varphi_\omega^*(j(m))} < +\infty. \tag{2.6}$$

By Lemma 2.2.1 we get

$$\begin{aligned}|h_m^{(j(m))}(y_m)| &= \left(T_{x_m} g_m\right)^{(j(m))}(x_m) \left|(\psi'(y_m))^{j(m)}\right| \\ &\geq \exp\left(m\varphi_\omega^*(\frac{j(m)}{m})\right) 2^{mdj(m)},\end{aligned}$$

hence, by (2.6),

$$\sup_{m\in\mathbb{N}} 2^{mj(m)d} \exp\left(m\varphi_\omega^*(\frac{j(m)}{m}) - m\omega(x_{m,j(m)+1}) - \varphi_\omega^*(j(m))\right) < +\infty.$$

From (2.5) we conclude

$$\sup_{m\in\mathbb{N}} \left(\frac{2^m}{me}\right)^{j(m)\,d} < +\infty,$$

which clearly is a contradiction. □

Our next result is an immediate consequence of the previous one plus Liouville's theorem and can be considered as a version of Beurling-Helson theorem (see for instance [86]).

Corollary 2.3.12. *Let* ψ *be an entire function with* $\psi(\mathbb{R}) \subset \mathbb{R}$. *If* $C_\psi(\Sigma_d(\mathbb{R})) \subset \Sigma_d(\mathbb{R})$, *then* ψ *is a polynomial of degree 1.*

The next result shows that condition (a) in Proposition 2.4.1 is reasonable. Its proof follows the steps of [55, Theoren 2.3] and is due to V. Asensio.

Proposition 2.3.13. *Given two weights* σ, ω *and* $\psi \in C^\infty(\mathbb{R})$ *such that* $C_\psi(\mathcal{S}_{(\omega)}(\mathbb{R})) \subset \mathcal{S}_{(\sigma)}(\mathbb{R})$, *there exists* C *such that* $\sigma(x) \leq C(1 + \omega(\psi(x)))$ *for every* $x \in \mathbb{R}$.

Proof. It suffices to prove that $\sigma(x) \leq C\omega(\psi(x))$ when x is big enough. Proceeding by contradiction we suppose that there exists $x_n \to +\infty$ such that $\sigma(x_n) > n\omega(\psi(x_n))$. Passing to a subsequence if necessary we may assume $|\psi(x_n)| + 1 < |\psi(x_{n+1})|$. We take $\rho \in \mathcal{S}_{(\omega)}(\mathbb{R})$ with support contained in $[-\frac{1}{2}, \frac{1}{2}]$ and $\rho(0) = 1$ and define

$$f(x) = \sum_{n=1}^{\infty} \frac{\rho(x - \psi(x_n))}{\exp(n\omega((\psi(x_n)))}.$$

Clearly $f \in C^\infty(\mathbb{R})$. We see that $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$. Since f vanishes outside of the union of the intervals $[\psi(x_n) - \frac{1}{2}, \psi(x_n) + \frac{1}{2}]$ we fix $m, \ell \in \mathbb{N}$ and take $x \in [\psi(x_n) - \frac{1}{2}, \psi(x_n) + \frac{1}{2}]$ with $n > Km$, K as in Definition 1.4.7(α). Then

$$\begin{aligned} |f^{(j)}(x)|e^{m\omega(x) - \ell\varphi^*_\omega(\frac{j}{\ell})} &\leq \frac{|\rho^{(j)}(x - \psi(x_n))|e^{Km\omega(x - \psi(x_n)) - \ell\varphi^*_\omega(\frac{j}{\ell})}}{e^{n\omega(\psi(x_n))}} e^{Km(1 + \omega(\psi(x_n)))} \\ &\leq \pi_{\ell, Km}(\rho) e^{Km} e^{(Km - n)\omega(\psi(x_n))} \\ &\leq \pi_{\ell, Km}(\rho) e^{Km}. \end{aligned}$$

On the other hand,

$$(f \circ \psi)(x_n) e^{m\sigma(x_n)} = e^{m\sigma(x_n) - n\omega(\psi(x_n))} > e^{(m-1)\sigma(x_n)}$$

which is unbounded since $\lim_n |x_n| = \infty$. □

The next corollary gives necessary conditions for weights ω that either satisfy condition (2.4) or are a power of logarithm:

Corollary 2.3.14. *Let $\psi \in C^\infty(\mathbb{R})$ be given and assume that $C_\psi(\mathcal{S}_{(\omega)}(\mathbb{R})) \subset \mathcal{S}_{(\omega)}(\mathbb{R})$.*

(a) If ω satisfies condition (2.4) then $|x| \leq C(1+|\psi(x)|)$ for some $C>0$.

(b) If $\omega(t) = \max(0, \log^s x)$, $s>1$, then there exist $C, a > 0$ such that $|x| \leq C(1+|\psi(x)|)^a$.

Proof. Only (a) needs a proof. We already know that $\lim\limits_{|x|\to\infty} |\psi(x)| = +\infty$. From Proposition 2.3.13 there is K such that $\omega(x) < K\omega(\psi(x))$ for $|x| > K$. Take $\ell \in \mathbb{N}$ so that $K \leq 2^\ell$ and use (1.16) ℓ times to find $C > 0$ with $K\omega(\psi(x)) \leq \omega(C\psi(x))$ for $|x| > C$. From $\omega(|x|) < \omega(C|\psi(x)|)$ for all $|x| > C$ and the fact that $\omega : [0,\infty) \to [0,\infty)$ is increasing we conclude $|x| \leq C|\psi(x)|$ for $|x| > C$.

□

2.4 Some sufficient conditions

In this section, we present some conditions on $\psi \in C^\infty(\mathbb{R})$, ω and σ, that are sufficient to guarantee that

$$C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\sigma)}(\mathbb{R}), f \mapsto f \circ \psi,$$

is a well-defined continuous operator.

Proposition 2.4.1. *Let ω be a subadditive weight, $a \geq 1$, and $\sigma(t) = \omega(t^{\frac{1}{a}})$. We assume*

(a) $|x| \leq C_0(1+|\psi(x)|)^a$ for every $x \in \mathbb{R}$.

(b) For every $m \in \mathbb{N}$ there is $C_m > 0$ such that

$$|\psi^{(j)}(x)| \leq C_m \exp(m\varphi^*_\sigma(\frac{j}{m}))\,(1+|\psi(x)|)^p \ \forall j \in \mathbb{N}\ \forall x \in \mathbb{R},$$

where $p = a-1$.

Then $C_\psi\left(\mathcal{S}_{(\omega)}(\mathbb{R})\right) \subset \mathcal{S}_{(\sigma)}(\mathbb{R})$.

Proof. It holds that $\varphi^*_\sigma(s) = \varphi^*_\omega(as),\ s \geq 0$. By Remark 1.4.14, σ is subadditive. Proceeding as in [52, Page 403], for each $m \in \mathbb{N}$ there is $B_m > 0$ such that

$$\prod_{\ell=1}^{j} \left| \frac{\psi^{(\ell)}(x)}{\ell!} \right|^{k_\ell} \leq G_m(j,k,x) := B_m^k \left(1 + |\psi(x)|\right)^{kp} \frac{\exp(m\varphi^*_\sigma(\frac{j-k}{m}))}{(j-k)!},$$

where $\sum_{\ell=1}^{j} \ell k_\ell = j$ and $k = \sum_{\ell=1}^{j} k_\ell$.

As usual, we denote $I_j := \left\{ \boldsymbol{k} = (k_1, k_2, \ldots, k_j) : \sum_{\ell=1}^{j} \ell k_\ell = j \right\}$. We now fix $m \in \mathbb{N}$ and take $M \in \mathbb{N}$ and $D > 0$ satisfying

$$C_0^q B_m^k \exp(M\varphi^*_\sigma(\frac{k+q}{M})) \leq D \exp(m\varphi^*_\sigma(\frac{k+q}{m}))$$

for every $k, q \in \mathbb{N}_0$ (Lemma 2.3.3). Then, there is $A_m > 0$ such that

$$\begin{aligned}
|x^q (f\circ\psi)^{(j)}(x)| &\leq C_0^q \left(1 + |\psi(x)|\right)^{aq} \sum_{\boldsymbol{k}\in I_j} \frac{j!}{k_1!\ldots k_j!} |f^{(k)}(\psi(x))| G_M(j,k,x) \\
&\leq A_m C_0^q B_m^k \sum_{\boldsymbol{k}\in I_j} \frac{j!}{k_1!\ldots k_j!} e^{\left(M\varphi^*_\omega(\frac{a(k+q)}{M})\right)} \cdot \frac{\exp(m\varphi^*_\sigma(\frac{j-k}{m}))}{(j-k)!} \\
&\leq A_m \sum_{\boldsymbol{k}\in I_j} \frac{j!}{k_1!\ldots k_j!} \frac{\exp\left(m\varphi^*_\sigma(\frac{j+q}{m})\right)}{(j-k)!} \\
&\leq A_m 4^j \exp\left(m\varphi^*_\sigma(\frac{j+q}{m})\right),
\end{aligned}$$

for all $x \in \mathbb{R}$, $q \in \mathbb{N}_0$ and $j \in \mathbb{N}$. In the last inequality we used the combinatorial identity (1.5). For every $\ell \in \mathbb{N}$ we apply Lemma 2.3.3 to find $m \in \mathbb{N}$ and $D_1 > 0$ such that

$$4^j \exp\left(m\varphi^*_\sigma(\frac{j+q}{m})\right) \leq D_1 \exp\left(\ell\varphi^*_\sigma(\frac{j+q}{\ell})\right)$$

for all $j, q \in \mathbb{N}_0$. Then

$$\sup_{x\in\mathbb{R}} \sup_{q,j\in\mathbb{N}_0} \left|x^q (f\circ\psi)^{(j)}(x)\right| \exp\left(-\ell\varphi_\sigma^*(\frac{j+q}{\ell})\right) < +\infty$$

and the proof is complete. □

Remark 2.4.2. *Under the hypotheses of Proposition 2.4.1, by Theorem 1.4.17, we conclude that the composition operator $C_\psi : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is also continuous.*

Corollary 2.4.3. *Let ω be a subadditive weight and let us assume that $\psi \in C^\infty(\mathbb{R})$ satisfies*

(a) $|x| \le C_0(1 + |\psi(x)|)$ *for every* $x \in \mathbb{R}$.

(b) For every $m \in \mathbb{N}$ there is $C_m > 0$ such that

$$\left|\psi^{(j)}(x)\right| \le C_m \exp\left(m\varphi_\omega^*(\frac{j}{m})\right) \ \forall j \in \mathbb{N}\ \forall x \in \mathbb{R}.$$

Then $C_\psi\left(\mathcal{S}_{(\omega)}(\mathbb{R})\right) \subset \mathcal{S}_{(\omega)}(\mathbb{R})$.

Condition (b) is related to the following.

Definition 2.4.4 ([18])**.** *Let ω be a weight. Then $\mathcal{B}_{\infty,\omega}$ consists of those functions f such that for every $m \in \mathbb{N}$ there exists C_m satisfying*

$$\left|f^{(j)}(x)\right| \le C_m \exp\left(m\varphi_\omega^*(\frac{j}{m})\right) \tag{2.7}$$

for all $j \in \mathbb{N}_0$ and $x \in \mathbb{R}$.

Remark 2.4.5. If in Proposition 2.4.1 we do not require that ψ satisfies inequality (a) then we still conclude that $C_\psi\left(\mathcal{S}_{(\omega)}(\mathbb{R})\right) \subset \mathcal{B}_{\infty,\sigma}$.

The following result means that the class $\mathcal{S}_{(\omega)}(\mathbb{R})$ is stable under composition with functions that are appropriate perturbations of a multiple of the identity.

Corollary 2.4.6. *Let ω be a subadditive weight and let us assume that $\psi(x) = Ax + \lambda(x)$ for some $A \neq 0$ and $\lambda \in \mathcal{B}_{\infty,\omega}$. Then $C_\psi\left(\mathcal{S}_{(\omega)}(\mathbb{R})\right) \subset \mathcal{S}_{(\omega)}(\mathbb{R})$.*

Theorem 2.4.7. *Let ω be a subadditive weight, $\sigma(t)=\omega(t^{\frac{1}{2}})$ and ψ a non constant polynomial. Then $f\circ\psi\in\mathcal{S}_{(\sigma)}(\mathbb{R})$ for every $f\in\mathcal{S}_{(\omega)}(\mathbb{R})$.*

Proof. Let N be the degree of the polynomial ψ. The result is trivial if $N=1$, hence we assume that $N\geq 2$. Then condition (b) in Proposition 2.4.1 holds with $p=\frac{N-1}{N}$ hence it suffices to take $a=\frac{2N-1}{N}<2$. $\square$

Corollary 2.3.9 means that Theorem 2.4.7 is, in some sense, optimal.

Proposition 2.4.8. Given a weight ω, not necessarily subadditive, the inclusion

$$C_\psi(\mathcal{S}_{(\omega)}(\mathbb{R}))\subset\mathcal{S}_{(\sigma)}(\mathbb{R})$$

holds for $\sigma(t)=\omega(t^{\frac{1}{3}})$ and ψ a non constant polynomial.

Proof. In fact, we can proceed as in Proposition 2.4.1, but instead of applying [52, Page 403] and (1.5) we use the identity

$$\sum_{\boldsymbol{k}\in I_j}\frac{j!}{k_1!\ldots k_j!}=\sum_{k=1}^{j}\binom{j-1}{k-1}\frac{j!}{k!}, \tag{2.8}$$

which follows from the main theorem in [47].

It holds that $\varphi^*_\sigma(s)=\varphi^*_\omega(3s),\ s\geq 0$. As usual, we denote

$$I_j:=\left\{\boldsymbol{k}=(k_1,k_2,\ldots,k_j):\sum_{\ell=1}^{j}\ell k_\ell=j\right\}.$$

We now fix $m\in\mathbb{N}$ and take $M\in\mathbb{N}$ and $D_0>0$ satisfying

$$j!\,C_0^q\,2^{j-1}\exp(M\varphi^*_\omega(\frac{j+q}{M}))\leq D_0\exp(m\varphi^*_\omega(\frac{2j+q}{m}))$$

for every $k,q\in\mathbb{N}_0$ (Lemma 2.3.3). Obviously, there is $C_0>0$ such that $|x|\leq C_0(1+|\psi(x)|)$, for all $x\in\mathbb{R}$. Then, there is $A_m,B_m>0$ such that

$|x^q\,(f\circ\psi)^{(j)}(x)|$ is less or equal to

$$B_m^j\,\exp\left(m\varphi_\omega^*(\frac{j}{m})\right)\,C_0^q\,(1+|\psi(x)|)^q\sum_{\boldsymbol{k}\in I_j}\frac{j!}{k_1!\ldots k_j!}|f^{(k)}(\psi(x))|$$

$$\le B_m^j\,\exp\left(m\varphi_\omega^*(\frac{j}{m})\right)\,A_m\,C_0^q\sum_{\boldsymbol{k}\in I_j}\frac{j!}{k_1!\ldots k_j!}\,\exp\left(M\varphi_\omega^*(\frac{j+q}{M})\right)$$

$$= B_m^j\,j!\,\exp\left(m\varphi_\omega^*(\frac{j}{m})\right)\,A_m\,C_0^q\,\exp\left(M\varphi_\omega^*(\frac{j+q}{M})\right)\sum_{k=1}^{j}\binom{j-1}{k-1}\frac{1}{k!}$$

$$\le B_m^j\,j!\,\exp\left(m\varphi_\omega^*(\frac{j}{m})\right)\,A_m\,C_0^q\,\exp\left(M\varphi_\omega^*(\frac{j+q}{M})\right)\,2^{j-1}\,e$$

$$\le A_m\,B_m^j\,D_0\,e\,2^{j-1}\,\exp\left(m\varphi_\omega^*(\frac{3j+q}{m})\right)$$

for all $x\in\mathbb{R}$, $q\in\mathbb{N}_0$ and $j\in\mathbb{N}$. In the last inequality we used the combinatorial identity (2.8). For every $\ell\in\mathbb{N}$ we apply Lemma 2.3.3 to find $m\in\mathbb{N}$ and $D_1>0$ such that

$$B_m^j\,2^{j-1}\,\exp\left(m\varphi_\sigma^*(\frac{j+q}{m})\right)\le D_1\,\exp\left(\ell\varphi_\sigma^*(\frac{j+q}{\ell})\right)$$

for all $j,q\in\mathbb{N}_0$. Then

$$\sup_{x\in\mathbb{R}}\sup_{q,j\in\mathbb{N}_0}\left|x^q(f\circ\psi)^{(j)}(x)\right|\,\exp\left(-\ell\varphi_\sigma^*(\frac{j+q}{\ell})\right)<+\infty$$

and the proof is complete. □

There are non-linear examples of symbols for $\mathcal{S}_\omega$, at least when ω is subadditive:

Example 2.4.9. *Let* $\psi(x)=\sqrt{1+x^2}$*. Then* $C_\psi\left(\mathcal{S}_{(\omega)}(\mathbb{R})\right)\subset\mathcal{S}_{(\omega)}(\mathbb{R})$ *for every subadditive weight* ω*.*

Proof. It is evident that $\psi'(x)=\frac{x}{(1+x^2)^{\frac{1}{2}}}$ and that $|\psi(x)|\ge|x|$ for all $x\in\mathbb{R}$. Set $p_0(x):=1$ for all $x\in\mathbb{R}$. Inductively, we set for each $j\ge 1$ the function p_j in the following way:

$$p_j(x):=(1+x^2)p'_{j-1}(x)-(2j+1)x\,p_{j-1}(x)$$

for $x \in \mathbb{R}$. Clearly, p_j is a polynomial of degree j. In general, if $a_h^{(\ell)}$ is the $h-$th coefficient of $p_\ell(x) = a_\ell x^\ell + ... + a_h x^h + ... + a_1 x + a_0$ the following identity holds:

$$a_q^{(j)} = (q - 2(j+1))\, a_{q-1}^{(j-1)} + (q+1) a_{q+1}^{(j-1)} \tag{2.9}$$

for all $q \in \mathbb{N}_0$ and $j \in \mathbb{N}$. We set $a_h^{(\ell)} = 0$ whenever $h < 0$. It is obvious that $a_h^{(\ell)} = 0$ whenever $h > \ell$. By induction, we check that for all $j \geq 2$ it holds that

$$\psi^{(j)}(x) = \frac{p_{j-2}(x)}{(1+x^2)^{\frac{2j-1}{2}}}$$

for all $x \in \mathbb{R}$. Observe that $a_j^{(j)} = (j-1)\, a_{j-1}^{(j-1)} - (2j+1)\, a_{j-1}^{(j-1)} = -(j+2)\, a_{j-1}^{(j-1)}$, for all $j \geq 2$, and hence, $|a_j^{(j)}| = \frac{(j+2)!}{2}$, for all $j \in \mathbb{N}$. We now need some estimates for the other coefficients of the polynomial p_j. From (2.9) we obtain that

$$\begin{aligned} |a_q^{(j)}| &\leq 2(j+1) \max\{|a_{q-1}^{(j-1)}|, |a_{q+1}^{(j-1)}|\} \\ &\leq 2(j+1) \max\{|a_h^{(j-1)}| : 0 \leq h \leq j-1\} \\ &\leq 2^2 (j+1) j \max\{|a_h^{(j-2)}| : 0 \leq h \leq j-2\} \\ &\vdots \\ &\leq 2^j (j+1)!\, |a_0^{(0)}| = 2^j (j+1)! \end{aligned}$$

for all $j, q \in \mathbb{N} \cup \{0\}$. It clearly holds that

$$\begin{aligned} |\psi^{(j)}(x)| &\leq \frac{2^{j-2}(j-1)!\,(j-1)\,|x|^{j-2}}{(1+x^2)^{\frac{2j-1}{2}}} \leq \frac{2^{j-2}\, j!\, |x|^{j-2}}{(1+x^2)^{\frac{2j-1}{2}}} \\ &\leq 2^{j-2}\, j!\, |x|^{-(j+1)} \leq 2^{j-2}\, j! \end{aligned} \tag{2.10}$$

for all $|x| \geq 1,\ j \in \mathbb{N}$ and also that

$$|\psi^{(j)}(x)| \leq \frac{2^{j-2}(j-1)!\,(j-1)}{(1+x^2)^{\frac{2j-1}{2}}} \leq 2^{j-2}\, j! \tag{2.11}$$

for all $|x| \leq 1,\ j \in \mathbb{N}$. We have just proved that

$$|\psi^{(j)}(x)| \leq 2^{j-2}\, j!$$

for all $x \in \mathbb{R}$ and $j \in \mathbb{N}$. As usual, by Lemma 1.4.11, we conclude that for each $m \in \mathbb{N}$ there is $D_m > 0$ such that

$$|\psi^{(j)}(x)| \leq D_m \exp\left(m\varphi^*_\omega(\frac{j}{m})\right)$$

for all $x \in \mathbb{R}$ and $j \in \mathbb{N}$. Using Corollary 2.4.3 we derive that the composition operator $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\omega)}(\mathbb{R})$ is continuous, as we wanted to show. ☐

For the weights $\omega(t) = \max\{0, \log \mathrm{t}\}^{\mathrm{s}}$ $(s > 1)$ we have the inclusion $C_\psi\left(\mathcal{S}_{(\omega)}(\mathbb{R})\right) \subset \mathcal{S}_{(\omega)}(\mathbb{R})$ under mild assumptions.

Corollary 2.4.10. *Let* $\omega(t) := \max\{0, \log \mathrm{t}\}^{\mathrm{s}}$ $(s > 1)$ *and let us assume that* $\psi \in C^\infty(\mathbb{R})$ *satisfies for*

(a) $|x| \leq C_0 (1 + |\psi(x)|)^{a_1}$ *for every* $x \in \mathbb{R}$.

(b) For every $m \in \mathbb{N}$ *there is* $C_m > 0$ *such that*

$$\left|\psi^{(j)}(x)\right| \leq C_m \exp\left(m\varphi^*_\omega(\frac{j}{m})\right) (1 + |\psi(x)|)^{a_2}\ \forall j \in \mathbb{N}\ \forall x \in \mathbb{R},$$

for some $a_1,\ a_2 > 0$. *Then,* $C_\psi\left(\mathcal{S}_{(\omega)}(\mathbb{R})\right) \subset \mathcal{S}_{(\omega)}(\mathbb{R})$.

Proof. Take $a := \max(a_1, a_2+1)$ and $\sigma(t) = \omega(t^{\frac{1}{a}})$. Since $\varphi^*_\omega(s) = \varphi^*_\sigma(\frac{s}{a})$ then ψ satisfies (a) and (b) in Proposition 2.4.1 and hence, since ω is equivalent to a subadditive weight, $C_\psi\left(\mathcal{S}_{(\omega)}(\mathbb{R})\right) \subset \mathcal{S}_{(\sigma)}(\mathbb{R})$. But $\mathcal{S}_{(\sigma)}(\mathbb{R}) = \mathcal{S}_{(\omega)}(\mathbb{R})$ since $\sigma(t) = \frac{1}{a^s}\,\omega(t)$. ☐

Notice that polynomials ψ automatically verify condition (b).

For powers of logarithm we also have the following result:

Corollary 2.4.11. *Let* $\omega(t) := \max\{0, \log \mathrm{t}\}^{\mathrm{s}}$ $(s > 1)$ *and* $\psi \in C^\infty(\mathbb{R})$ *such that for every* $m \in \mathbb{N}$ *there is* $C_m > 0$ *such that*

$$\left|\psi^{(j)}(x)\right| \leq C_m \exp\left(m\varphi^*_\omega(\frac{j}{m})\right) (1 + |\psi(x)|)\ \forall j \in \mathbb{N}\ \forall x \in \mathbb{R}.$$

If $\varphi(x) := \exp(\psi(x)) \geq |x|^d$ *for some* $d > 0$, *then* $C_\varphi\left(\mathcal{S}_{(\omega)}(\mathbb{R})\right) \subset \mathcal{S}_{(\omega)}(\mathbb{R})$.

Proof. We only have to verify condition (b) in the previous result. From the fact that $\psi(x) \geq 0$ for $|x| \geq 1$ we may find $C > 0$ such that $|\psi(x)| \leq C + \psi(x)$ for all $x \in \mathbb{R}$. Set $\sigma(t) = \omega(t^{1/2})$, for $t \geq 0$. By Faà di Bruno's formula (1.4), Lemma 1.4.11 and arguing as in the proof of Proposition 2.4.1 we obtain for every $m \in \mathbb{N}$ a constant $B_m > 0$ such that for every $x \in \mathbb{R}$,

$$\begin{aligned}
\left|\varphi^{(j)}(x)\right| &\leq \phi(x)\, e^{m\varphi^*_\omega(\frac{j}{m})}\, B_m^j \sum_{\boldsymbol{k}\in I_j} \frac{j!}{k_1!\ldots k_j!}\, \frac{(C+\psi(x))^k}{(j-k)!} \\
&\leq \phi(x)\, B_m^j\, e^{m\varphi^*_\omega(\frac{j}{m})} \exp(C+\psi(x)) \sum_{\boldsymbol{k}\in I_j} \frac{j!}{k_1!\ldots k_j!}\, \frac{k!}{(j-k)!} \\
&\leq \phi^2(x)\, e^{m\varphi^*_\omega(\frac{j}{m})}\, e^C j!\, B_m^j \sum_{\boldsymbol{k}\in I_j} \frac{j!}{k_1!\ldots k_j!}\, \frac{1}{(j-k)!} \\
&\leq \phi^2(x)\, e^{m\varphi^*_\omega(\frac{j}{m})}\, e^C\, (4B_m)^j\, j! \leq D_m\, \phi^2(x)\, e^{m\varphi^*_\sigma(\frac{j}{m})}
\end{aligned}$$

for some $D_m > 0$, where as usual $k = \sum_{\ell=1}^{j} k_\ell$ and

$$I_j := \left\{ \boldsymbol{k} = (k_1, k_2, \ldots, k_j) : \sum_{\ell=1}^{j} \ell k_\ell = j \right\}.$$

Now the conclusion follows from the fact that $\mathcal{S}_{(\sigma)}(\mathbb{R}) = \mathcal{S}_{(\omega)}(\mathbb{R})$. □

Example 2.4.12. Taking $\psi(x) = x^2$ we get that $\phi(x) = e^{x^2}$ defines a continuous composition operator on $\mathcal{S}_{(\omega)}(\mathbb{R})$ when $\omega(x) = \max\{0, \log \mathrm{t}\}^{\mathrm{s}}$ $(s > 1)$.

To provide examples of functions ψ satisfying the conditions of corollaries 2.4.3 and 2.4.10 the next two results are useful.

Corollary 2.4.13. *Let ψ be a holomorphic function on an open set containing the cone $C = \{z \in \mathbb{C} : |Imz| \leq L|Rez|\}$. Assume that $\psi(\mathbb{R}) \subset \mathbb{R}$ and that there is $A > 0$ such that*

(a) $|x| \leq A(1 + |\psi(x)|)$ for every $x \in \mathbb{R}$,

(b) $|\psi(z)| \leq A(1 + |z|)$ for every $z \in C$.

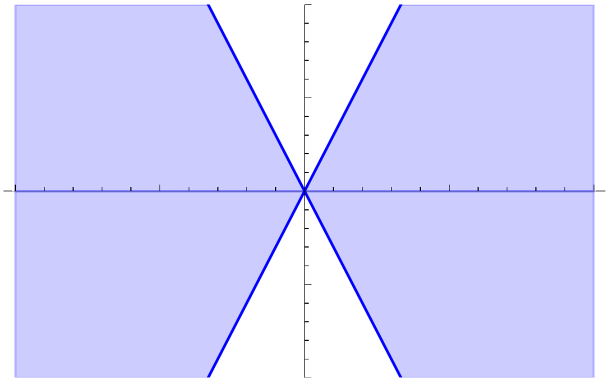

Figure 2.1: The set C.

Then, for every subadditive weight ω, $C_\psi\left(\mathcal{S}_{(\omega)}(\mathbb{R})\right) \subset \mathcal{S}_{(\omega)}(\mathbb{R})$.

Proof. There is $\delta > 0$ such that, for every $x \in \mathbb{R}$ with $|x| \geq 1$, the ball centered at x with radius $r_x := \delta|x|$ is contained in C. By the Cauchy inequalities

$$\left|\psi^{(j)}(x)\right| \leq j!\,\frac{1+|x|+r_x}{r_x^j},$$

for each $j \in \mathbb{N}$. Then, there is a constant $C > 0$ such that whenever $|x| \geq 1$ one has

$$|\psi^{(j)}(x)| \leq C\,j!\,\left(\frac{1}{\delta}\right)^{j-1}.$$

Since ψ is real analytic in $[-1,1]$ we conclude the existence of $B > 0$ such that

$$|\psi^{(j)}(x)| \leq j!\,B^{j+1} \text{ for each } x \in \mathbb{R}.$$

Therefore, using Lemma 1.4.11 we conclude that ψ fulfills (b) in Corollary 2.4.3. □

In the case of ω being a power of logarithm, the hypotheses of Corollary 2.4.13 may be relaxed:

Corollary 2.4.14. *Let* $\omega(t) := \max\{0, \log \mathrm{t}\}^{\mathrm{s}}$ *(*$s > 1$*) and let us assume that* $\psi \in C^\infty(\mathbb{R})$ *admits a holomorphic extension to the strip* $H := \{z \in \mathbb{C} : |Imz| < L\}$ *for some* $L > 0$ *and, for some* a_1, $a_2 > 0$,

(a) $|x| \leq C_0(1+|\psi(x)|)^{a_1}$ *for every* $x \in \mathbb{R}$.

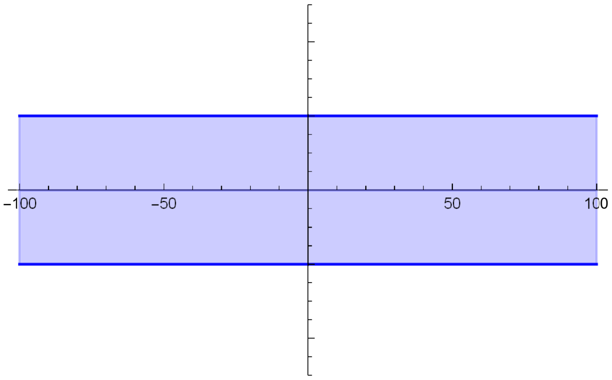

Figure 2.2: The set H.

(b) For every $m \in \mathbb{N}$ *there is* $C_m > 0$ *such that*

$$|\psi(z)| \leq C(1+|z|)^{a_2} \ \forall z \in H.$$

Then, $C_\psi\left(\mathcal{S}_{(\omega)}(\mathbb{R})\right) \subset \mathcal{S}_{(\omega)}(\mathbb{R})$.

Proof. It is enough to use the Cauchy inequalities to check that Corollary 2.4.10 applies. $\square$

Example 2.4.15. (a) Let $\psi(x) = \sqrt{1+x^2}$. Then $C_\psi\left(\mathcal{S}_{(\omega)}(\mathbb{R})\right) \subset \mathcal{S}_{(\omega)}(\mathbb{R})$ for every subadditive weight ω.

In fact, if $\log_0$ denotes the branch of the logarithm whose imaginary part takes values in $(-\pi,\pi)$, then $\psi(z) := \exp(\frac{1}{2}\log_0(1+z^2))$ is holomorphic in a neighbourhood of $\{z \in \mathbb{C} : |Imz| \leq |Rez|\}$, and clearly satisfies conditions (a) and (b) in Corollary 2.4.13.

(b) The functions $\psi(x) = (1+x^2)^a$, $a > 0$ or $\psi(x) = \frac{P(x)}{Q(x)}$, P and Q non-constant polynomials with $Q(x) \neq 0$ for every $x \in \mathbb{R}$ and P having greater degree than Q, satisfy the hypothesis in Corollary 2.4.14.

Proof. Example 2.4.15(a) is clear.

Example 2.4.15(b) follows from the fact that

$$\psi(z) := \exp\left(\frac{1}{2}\log_0\left((1+z^2)^a\right)\right)$$

is holomorphic also in a neighbourhood of $\{z \in \mathbb{C} : |\mathrm{Im}\ z| \leq \frac{1}{2}\}$, and clearly satisfies conditions (a) and (b) in Corollary 2.4.14. The rest is obvious.

$\square$

Chapter 3

Dynamics of composition operators on Gelfand-Shilov classes

3.1 Introduction

This chapter is an adaptation of the article:

-Ariza, H., Fernández, C., Galbis, A.: *Iterates of composition operators on global spaces of ultradifferentiable functions*. Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. 119, 9 (2025).

We continue the research started in the previous chapter where we initiated the study of the composition operators in Gelfand-Shilov spaces. In general, the Gelfand-Shilov classes are not composition invariant with a polynomial of degree greater than one (see for example Theorem 2.3.11), however for every weight function ω that is subadditive a new weight σ can be found so that $f \circ \psi \in \mathcal{S}_{(\sigma)}(\mathbb{R})$ whenever $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$ and ψ is a polynomial (see Theorem 2.4.7). Since the choice of σ does not depend on the polynomial ψ it makes sense to analyze the dynamics of the operator $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\sigma)}(\mathbb{R}), \quad f \mapsto f \circ \psi$.

We analyze the behaviour of the iterates of composition operators defined by polynomials acting on global classes of ultradifferentiable functions of Beurling type and being invariant under Fourier transform. We characterize the polynomials ψ for which the sequence of iterates is equicontinuous between two different Gelfand-Shilov spaces. For the particular case in which the weight ω is equivalent to a power of the logarithm, the result obtained characterizes the polynomials ψ for which the composition operator C_ψ is power bounded in $\mathcal{S}_{(\omega)}(\mathbb{R})$. Unlike the composition operators in the Schwartz class, the Waelbroek spectrum of an operator C_ψ, ψ being a polynomial of degree greater than one lacking fixed points is never compact. We focus on the problem of the convergence of Neumann series. We deduce the continuity of the resolvent operator between two different Gelfand-Shilov classes for polynomials ψ lacking fixed points. Concerning polynomials of second degree the most interesting case is the one in which the polynomial only has one fixed point: we provide some restrictions on the indices d, d' that are necessary for the resolvent operator to be continuous between the Gelfand-Shilov classes Σ_d and $\Sigma_{d'}$.

If $C_{\psi_m} : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\sigma)}(\mathbb{R})$, for all $m \in \mathbb{N}$, then there is no element $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$ such that $\left(C_{\psi_m} f\right)_m$ is dense in $\mathcal{S}_\sigma(\mathbb{R})$. This is so because $\left(C_{\psi_m} f\right)(\mathbb{R}) \subset f(\mathbb{R})$ for every $m \in \mathbb{N}$ and $f(\mathbb{R})$ is not dense in $\mathbb{R}$. Therefore, it makes no sense to study hypercyclicity in the context of Gelfand-Shilov spaces. As for broader concepts, such as supercyclicity, let us recall the following result.

Theorem 3.1.1 ([53, Theorem 2.1.]). Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a proper continuous function. Then

$$C_\varphi : C_0(\mathbb{R}) \to C_0(\mathbb{R})$$

is not supercyclic. If in addition X is a lcs, $X \hookrightarrow C_0(\mathbb{R})$ is continuously embedded with dense range and $C_\varphi(X) \subseteq X$, then $C_\varphi : X \to X$ is not supercyclic.

It follows from this that, in the case where the composition operator C_ψ maps $\mathcal{S}_{(\omega)}(\mathbb{R})$ into $\mathcal{S}_{(\omega)}(\mathbb{R})$, $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\omega)}(\mathbb{R})$ can never be supercyclic.

3.2 Some estimates on polynomials

The aim of this section is to obtain lower bounds for the iterations of a polynomial lacking fixed points (Proposition 3.2.4) and upper bounds for the derivatives of the iterates of such a polynomial (Proposition 3.2.5). First we need some auxiliary lemmas.

Lemma 3.2.1. *For every* $\boldsymbol{k} = (k_1, ..., k_n) \in \mathbb{N}_0^n$ *such that* $\sum_{\ell=1}^n \ell k_\ell = n$ *one has*

$$\prod_{\ell=1}^n \ell^{\ell k_\ell} \leq \frac{n^n}{n!} \prod_{\ell=1}^n \ell!^{k_\ell}.$$

Proof. Without loss of generality we can assume $k_\ell \geq 1$ for all $1 \leq \ell \leq n$. According to the multinomial theorem, for every $\alpha, \beta \in \mathbb{N}_0^q$ such that $|\alpha| = |\beta| = N$ we have

$$N^N = (\alpha_1 + \ldots + \alpha_q)^N \geq \frac{N!}{\beta_1! \ldots \beta_q!} \alpha_1^{\beta_1} \ldots \alpha_q^{\beta_q}.$$

After choosing $\beta = \alpha$ we obtain

$$\prod_{p=1}^q \frac{\alpha_p^{\alpha_p}}{\alpha_p!} \leq \frac{N^N}{N!}. \tag{3.1}$$

In particular, for $k_\ell \geq 1$ and $\alpha_p = \ell,\ 1 \leq p \leq k_\ell$,

$$\left(\frac{\ell^\ell}{\ell!}\right)^{k_\ell} \leq \frac{(\ell k_\ell)^{\ell k_\ell}}{(\ell k_\ell)!}. \tag{3.2}$$

Finally, after taking $\alpha_\ell = \ell k_\ell$ and $q = n$ in (3.1) we conclude

$$\prod_{\ell=1}^n \frac{(\ell k_\ell)^{\ell k_\ell}}{(\ell k_\ell)!} \leq \frac{n^n}{n!}. \tag{3.3}$$

The conclusion follows from (3.2) and (3.3). □

Lemma 3.2.2. *For every* $n \in \mathbb{N}$, *it holds that*

$$\sum_{k=0}^n \binom{n}{k}^{-1} = \frac{n+1}{2^n} \sum_{k=0}^n \frac{2^k}{k+1} \leq 3. \tag{3.4}$$

Proof. We denote $S_n := \sum_{k=0}^n \binom{n}{k}^{-1}$. From the relation between Gamma and

Beta functions we have

$$\begin{aligned}
S_n &= (n+1)\sum_{k=0}^{n}\frac{\Gamma(k+1)\Gamma(n-k+1)}{\Gamma(n+2)} = (n+1)\sum_{k=0}^{n}B(k+1,n-k+1)\\
&= (n+1)\sum_{k=0}^{n}\int_0^1 (1-t)^{n-k}t^k\,dt = (n+1)\int_0^1 (1-t)^n\sum_{k=0}^{n}\left(\frac{t}{1-t}\right)^k dt\\
&= (n+1)\int_0^1 \frac{t^{n+1}-(1-t)^{n+1}}{2t-1}dt = \frac{n+1}{2^{n+2}}\int_{-1}^1 \frac{(s+1)^{n+1}-(1-s)^{n+1}}{s}\,ds\\
&= \frac{n+1}{2^{n+1}}\int_0^1 \frac{(s+1)^{n+1}-(1-s)^{n+1}}{s}\,ds\\
&= \frac{n+1}{2^{n+1}}\left(\int_0^1\sum_{k=0}^{n}(1+s)^k ds + \int_0^1\sum_{k=0}^{n}(1-s)^k ds\right)\\
&= \frac{n+1}{2^{n+1}}\sum_{k=0}^{n}\left(\frac{2^{k+1}-1}{k+1}+\frac{1}{k+1}\right).
\end{aligned}$$

Hence,

$$S_n = \frac{n+1}{2^n}\sum_{k=0}^{n}\frac{2^k}{k+1},$$

for all $n \in \mathbb{N}$, as we wanted. Now, we can check that

$$S_n = 1 + \frac{n+1}{2n}S_{n-1},$$

for all $n \geq 2$. By an easy computation we get $S_n \leq 3$ whenever $n \leq 6$. Finally, assuming that $S_{n-1} \leq 3$ for some $n \geq 7$ we conclude that

$$S_n = 1 + \frac{n+1}{2n}S_{n-1} < 1 + \left(\frac{1}{2}+\frac{1}{2n}\right)3 < 3.$$

□

Given two natural numbers n and k we consider the set

$$H_{n,k} := \left\{\boldsymbol{k} = (k_1, ..., k_n) \in \mathbb{N}_0^n : \sum_{\ell=1}^{n} k_\ell = k, \quad \sum_{\ell=1}^{n} \ell k_\ell = n\right\}.$$

Observe that $H_{n,k} = \emptyset$ whenever $n < k$.

Lemma 3.2.3. *For every $n \geq k$ we have*

$$\sum_{\boldsymbol{k} \in H_{n,k}} \prod_{\ell=1}^{n} \ell!^{k_\ell} \leq n! \tag{3.5}$$

Proof. We first observe that for $n = k$ the set $H_{n,n}$ consists of the only element $\boldsymbol{k} = (n, 0, \ldots, 0)$. That is, $k_1 = n$ while $k_\ell = 0$ for every $\ell \geq 2$. Therefore, in this case the inequality established by the lemma is obvious, so we will assume from now on that $n > k$. We will proceed by induction on k.

We start with the case $k = 1$ by noticing that $H_{n,1}$ consists of the only element $\boldsymbol{k} = (0, ..., 0, 1) \in \mathbb{N}_0^n$ for all $n > 1$. Hence in this case

$$\sum_{\boldsymbol{k} \in H_{n,k}} \prod_{\ell=1}^{n} \ell!^{k_\ell} = n!$$

We now fix $n > k \geq 2$ and assume that inequality (3.5) holds for $H_{q,k-1}$ whenever $q \geq k - 1$. We note that the first non-zero coordinate of $\boldsymbol{k} \in H_{n,k}$ cannot be the nth coordinate, since in that case we would have $\boldsymbol{k} = (0, \ldots, 0, 1)$ and therefore $k = 1$, which contradicts the condition $k \geq 2$. Hence we can decompose

$$H_{n,k} = \bigcup_{j=1}^{n-1} I_j,$$

where I_j denotes the subset of $H_{n.k}$ consisting of those $\boldsymbol{k}$ for which the first non-zero coordinate is precisely the jth coordinate. Obviously, the sets $I_1, ..., I_{n-1}$ are pairwise disjoint. Some of the sets I_j could be empty, as follows from the discussion below.

Claim: Suppose $I_j \neq \emptyset, 1 \leq j \leq n - 1$. Then $n \geq 2j$ and $k_\ell = 0$ for every $\ell > n - j$ whenever $\boldsymbol{k} = (k_1, \ldots, k_n) \in I_j$.

In fact, let us first assume $2j > n$. Then for every $\ell > j$ we have $jk_j + \ell k_\ell \leq n < 2j$, which implies $k_\ell = 0$ since $k_j \geq 1$. Hence $k_\ell \neq 0$ only when $\ell = j$. Thus $k_j = k \geq 2$ and $n = jk_j \geq 2j$, a contradiction. Hence $n \geq 2j$. Let us now assume $\ell > n - j \geq j$. Then

$$n - j \geq jk_j + \ell k_\ell - j = j(k_j - 1) + \ell k_\ell.$$

Since $\ell > n-j$ and $k_j \geq 1$ we conclude $k_\ell = 0$, which proves the claim as desired.

To each element $\boldsymbol{k} \in I_j$ $(1 \leq j \leq n-1)$ we associate $\tilde{\boldsymbol{k}} \in H_{n-j,k-1}$ obtained from $\boldsymbol{k}$ by removing all coordinates after the coordinate $n-j$ (which are null according to the Claim) and replacing k_j (non-zero) by $k_j - 1$. In fact,

$$\sum_{\ell=1}^{n-j} \tilde{k}_\ell = \sum_{\ell=1}^{n} k_\ell - 1 = k-1 \quad \text{while} \quad \sum_{\ell=1}^{n-j} \ell \tilde{k}_\ell = \sum_{\ell=1}^{n} \ell k_\ell - j = n-j.$$

Observe that

$$\prod_{\ell=1}^{n} \ell!^{k_\ell} = j! \prod_{\ell=1}^{n-j} \ell!^{\tilde{k}_\ell}.$$

Finally, from the induction hypothesis we obtain

$$\begin{aligned} \sum_{\boldsymbol{k} \in H_{n,k}} \prod_{\ell=1}^{n} \ell!^{k_\ell} &= \sum_{j=1}^{n-1} \sum_{\boldsymbol{k} \in I_j} \prod_{\ell=1}^{n} \ell!^{k_\ell} \leq \sum_{j=1}^{n-1} j! \sum_{\tilde{\boldsymbol{k}} \in H_{n-j,k-1}} \prod_{\ell=1}^{n-j} \ell!^{\tilde{k}_\ell} \\ &\leq \sum_{j=1}^{n-1} j!(n-j)! = n! \sum_{j=1}^{n-1} \binom{n}{j}^{-1} \leq n! \end{aligned}$$

The last inequality follows from Lemma 3.2.2. □

We recall that two polynomials ψ, ϕ are said to be linearly equivalent if there exists $\ell(x) = \alpha x + \beta$, with $x \in \mathbb{R}$, $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$, such that $\phi(x) = (\ell \circ \psi \circ \ell^{-1})(x)$ for all $x \in \mathbb{R}$. Every polynomial of even degree is linearly equivalent to a monic polynomial: if $\psi(x) = ax^{2p} + r(x)$ with $a \neq 0$ and $r(x)$ a polynomial of degree less than $2p$, we choose b satisfying $b^{2p-1} = a$. Then, for $\ell(x) = bx$, the polynomial $\ell \circ \psi \circ \ell^{-1}$ is monic.

Below we obtain lower bounds for the iterations of a polynomial lacking fixed points. From now on, $\psi_n = \psi \circ \ldots \circ \psi$ denotes the n-th iteration of ψ. We note that a polynomial of degree greater than 1 and without fixed points has to have necessarily even degree.

Proposition 3.2.4. *Let ψ be a polynomial of degree greater than 1 and without fixed points. Then, for each $b > 1$ there is $m_0 \in \mathbb{N}$ such that*

$$|\psi_{m_0+k}(x)| \geq b^{2^k}$$

for all $x \in \mathbb{R}$ and for all $k \in \mathbb{N}$.

Proof. By hypothesis, ψ has even degree, say $2p$. In the case of $p = 1$, ψ is linearly equivalent to $\phi(x) = x^2 + c$ with $c > \frac{1}{4}$ (see [54, Section 3]), while ψ is linearly equivalent to $\phi(x) = x^{2p} + r(x)$ with r being a polynomial of degree less than $2p$ for $p \geq 2$. Indeed, we write $\psi(x) = a_{2p}x^{2p} + a_{2p-1}x^{2p-1} + \ldots + a_1x + a_0$ with $a_{2p} \neq 0$ and observe that if $\ell^{-1}(x) = a_{2p}^{\frac{1}{2p-1}}x$ we have that

$$(\ell^{-1} \circ \psi \circ \ell)(x) = a_{2p}^{\frac{1}{2p-1}} \psi(\frac{x}{a_{2p}^{\frac{1}{2p-1}}}) = x^{2p} + r(x)$$

with r being a polynomial of degree less than $2p$, as it was required. Since ϕ lacks fixed points we have $\phi(x) > x$ for all $x \in \mathbb{R}$ and there is $B > 1$ (to be chosen later) so that $\phi(x) \geq x^2$ for all $|x| \geq B$. Hence, $\phi_m(x) \geq x^{2^m} \geq B^{2^m}$ for all $|x| \geq B$. Since $\phi(x) > x$ for all $x \in \mathbb{R}$ and $\lim_{|x|\to\infty}(\phi(x) - x) = +\infty$, there is $a > 0$ such that $\phi(x) > x + a$ for all $x \in \mathbb{R}$ and therefore, $\phi_m(x) > x + ma$ for all $x \in \mathbb{R}$ and $m \in \mathbb{N}$. In particular, $\phi_m(x) > \min\{\phi(x) : x \in \mathbb{R}\} + (m-1)a$ for all $m \geq 2$ and $x \in \mathbb{R}$. Choose $m_0 \geq 2$ so that $\phi_{m_0}(x) \geq B$ for all $x \in \mathbb{R}$. By induction on k we deduce that

$$\phi_{m_0+k}(x) \geq B^{2^k}$$

for all $x \in \mathbb{R}$, $k \in \mathbb{N}$. Now, take $\ell(x) = ex + d$ so that $\psi = \ell \circ \phi \circ \ell^{-1}$. Fix $b > 0$. After choosing $B > \max\{b, \frac{|d|+1}{|e|}b\}$ we get that

$$\left(\frac{B}{b}\right)^{2^k} |e| - 1 > |d|$$

for all $k \in \mathbb{N}_0$. Then,

$$|\psi_{m_0+k}(x)| = \left|\ell(\phi_{m_0+k}(\ell^{-1}(x)))\right| \geq |e|\,B^{2^k} - |d| \geq b^{2^k}$$

for all $x \in \mathbb{R}$, $k \in \mathbb{N}$. $\square$

It follows from the previous proof that every polynomial without fixed points is linearly equivalent to a monic polynomial without fixed points. Now we obtain an upper bound for the derivatives of the successive iterations of a polynomial without fixed points.

Proposition 3.2.5. *Let ψ be a polynomial of degree greater than 1 without fixed points. For every $\alpha > 1$ there exist $C > 0$ and $r > 1$ such that*

$$\left|\psi_m^{(n)}(x)\right| \leq C\, r^n\, n!^2\, (1 + |\psi_m(x)|)^\alpha$$

for all $x \in \mathbb{R}$, $n \in \mathbb{N}, m \in \mathbb{N}$.

Proof. Without loss of generality we can assume that ψ is a monic polynomial of even degree $2p$. Choose $0 < c < \left(\frac{1}{(2p)!}\right)^{\frac{1}{2p-1}}$ and let $\alpha > 1$. Then there exists $x_0 > 0$ such that

$$\sum_{k=1}^{2p} c^{k-1}\,(1 + |x|)^{\alpha k} \cdot \left|\psi^{(k)}(x)\right| \leq (1 + |\psi(x)|)^\alpha \quad \forall\ x \geq x_0. \tag{3.6}$$

In fact, since $\alpha > 1$, the term in the left hand side behaves like $\lambda\,(1 + |x|)^{2p\alpha}$ as $|x| \to \infty$, where $\lambda = (2p)!\,c^{2p-1} < 1$, while the term in the right hand side behaves like $(1 + |x|)^{2p\alpha}$. From Proposition 3.2.4 there is $m_0 \in \mathbb{N}$ such that

$$\psi_m(x) \geq x_0 \ \ \forall x \in \mathbb{R},\ m \geq m_0.$$

Take $D > 1$ such that

$$\left|\psi_m^{(n)}(x)\right| \leq D\,(1 + |\psi_m(x)|)^\alpha \quad \forall\ 1 \leq m \leq m_0,\ x \in \mathbb{R},\ n \in \mathbb{N}.$$

Finally we put $r := \frac{D}{c} > 1$ and prove by induction that

$$\left|\psi_m^{(n)}(x)\right| \leq c n! r^n n^n\,(1 + |\psi_m(x)|)^\alpha \tag{3.7}$$

for all $x \in \mathbb{R}$, $n \in \mathbb{N}$, $m \in \mathbb{N}$. Due to the choice of constants, the inequality (3.7) is satisfied for $1 \leq m \leq m_0$, $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Let us now assume that (3.7) is satisfied for some $m \geq m_0$ and every $n \in \mathbb{N}$. Then, for every $n \geq k$ and $\boldsymbol{k} \in H_{n,k}$ we have

$$\begin{aligned}\prod_{\ell=1}^{n}\left|\frac{\psi_m^{(\ell)}(x)}{\ell!}\right|^{k_\ell} &\leq c^k\, r^n\,(1 + |\psi_m(x)|)^{\alpha k}\prod_{\ell=1}^{n}\ell^{\ell k_\ell}\\ &\underset{\text{Lemma 3.2.1}}{\leq} c^k\, r^n\,(1 + |\psi_m(x)|)^{\alpha k}\,\frac{n^n}{n!}\prod_{\ell=1}^{n}\ell!^{k_\ell}.\end{aligned}$$

Consequently, by Faà Di Bruno's formula, we obtain:

$$\begin{aligned}\left|\psi_{m+1}^{(n)}(x)\right| &= \left|(\psi\circ\psi_m)^{(n)}(x)\right| \leq \sum_{k=1}^{2p}\sum_{\boldsymbol{k}\in H_{n,k}}\frac{n!}{k_1!\dots k_n!}\left|\psi^{(k)}(\psi_m(x))\right|\cdot\prod_{\ell=1}^{n}\left|\frac{\psi_m^{(\ell)}(x)}{\ell!}\right|^{k_\ell}\\ &\leq cr^n n^n\sum_{k=1}^{2p}c^{k-1}\left(1+|\psi_m(x)|\right)^{\alpha k}\left|\psi^{(k)}(\psi_m(x))\right|\sum_{\boldsymbol{k}\in H_{n,k}}\prod_{\ell=1}^{n}\ell!^{k_\ell}\\ &\underset{\text{Lemma 3.2.3}}{\leq} cr^n n^n n!\sum_{k=1}^{2p}c^{k-1}\left(1+|\psi_m(x)|\right)^{\alpha k}\left|\psi^{(k)}(\psi_m(x))\right|\end{aligned}$$

Since $\psi_m(x)\geq x_0$ we can apply inequality (3.6) to finally conclude that

$$\left|\psi_{m+1}^{(n)}(x)\right|\leq cr^n n^n n!\left(1+|\psi_{m+1}(x)|\right)^{\alpha}$$

and the induction process is complete. The conclusion now follows from Stirling's formula. □

3.3 Power bounded composition operators

We recall that a continuous linear operator $T: X\to X$ on a Fréchet space X is said to be power bounded if the sequence of iterates $(T^m)_m$ is equicontinuous. A related concept to power boundedness is that of mean ergodicity. The operator $T: X\to X$ is mean ergodic if the limit

$$Px := \lim_n\frac{1}{n}\sum_{k=1}^{n}T^k(x)$$

exists for every $x\in X$ and $P\in L(X)$. Every power bounded operator T on a reflexive Fréchet space X is mean ergodic (see [4]). We will use that mean ergodicity implies that $\lim_n\frac{T^n(x)}{n}=0$, for every $x\in X$. There are many articles dedicated to the study of the behaviour of the sequence of iterations of a continuous operator in a function space, and especially when said operator is a composition operator.

The problem we address here is different. It is obvious that if ψ is a non-constant polynomial then the composition operator $C_\psi : f \mapsto f \circ \psi$ maps the Schwartz class $\mathcal{S}(\mathbb{R})$ into itself. However, as we have seen in Corollary 2.3.9, C_ψ does not leave invariant the classical Gelfand-Shilov spaces $\Sigma_d(\mathbb{R})$ unless ψ has degree one. On the other hand, according to Proposition 2.4.1 and Theorem 2.4.7, when ω is a sub-additive weight and ψ is a polynomial of degree N we have $C_\psi\left(\mathcal{S}_{(\omega)}(\mathbb{R})\right) \subset \mathcal{S}_{\omega_N}(\mathbb{R})$ with $\omega_N(t) = \omega(t^{N/(2N-1)})$. Since $\frac{2N-1}{N} < 2$ for all $N \geq 1$, the weight $\sigma(t) = \omega(t^{\frac{1}{2}})$ satisfies $C_\psi\left(\mathcal{S}_{(\omega)}(\mathbb{R})\right) \subset \mathcal{S}_{(\sigma)}(\mathbb{R})$ for every polynomial ψ. This makes the new weight σ optimal for studying iterations and dynamics of composition operators. In fact, since each iterate C_ψ^m can be written as $C_\psi^m = C_{\psi_m}$, where ψ_m is a polynomial, we conclude that $\left(C_\psi^m\right)_m$ is a sequence of continuous operators from $\mathcal{S}_{(\omega)}(\mathbb{R})$ into $\mathcal{S}_{(\sigma)}(\mathbb{R})$. So it is natural to ask whether this family of operators is equicontinuous or not.

Definition 3.3.1. *The operator* $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\sigma)}(\mathbb{R})$ *is said to be power bounded if the sequence of iterates* $C_\psi^m : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\sigma)}(\mathbb{R})$, $m \in \mathbb{N}$, *is equicontinuous.*

Since the sequence of operators acts between two different Fréchet spaces, most of the known results are not useful for our purposes, in particular the connection with mean ergodicity.

Now, we focus our attention to the properties of power boundedness and mean ergodicity in the case where ψ is a monotonic function. From now on, unless otherwise stated, ω and σ are weight functions. Let us start with the following useful general result:

Lemma 3.3.2. *It holds that*

1. *If there exists an unbounded sequence* $(x_n)_n$ *such that* $(\psi_n(x_n))_n$ *is bounded then* $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ *is not power bounded.*
2. *If* $|x_n|^k \geq n$ *for some* $k \in \mathbb{N}$ *and* $(\psi_n(x_n))_n$ *is bounded then* $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ *is not mean ergodic.*

Proof. See the proof of [53, Lemma 3.1] and observe that the function f used there can be taken to be in $\mathcal{S}_{(\omega)}(\mathbb{R})$. □

We have the following result:

Corollary 3.3.3. *Suppose that the function* ψ *verifies* $\psi(\mathbb{R}) = \mathbb{R}$ *and that there exists* $\delta > 0$ *such that* $|\psi(x) - x| > \delta$ *for all* $x \in \mathbb{R}$ *then,* $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ *is not mean ergodic.*

Proof. Same proof of [53, Corollary 3.2]. $\square$

Lemma 3.3.4. Let ψ be an increasing function. Then, for every $x \in \mathbb{R}$ there exists

$$\psi^*(x) = \lim_{n\to\infty} \psi_n(x) \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}\,.$$

If $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_\square(\mathbb{R})$ is mean ergodic then $f \circ \psi^* \in \mathcal{S}_\square(\mathbb{R})$ for all $f \in \mathcal{S}_\omega(\mathbb{R})$, where $\mathcal{S}_\square(\mathbb{R})$ can be either $\mathcal{S}(\mathbb{R})$ or $\mathcal{S}_\sigma(\mathbb{R})$.

Proof. Same proof of [53, Lemma 3.5]. $\square$

It can be shown that

Proposition 3.3.5. Let ψ be an increasing function with some fixed point. Then, it holds that either $\psi(x) = x$ for every $x \in \mathbb{R}$ or $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is not mean ergodic.

Proof. Same proof of [53, Proposition 3.6], noticing that the functions f that appear in the proof can be also taken to be in $\mathcal{S}_{(\omega)}(\mathbb{R})$. $\square$

Similarly, we can prove that

Proposition 3.3.6. Let ψ be a decreasing function. Then, it holds that either $\psi_2(x) = x$ for every $x \in \mathbb{R}$ (in which case C_ψ is power bounded and mean ergodic in any setting) or $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is not mean ergodic.

Proof. Same proof of [53, Proposition 3.6], noticing that the functions f that appear in the proof can be also taken to be in $\mathcal{S}_{(\omega)}(\mathbb{R})$. $\square$

For convenience, $\mathcal{S}_\square(\mathbb{R})$ can be either $\mathcal{S}(\mathbb{R})$ or $\mathcal{S}_{(\sigma)}(\mathbb{R})$ from now on.

In the case where ψ is increasing, it holds the following result:

Proposition 3.3.7. *Let* ψ *be an increasing smooth function. If* $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_\square(\mathbb{R})$ *is power bounded then,* $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_\square(\mathbb{R})$ *is mean ergodic.*

Proof. Fix $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$. We suppose that $\{C_{\psi_m} f : m \in \mathbb{N}\}$ is bounded in $\mathcal{S}_\square(\mathbb{R})$. It suffices to show that $(C_{\psi_m} f)_m$ is convergent in $\mathcal{S}_\square(\mathbb{R})$. As $\mathcal{S}_\square(\mathbb{R})$ is a Fréchet space it suffices to show that there is $h \in \mathcal{S}_\square(\mathbb{R})$ such that for any subsequence $(C_{\psi_{m_k}} f)_k$ of $(C_{\psi_m} f)_m$ there is another subsequence $(C_{\psi_{m_{r_k}}} f)_k$ of $(C_{\psi_{m_k}} f)_k$ for which $\lim_{k\to\infty} C_{\psi_{m_{r_k}}} f = h$ in $\mathcal{S}_\square(\mathbb{R})$. Since $\{C_{\psi_m} f : m \in \mathbb{N}\}$ is bounded in $\mathcal{S}_\square(\mathbb{R})$. So is any subsequence $(C_{\psi_{m_k}} f)_k$. We claim that the limit (it might be $\pm\infty$) of $\psi_m(x)$ as $m \to \infty$ exists, for every $x \in \mathbb{R}$. This is obvious when x is a fixed point for ψ. If x is not a fixed point for ψ, we either have that (i) $\psi(x) > x$ or (ii) $\psi(x) < x$. We deal with (i), (ii) can be done similarly. Since ψ is increasing, $\psi_2(x) > \psi(x) > x$. Inductively, we see that $\{\psi_m(x)\}_m$ is increasing and we are done. Let us write $\psi^*(x) = \lim_{n\to\infty} \psi_n(x) \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. As $\mathcal{S}_\square(\mathbb{R})$ is a Montel space, there is a subsequence $(C_{\psi_{m_{r_k}}} f)_k$ of $(C_{\psi_{m_k}} f)_k$ and $g \in \mathcal{S}_\square(\mathbb{R})$ such that $g = \lim_{k\to\infty} C_{\psi_{m_{r_k}}} f$ in $\mathcal{S}_\square(\mathbb{R})$. We claim that $g(x) = f \circ \psi^*(x)$ for all $x \in \mathbb{R}$. Indeed, for all $x \in \mathbb{R}$ we have that

$$g(x) = \lim_k (C_{\psi_{m_{r_k}}} f)(x) = \lim_k f(\psi_{m_{r_k}}(x)) = f(\psi^*(x)).$$

$\square$

Remark 3.3.8. *We observe the following:*

1. *In the proof of Proposition 3.3.7, we only used the fact that ψ is increasing to conclude that the limits $\lim_{m\to\infty} \psi_m(x)$ exists (they might be $\pm\infty$), for each $x \in \mathbb{R}$. So Proposition 3.3.7 remains true for any smooth function ψ such that the limits $\lim_{m\to\infty} \psi_m(x)$ exists (they might be $\pm\infty$), for each $x \in \mathbb{R}$.*

2. *If ψ is decreasing, Proposition 3.3.7 also holds, despite of the fact that $\lim_{m\to\infty} \psi_m(x)$ might not exist for some $x \in \mathbb{R}$.*

Proof. We only need to prove 2. Clearly, ψ_2 is increasing and the set $\{C_{\psi_{2m+1}} f : m \in \mathbb{N}\}$ is bounded in $\mathcal{S}_\square(\mathbb{R})$, for each $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$. We proceed as in the proof of Proposition 3.3.7 to conclude that the limit $g := \lim_m C_{\psi_{2m}} f$ exists in $\mathcal{S}_\square(\mathbb{R})$, for every $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$. Since $(C_{\psi_{2m+1}} f)(x) \to (C_\psi g)(x)$ as $m \to \infty$, we conclude that the limits

$$\lim_{m\to\infty} C_{\psi_{2m+1}} f = g \circ \psi$$

exists in $\mathcal{S}_\square(\mathbb{R})$ and hence, we get that

$$\lim_N \frac{1}{N} \sum_{m=1}^{N} C_{\psi_m} f = \frac{g + g \circ \psi}{2} \text{ in } \mathcal{S}(\mathbb{R}).$$

□

We are ready to prove the next result:

Theorem 3.3.9. The following results hold:

(a) If ψ is increasing then, $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is power bounded if and only if $\psi(x) = x$ for each $x \in \mathbb{R}$ if and only if $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\sigma)}(\mathbb{R})$ is power bounded.

(b) If ψ is decreasing then, $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\omega)}(\mathbb{R})$ is power bounded if and only if $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is power bounded if and only if $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is mean ergodic if and only if $\varphi_2(x) = x$ for each $x \in \mathbb{R}$ if and only if $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_\sigma(\mathbb{R})$ is mean ergodic.

Proof. (a) By Proposition 3.3.7, we can proceed as in [53, Theorem 3.8(a)].

(b) Due to Proposition 3.3.7, it is the content of Proposition 3.3.6. Obviously, ψ_2 is increasing and $C_{\psi_2} : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is also power bounded. Let us fix $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$. By repeating the argument from Proposition 3.3.7, we conclude that there is $g \in \mathcal{S}(\mathbb{R})$ such that $C_{\psi_{2m}} f \to g$ as $m \to \infty$ in $\mathcal{S}(\mathbb{R})$. Now, we observe that $(C_{\psi_{2m+1}} f)(x) = (C_{\psi_{2m}} f)(\psi(x)) \to (g \circ \psi)(x)$ as $m \to \infty$, for each $x \in \mathbb{R}$. We also have that $g \circ \psi \in \mathcal{S}(\mathbb{R})$ and $C_{\psi_{2m+1}} f \to g \circ \psi$ as $m \to \infty$ in $\mathcal{S}(\mathbb{R})$. We easily conclude:

$$\lim_N \frac{1}{N} \sum_{m=1}^{N} C_{\psi_m} f = \frac{g + g \circ \psi}{2} \text{ in } \mathcal{S}(\mathbb{R})$$

and we are done. □

Let us begin considering the case that ψ is a polynomial of degree 1. Then $C_\psi\left(\mathcal{S}_{(\omega)}(\mathbb{R})\right) \subset \mathcal{S}_{(\omega)}(\mathbb{R})$.

Proposition 3.3.10. *Let $\psi(x) = ax + b$, with $a \neq 0$. The following are equivalent:*

1. *$C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\omega)}(\mathbb{R})$ is power bounded.*
2. *$C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\omega)}(\mathbb{R})$ is mean ergodic.*
3. *$(C_{\psi_m})_m$ is equicontinuous in $\mathcal{L}(\mathcal{S}_{(\omega)}(\mathbb{R}), \mathcal{S}(\mathbb{R}))$.*

4. *$\psi(x) = x$ or $\psi(x) = -x + b$.*

Proof. (1) $\Rightarrow$ (3) is obvious. (1) $\Rightarrow$ (2) is clear because $\mathcal{S}_{(\omega)}(\mathbb{R})$ is reflexive and (4) $\Rightarrow$ (1) holds since $\{\psi_m : m \in \mathbb{N}\}$ is a finite set of affine functions. We now assume that (4) is not satisfied. Take $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$ such that $f'(-1) = f'(1) = f'(0) = 1$. The existence of such a function f can be easily obtained using bump functions and multiplying them by appropriate translations of $g(x) = x$.

We will check that $\left(\frac{1}{m} C_{\psi_m} f\right)_m$ is unbounded in $\mathcal{S}(\mathbb{R})$, which implies that neither (2) nor (3) are satisfied. To this end we distinguish several cases.

If $a = 1$ then $\psi_m(x) = x + mb$ with $b \neq 0$. For $x_m := -bm$ we have

$$\lim_m \frac{x_m^2}{m} \left|\left(C_{\psi_m} f\right)'(x_m)\right| = \infty$$

and we are done.

If $a \neq 1$ then $\psi(x)$ is linearly equivalent to $\phi(x) = ax$. We take $x_m = |a|^{-m}$. In the case $0 < |a| < 1$ we have

$$\lim_m \frac{x_m^2}{m} \left|\left(C_{\phi_m} f\right)'(x_m)\right| = \lim_m \frac{x_m}{m} = \infty,$$

while for $|a| > 1$ we have

$$\lim_m \frac{1}{m} \left|\left(C_{\phi_m} f\right)'(x_m)\right| = \lim_m \frac{|a|^m}{m} = \infty.$$

The proof is complete. $\square$

We will need the following result:

Proposition 3.3.11. *Let ω be a sub-additive weight, $a \geq 1$, $\sigma(t) = \omega(t^{\frac{1}{a}})$ for all $t \geq 0$. Assume that $\psi \in C^\infty(\mathbb{R})$ satisfies:*

1. *there is $C_0 > 0$ such that $|x| \leq C_0 (1 + |\psi_m(x)|)^a$ for all $x \in \mathbb{R}$, $m \in \mathbb{N}$,*

2. *for all $\lambda > 0$ there is $C_\lambda > 0$ such that*

$$\left|\psi_m^{(n)}(x)\right| \leq C_\lambda \exp\left(\lambda \varphi_\sigma^*(\frac{n}{\lambda})\right) (1 + |\psi_m(x)|)^p$$

for all $n \in \mathbb{N}$, $x \in \mathbb{R}$, $m \in \mathbb{N}$, where $p = a - 1$.

Then, the sequence of iterates of $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_\sigma(\mathbb{R})$ *is equicontinuous.*

Proof. We can adapt the proof of Proposition 2.4.1 to prove that $(C_{\psi_m} f)_m$ is a bounded sequence in $\mathcal{S}_{(\sigma)}(\mathbb{R})$ for each $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$. Indeed, Note that $\varphi^*_\sigma(s) = \varphi^*_\omega(as)$ for all $s \geq 0$ and that σ is also sub-additive. For each $j \in \mathbb{N}$, we denote

$$I_j := \left\{ \boldsymbol{k} = (k_1, k_2, \ldots, k_j) \in \mathbb{N}_0^j : \sum_{\ell=1}^{j} \ell k_\ell = j \right\}.$$

We know [52, Proposition 2.1.] that for every $\lambda > 0$ there is $B_\lambda > 0$ such that

$$\prod_{\ell=1}^{j} \left| \frac{\psi_m^{(\ell)}(x)}{\ell!} \right|^{k_\ell} \leq B_\lambda^k (1 + |\psi_m(x)|)^{kp} \frac{\exp(\lambda \varphi^*_\sigma(\frac{j-k}{\lambda}))}{(j-k)!} \tag{3.8}$$

for all $m \in \mathbb{N}$, $x \in \mathbb{R}$, where $k := \sum_{\ell=1}^{j} k_\ell$ and $(k_1, ..., k_j) \in I_j$. Fix $\lambda > 0$ and take $M_\lambda > 0$, $D_\lambda > 0$ such that

$$C_0^q B_\lambda^k \exp\left(M_\lambda \varphi^*_\sigma(\frac{k+q}{M_\lambda}) \right) \leq D_\lambda \exp\left(\lambda \varphi^*_\sigma(\frac{k+q}{\lambda}) \right) \tag{3.9}$$

for every $q, k \in \mathbb{N}_0$. Fix $f \in \mathcal{S}_\omega(\mathbb{R})$. There is $A_\lambda \equiv A_{M_\lambda} > 0$ such that

$$\begin{aligned} (1 + |\psi_m(x)|)^{aq+kp} \left| f^{(k)}(\psi_m(x)) \right| &\leq A_\lambda \exp\left(M_\lambda \varphi^*_\omega(\frac{aq+kp+k}{M_\lambda}) \right) \\ &= A_\lambda \exp\left(M_\lambda \varphi^*_\omega(\frac{a(q+k)}{M_\lambda}) \right) \\ &= A_\lambda \exp\left(M_\lambda \varphi^*_\sigma(\frac{q+k}{M_\lambda}) \right) \end{aligned} \tag{3.10}$$

for all $q, k \in \mathbb{N}_0$, $m \in \mathbb{N}$ and $x \in \mathbb{R}$. Set $G(j,k,\lambda,m)(x) := |f^{(k)}(\psi_m(x))| \frac{\exp(\lambda\varphi^*_\sigma(\frac{j-k}{\lambda}))}{(j-k)!}$.

Now, using (3.8), (3.9), (3.10), the fact that $\varphi^*_\sigma(s) + \varphi^*_\sigma(t) \leq \varphi^*_\sigma(s+t)$, for all $s, t \geq 0$, (see Proposition 1.4.11(ii)) and Lemma 1.3.1 we obtain that

$$\begin{aligned}
\left|x^q(f\circ\psi_m)^{(j)}(x)\right| &\leq C_0^q(1+|\psi_m(x)|)^{aq}\sum_{\boldsymbol{k}\in I_j}\frac{j!}{k_1!\dots k_j!}|f^{(k)}(\psi_m(x))|\prod_{\ell=1}^{j}\left|\frac{\psi_m^{(\ell)}(x)}{\ell!}\right|^{k_\ell}\\
&\leq \sum_{(k_1,\dots,k_j)\in I_j}\frac{j!C_0^qB_\lambda^k(1+|\psi_m(x)|)^{aq+kp}}{k_1!\dots k_j!}G(j,k,\lambda,m)(x)\\
&\leq A_\lambda\sum_{k=1}^{j}\binom{j}{k}C_0^q\,B_\lambda^k\exp\left(M_\lambda\varphi_\sigma^*(\frac{q+k}{M_\lambda})\right)\exp\left(\lambda\varphi_\sigma^*(\frac{j-k}{\lambda})\right)2^{j-1}\\
&\leq D_\lambda A_\lambda\sum_{k=1}^{j}2^{j-1}\binom{j}{k}\exp\left(\lambda\varphi_\sigma^*(\frac{q+k}{\lambda})\right)\exp\left(\lambda\varphi_\sigma^*(\frac{j-k}{\lambda})\right)\\
&\leq D_\lambda A_\lambda\exp\left(\lambda\varphi_\sigma^*(\frac{q+j}{\lambda})\right)4^j
\end{aligned}\tag{3.11}$$

for all $j \in \mathbb{N}$, $q \in \mathbb{N}_0$, $x \in \mathbb{R}$ and $m \in \mathbb{N}$. It is left to deal with the case $j = 0$. We also know that there is $Q_\lambda > 0$ such that

$$|(1+|y|)^{aq}f(y)| \leq Q_\lambda\exp\left(\lambda\varphi_\omega^*(\frac{aq}{\lambda})\right) = Q_\lambda\exp\left(\lambda\varphi_\sigma^*(\frac{q}{\lambda})\right)$$

for all $y \in \mathbb{R}$ and $q \in \mathbb{N}_0$. Now, by the hypothesis, it holds that

$$\begin{aligned}
|x^q(f\circ\psi_m)(x)| &= \frac{|x|^q}{(1+|\psi_m(x)|)^{aq}}\,|(1+|\psi_m(x)|)^{aq}f(\psi_m(x))|\\
&\leq C_0\,|(1+|\psi_m(x)|)^{aq}f(\psi_m(x))|\\
&\leq C_0Q_\lambda\exp\left(\lambda\varphi_\sigma^*(\frac{q}{\lambda})\right)
\end{aligned}\tag{3.12}$$

for all $x \in \mathbb{R}$, $q \in \mathbb{N}$, $m \in \mathbb{N}$. Using again Lemma 1.4.11, (3.11) and (3.12) we conclude that the set $\{C_{\psi_m}f : m \in \mathbb{N}\}$ is bounded in $\mathcal{S}_{(\sigma)}(\mathbb{R})$. Theorem 1.2.4) gives the conclusion. □

Compare the above result with

Proposition 3.3.12 ([53, Proposition 3.9.]). *Given ψ such that $C_\psi : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$, the composition operator $C_\psi : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is power bounded if and only if the following statements hold* *(i) For all $n \in \mathbb{N}_0$ there exist $C, p > 0$ such that*

$$\left|(\psi_\ell)^{(n)}(x)\right| \leq C(1+\psi_\ell(x)^2)^p$$

for every $x \in \mathbb{R}$ and every $\ell \in \mathbb{N}$.

(ii) There exists $k > 0$ such that $|\psi_\ell(x)| \geq |x|^{1/k}$ for all $|x| \geq k$ and every $\ell \in \mathbb{N}$.

From which we deduce the following:

Remark 3.3.13. *Under the hypotheses of Proposition 3.3.11, we also have that $C_\psi : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is power bounded.*

Concerning polynomials of degree greater than one we have the following characterization. To prove (2) ⇒ (3) we essentially follow the ideas of [53, Theorem 3.11]. We present the details of the proof because the argument of [53] is based on the fact that every power bounded operator $T : X \to X$ on a reflexive Fréchet space is mean ergodic. The proof of (3) ⇒ (1) is very different from that of [53] and is based on Proposition 3.2.5.

Theorem 3.3.14. *Let ω be any subadditive weight and $\sigma(t) = \omega(t^{\frac{1}{a}})$ for $a > 2$. Given a polynomial ψ of degree greater than one, the following statements are equivalent:*

1. *$(C_{\psi_m})_m$ is equicontinuous in $\mathcal{L}(\mathcal{S}_{(\omega)}(\mathbb{R}), \mathcal{S}_\sigma(\mathbb{R}))$.*
2. *$(C_{\psi_m})_m$ is equicontinuous in $\mathcal{L}(\mathcal{S}_{(\omega)}(\mathbb{R}), \mathcal{S}(\mathbb{R}))$.*
3. *ψ lacks fixed points.*

Proof. According to [11, Theorem 4.4], $C_{\psi_m}\big(\mathcal{S}_{(\omega)}(\mathbb{R})\big) \subset \mathcal{S}_\sigma(\mathbb{R})$ for every $m \in \mathbb{N}$.

(1) ⇒ (2) is obvious.

(2) ⇒ (3). Let us assume that ψ has some fixed point. There is $K > 0$ such that $|\psi(x)| \geq 2|x|$ for every $|x| \geq K$, hence $|\psi_m(x)| \geq 2^m|x|$ for all $m \in \mathbb{N}$ and $|x| \geq K$. In particular $\lim_m |\psi_m(x)| = \infty$ whenever $|x| \geq K$. We consider

$$A = \{x \in \mathbb{R} : \ (|\psi_m(x)|)_m \text{ does not diverge to } \infty\}.$$

Since ψ has some fixed point then the set A is non-empty and, due to the previous considerations, it is bounded. Take $b = \sup\{|x| : \ x \in A\}$ and let $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$ be given such that $f(x) = 1$ for all $|x| \leq b$. By condition (2), the sequence $(C_{\psi_m} f)_m$ is bounded in the Montel space $\mathcal{S}(\mathbb{R})$, hence it admits a subsequence that converges pointwise to some $g \in \mathcal{S}(\mathbb{R})$. For simplicity we denote the subsequence as the entire sequence. There is a sequence $(x_n)_n \subset A$ that converges to either b or $-b$. Moreover, $\psi_m(x_n) \in A \subset [-b, b]$ for every $m, n \in \mathbb{N}$, hence

$$g(x_n) = \lim_m f\left(\psi_m(x_n)\right) = 1 \quad \forall n \in \mathbb{N}.$$

If $(x_n)_n$ converges to b then $g(b) = 1$. However, for $|x| > b$ the sequence $(\psi_m(x))_m$ diverges to ∞, which implies

$$g(x) = \lim_m f\left(\psi_m(x)\right) = 0.$$

This is a contradiction. In the case that $(x_n)_n$ converges to $-b$ we argue similarly.

(3) $\Rightarrow$ (1). We only need to check that the two conditions appearing in Proposition 3.3.11 hold:

1. Since ψ has even degree it is equivalent to a monic polynomial lacking fixed points. Hence, we will assume that ψ itself is monic. As in the proof of (2) $\Rightarrow$ (3), there is $K > 1$ such that $|\psi_m(x)| \geq 2^m |x|$ for all $m \in \mathbb{N}$ and $|x| \geq K$. Then

$$|x| \leq 1 + |\psi_m(x)| \ \ \forall \ |x| \geq K, \ m \in \mathbb{N}$$

and condition 1 in Proposition 3.3.11 is satisfied with $C_0 = K$.

2. We recall that $\varphi^*_\sigma(s) = \varphi^*_\omega(as), \ s \geq 0$. Take $\alpha = a - 1$, then by Proposition 3.2.5 there exist $C > 0$ and $r > 1$ such that

$$|\psi_m^{(n)}(x)| \leq C r^n n!^2 \left(1 + |\psi_m(x)|\right)^\alpha$$

for all $x \in \mathbb{R}$, $n \in \mathbb{N}, m \in \mathbb{N}$. By Lemma 1.4.11, for every $\lambda > 0$ there exist $B_\lambda > 0$ such that for all n :

$$r^n n!^2 \leq B_\lambda \exp\left(\lambda \varphi^*_\omega\left(\frac{2n}{\lambda}\right)\right) \leq B_\lambda \exp\left(\lambda \varphi^*_\sigma\left(\frac{n}{\lambda}\right)\right).$$

Hence, condition 2 in Proposition 3.3.11 is satisfied with $C_\lambda = C B_\lambda$ and $p = \alpha$. The proof is complete. $\square$

Proposition 3.3.15. *Let ω be any subadditive weight and $\sigma(t) = \omega(t^{\frac{1}{a}})$ for $a > 2$. Given a polynomial ψ of degree greater than one, the following statements are equivalent:*

1. *ψ lacks fixed points.*
2. *$\lim_m f \circ \psi_m = 0$ in $\mathcal{S}_{(\sigma)}(\mathbb{R})$ for every $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$.*
3. *$\lim_n \frac{1}{n} \sum_{m=1}^{n} f \circ \psi_m$ exists in $\mathcal{S}_{(\sigma)}(\mathbb{R})$ for every $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$.*
4. *$\lim_n \frac{1}{n} \sum_{m=1}^{n} f \circ \psi_m$ exists in $\mathcal{S}(\mathbb{R})$ for every $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$.*

Proof. (1) ⇒ (2). Since ψ lacks fixed points then either $\psi(x) > x + b$ for every $x \in \mathbb{R}$ and some $b > 0$ or $\psi(x) < x - b$ for every $x \in \mathbb{R}$ and some $b > 0$. In either case we have $\lim_m |\psi_m(x)| = \infty$ for all $x \in \mathbb{R}$, which implies $\lim_m f(\psi_m(x)) = 0$ for every $x \in \mathbb{R}$. Now the conclusion follows from the fact that $(f \circ \psi_m)_m$ is a bounded sequence in the Fréchet Montel space $\mathcal{S}_{(\sigma)}(\mathbb{R})$ (Theorem 3.3.14) and its only possible accumulation point is the zero function.

(2) ⇒ (3) ⇒ (4) are obvious. To show (4) ⇒ (1) we can proceed as in the proofs of [53, Proposition 3.4 and Theorem 3.11 (2) ⇒ (3)] with the obvious changes. □

We do not know whether the above results are also true for $a = 2$.

It may be worth making the above results explicit in the case where the weight ω is a power of the logarithm. In this case, keeping the notation of the previous result, $\mathcal{S}_{(\omega)}(\mathbb{R}) = \mathcal{S}_{(\sigma)}(\mathbb{R})$. The limit case $p = 1$ corresponds to [53, Theorem 3.11], since in this case $\mathcal{S}_{(\omega)}(\mathbb{R}) = \mathcal{S}(\mathbb{R})$, despite of the fact that ω would not be strictly speaking a weight function (Definition 1.4.7(γ) does not hold).

Corollary 3.3.16. *Let $\omega(x) = \max\{0, \log(x)\}^p$ with $p > 1$. Given a polynomial ψ of degree greater than one, the following statements are equivalent:*

1. *$C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\omega)}(\mathbb{R})$ is power bonded.*
2. *ψ lacks fixed points.*

3. $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\omega)}(\mathbb{R})$ *is mean ergodic.*

Corollary 3.3.16 holds for every weight ω satisfying the following condition:

$$\exists \gamma > 1 \ \exists C \geq 1 \ \forall t \geq 0 : \ \omega(t^\gamma) \leq C\omega(t) + C. \tag{3.13}$$

Corollary 3.3.16 extends Theorem 1.1.2 from [53].

Compare Theorem 3.3.16 and Theorem 1.1.2 with the power boundedness of composition operators acting on $H(\mathbb{D})$, whose power boundedness exhibits nearly the exact opposite behavior:

Theorem 3.3.17 ([26, Corollary 1])**.** *Let* $\varphi : \mathbb{D} \to \mathbb{D}$ *be a holomorphic map. Then* $C_\varphi : H(\mathbb{D}) \to H(\mathbb{D})$ *is power bounded if and only if* φ *has a fixed point.*

Let $\psi(x) = \sqrt{x^2+1}$, for $x \in \mathbb{R}$. We know from Example 2.4.15(a) that the composition operator $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\omega)}(\mathbb{R})$ is continuous, for every sub-additive weight ω. Let us proceed to further prove:

Proposition 3.3.18. *Given any sub-additive weight* ω*, the composition operator* $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\omega)}(\mathbb{R})$ *is power bounded, where* $\psi(x) = \sqrt{1+x^2}$ *for all* $x \in \mathbb{R}$.

Proof. The proof relies on Proposition 3.3.11, with $a = 1$. Let us verify that the hypotheses of Proposition 3.3.11 hold. First observe that $\psi_m(x) = \sqrt{m+x^2} \geq |x|$, for all $x \in \mathbb{R}$, $m \in \mathbb{N}$. Now, by Faà Di Bruno's formula (1.4) with $f(x) = \sqrt{x}$, $g(x) = m + x^2$, for $x \in \mathbb{R}$, the fact that (due to Lemma 1.3.1)

$$\sum \frac{(k_1+k_2)!}{k_1!\,k_2!} \leq 2^{j-1},$$

for every $j \in \mathbb{N}$, where the sum extends over pairs $(k_1, k_2) \in \mathbb{N}_0^2$ such that $k_1 + 2k_2 = j$, and that $\frac{2|x|}{n+x^2} \leq 1$, for all $x \in \mathbb{R}$, $n \in \mathbb{N}$, we deduce the following inequality holds:

$$
\begin{aligned}
|\psi_m^{(j)}(x)| &\leq \sum \frac{j!}{k_1!k_2!} \frac{1\cdot 3 \cdot \ldots \cdot (2k-1)}{2^k} \frac{(2\,|x|)^{k_1}}{(n+x^2)^{k+\frac{1}{2}}} \\
&= j! \sum \frac{1}{k_1!k_2!} \frac{(2k)!}{2^k \cdot 2 \cdot 4 \cdot \ldots \cdot (2\,k)} \frac{(2|x|)^{k_1}}{(n+x^2)^{k+\frac{1}{2}}} \\
&= j! \sum \frac{1}{k_1!k_2!} \frac{(2k)!}{2^k\,k!\,2^k} \left(\frac{2|x|}{n+x^2}\right)^{k_1} \frac{1}{(n+x^2)^{k_2+\frac{1}{2}}} \\
&\leq j! \sum \frac{k!}{k_1!k_2!} \frac{(2k)!}{(2^k k!)^2} \\
&= j! \sum \frac{k!}{k_1!k_2!} \binom{2k}{k} \frac{1}{4^k} \\
&\leq j! \sum \frac{k!}{k_1!k_2!} \\
&\leq j!\, 2^{j-1},
\end{aligned}
\tag{3.14}
$$

for all $j \in \mathbb{N}$, $x \in \mathbb{R}$, $m \in \mathbb{N}$, where the sum extends over pairs $(k_1, k_2) \in \mathbb{N}_0^2$ such that $k_1 + 2k_2 = 2$ and we denote $k = k_1 + k_2$. Fix any sub-additive weight function ω. Now, using (3.14) and the fact that (see Lemma 1.4.11) for every $\lambda > 0$ there is $C_\lambda > 0$ such that

$$
j!\, 2^{j-1} \leq C_\lambda \exp\left(\lambda \varphi_\omega^*(\frac{j}{\lambda})\right)
$$

for all $j \in \mathbb{N}$, we see that all hypotheses of Proposition 3.3.11 (with $a = 1$) are fulfilled and then we are done. □

3.4 Spectrum and Neumann series

We start with some results that are obtained by simply adapting results already obtained in [53]. Later we will show that, with regard to the Waelbroeck spectrum, the behavior of the spectrum in Gelfand-Shilov classes compared to the Schwartz class can be different.

Proposition 3.4.1. *If* ψ *is such that* $C_\psi : \mathcal{S}_\omega(\mathbb{R}) \to \mathcal{S}_\omega(\mathbb{R})$, *then* $\sigma_p(C_\psi) \subset \overline{\mathbb{D}}$. *If all the orbits of* ψ *are unbounded then* $\sigma_p(C_\psi) \subset \mathbb{D}$.

Proof. See the proof of [53, Proposition 4.1]. ☐

For injective functions ψ, more can be said. If ψ is increasing we have the following result:

Proposition 3.4.2. *If ψ is an increasing function such that $C_\psi : \mathcal{S}_\omega(\mathbb{R}) \to \mathcal{S}_\omega(\mathbb{R})$ then, it holds:*

(a) $\sigma_p(C_\psi) \subset \{1\}$.

(b) $\sigma_p(C_\psi) = \{1\}$ *if and only if the set* $\{t : \psi(t) = t\}$ *has interior points.*

Proof. See the proof of [53, Proposition 4.2]. ☐

If ψ is decreasing we have the following result:

Proposition 3.4.3. Let φ be a decreasing function such that $C_\psi : \mathcal{S}_\omega(\mathbb{R}) \to \mathcal{S}_\omega(\mathbb{R})$. Then, the following conditions are equivalent:

(a) $\sigma_p(C_\psi) \neq \emptyset$.

(b) The set $\{t : \psi_2(t) = t\}$ has interior points.

(c) $\sigma_p(C_\psi) = \{-1, 1\}$.

Proof. See the proof of [53, Proposition 4.3]. ☐

We are ready to prove the following result:

Corollary 3.4.4. If $C_\psi : \mathcal{S}_\omega(\mathbb{R}) \to \mathcal{S}_\omega(\mathbb{R})$ is mean ergodic then $\sigma(C_\psi) \subset \overline{\mathbb{D}}$.

Proof. It is a consequence of [53, Proposition 4.4] and Proposition 3.4.1. ☐

We also have a similar result to [53, Corollary 4.6]:

Corollary 3.4.5. *If the following conditions hold*

1. *there is* $C_0 > 0$ *such that* $|x| \leq C_0(1 + |\psi_m(x)|)$ *for all* $x \in \mathbb{R}$, $m \in \mathbb{N}$,
2. *for all* $\lambda > 0$ *there is* $C_\lambda > 0$ *such that*

$$|\psi_m^{(j)}(x)| \leq C_\lambda \exp\left(\lambda \varphi_\omega^*(\frac{j}{\lambda})\right)$$

for all $j \in \mathbb{N}$, $x \in \mathbb{R}$, $m \in \mathbb{N}$, *and*

3.

$$\sup_{x\in\mathbb{R}}\sum_{n=1}^{\infty}\frac{1}{(1+|\psi_n(x)|)^p}<\infty,$$

for some $1\leq p<\infty$,

then $\sigma(C_\psi)\subset\mathbb{D}$.

Proof. We already knew from [53, Proposition 4.4] and conditions (1), (2) that $\sigma(C_\psi)\subset\overline{\mathbb{D}}$. Hence we only need to check that the operator $C_\varphi-\lambda I$ is an isomorphism whenever $|\lambda|=1$. Fix $\lambda\in\mathbb{C}$ with $|\lambda|=1$. Let us start by checking that $C_\varphi-\lambda I$ is surjective. To do so, we will see that the hypotheses imply, after proceeding as in the proof of Proposition 3.4.10 that the series

$$f:=-\sum_{n=0}^{\infty}\frac{1}{\lambda^{n+1}}\,g\circ\psi_n$$

is convergent in $\mathcal{S}_\omega(\mathbb{R})$ for every $g\in\mathcal{S}_\omega(\mathbb{R})$ and $C_\psi(f)-\lambda f=g$. Indeed, given $\mu\neq 0$, and $\lambda>0$ we will show that the series $\sum_{m=0}^{\infty}\frac{\|C_{\psi_m}f\|_{\omega,\mu}}{\lambda^m}$ converges. On the one hand, it obviously holds that

$$-\lambda f=g+\sum_{m=1}^{\infty}\frac{C_{\psi_m}g}{\lambda^m}=g-C_\psi f,$$

as we wanted. On the other hand, proceeding as in the proof of Proposition 4.1 in [11] we obtain that given $\rho>0$ there exist positive numbers ρ' and C such that

$$(1+|\psi_m(x)|)^q\left|(g\circ\psi_m)^{(n)}(x)\right|\leq C\,\|g\|_{\omega,\rho'}\exp\left(\rho\varphi^*_\sigma(\frac{n+q}{\rho})\right),$$

for all $x\in\mathbb{R}$, $n\in\mathbb{N}$, $m\in\mathbb{N}$, $q\in\mathbb{N}_0$. By the hypotheses and the fact that $g\in\mathcal{S}_\omega(\mathbb{R})$, we have that

$$\begin{aligned}(1+|x|)^q\left|(g\circ\psi_m)^{(n)}(x)\right| &\leq C_0^q\,(1+|\psi_m(x)|)^{q+p}\left|(g\circ\psi_m)^{(n)}(x)\right|\frac{1}{(1+|\psi_m(x)|)^p}\\ &\leq C_0^q\,C\,\|g\|_{\omega,\rho'}\exp\left(\rho\varphi^*_\sigma(\frac{n+q+p}{\rho})\right)\frac{1}{(1+|\psi_m(x)|)^p}.\end{aligned}$$

for all $m \in \mathbb{R}$, $x \in \mathbb{R}$.

Given $\mu > 0$ we choose $\rho > 0$ and $C' > 0$ such that

$$C_0^q C \exp\left(\rho\varphi_\sigma^*(\frac{n+q+p}{\rho})\right) \leq C' \exp\left(\mu\varphi_\sigma^*(\frac{n+q}{\mu})\right), \forall x \in \mathbb{R},\, \forall n, q \in \mathbb{N}_0.$$

This is possible by the convexity of φ_σ^* and Lemma 3.3 [11].

Putting everything together, we obtain, after some easy computations and using condition 3., the following inequality:

$$\sum_{m=0}^{\infty} \frac{\|C_{\psi_m}(g)\|_{\omega,\mu}}{|\lambda|^m} = \sum_{m=0}^{\infty} \|C_{\psi_m}(g)\|_{\omega,\mu} \leq C'\|g\|_{\omega,\rho'} \sup_{x\in\mathbb{R}} \sum_{m=0}^{\infty} \frac{1}{(1+|\psi_m(x)|)^p} < \infty,$$

from where the convergence of $\sum_{m=0}^{\infty} \frac{C_{\psi_m} g}{\lambda^m}$ in $\mathcal{S}_\omega(\mathbb{R})$ follows and hence, since $g \in \mathcal{S}_\omega(\mathbb{R})$ was arbitrary, the surjectivity of $C_\varphi - \lambda I : \mathcal{S}_\omega(\mathbb{R}) \to \mathcal{S}_\omega(\mathbb{R})$ holds.

To finish off the proof we also need to show that $C_\varphi - \lambda I : \mathcal{S}_\omega(\mathbb{R}) \to \mathcal{S}_\omega(\mathbb{R})$ is also injective, but since all the orbits of ψ are unbounded (by the condition 3.) we can apply Proposition 3.4.1 to derive it.

$\square$

As in the Schwartz space, we have:

Proposition 3.4.6. *The following statements hold:*

1. *Let* $\psi(x) = ax$, *where* $a \neq 0$ *and* $|a| \neq 1$. *Then*

   $$\sigma(C_\psi) = \mathbb{C} \setminus \{0\}.$$

   If $a = -1$ *then,* $\sigma(C_\varphi) = \sigma_p(C_\varphi) = \{-1, 1\}$.

2. *Let* $\psi(x) = x + 1$. *Then* $\sigma_p(C_\psi) = \emptyset$ *while*

   $$\sigma(C_\psi) = \{\lambda \in \mathbb{C} : \quad |\lambda| = 1\}.$$

Proof. To prove 1. see the proof of [53, Example 6].

Let us prove 2. Since ψ is increasing and $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\omega)}(\mathbb{R})$ for any ω weight, we have that $\sigma_p(C_\psi) = \emptyset$ (Proposition 3.4.2). From the part (c) of

[53, Example 5] we deduce that $\{\lambda \in \mathbb{C} : \ \ |\lambda| = 1\} \subset \sigma(C_\psi)$. Let us see that if $|\lambda| \neq 1$ then, $C_\psi - \lambda I : \mathcal{S}_\omega(\mathbb{R}) \to \mathcal{S}_\omega(\mathbb{R})$ is surjective. We first discuss the case $|\lambda| > 1$. It suffices to show that

$$\sum_m \frac{C_{\psi_m} f}{\lambda^m} = \sum_m \frac{f(\bullet + m)}{\lambda^m}$$

converges in $\mathcal{S}_\omega(\mathbb{R})$ for every $f \in \mathcal{S}_\omega(\mathbb{R})$ because $(C_\psi - \lambda I)^{-1}(f) = -\sum_m \frac{C_{\psi_m} f}{\lambda^{m+1}}$ for every $f \in \mathcal{S}_\omega(\mathbb{R})$. Take $A > 0$ so that $\delta := \frac{\exp(\frac{2}{A})}{|\lambda|} < 1$. We consider the following sub-cases:

(i) $x \in I_{1,m} := \{y \in \mathbb{R} : |y + m| \geq \frac{|y|}{2}\}$. Then $(1 + |x|)^q \leq 2^q \, (1 + |x + m|)^q$ for all $q \in \mathbb{N}$ and hence, by Lemma 1.4.11, we get that for every $\mu > 0$ there is $M_\mu > 0$ such that

$$\sup_{x \in I_{1,m}, q, j} (1 + |x|)^q |f^{(j)}(x + m)| \exp\left(-\mu \varphi_\omega^*\left(\frac{j + q}{\mu}\right) \right) \leq M_\mu,$$

for every $m \in \mathbb{N}_0$.

(ii) $x \in I_{2,m} := \{y \in \mathbb{R} : |y + \ell| < \frac{|y|}{2}\}$. Then $x < 0$ and $|x| - m < \frac{|x|}{2}$, hence $|x| < 2m$ and

$$\frac{(1 + |x|)^q}{A^q \, q! \, |\lambda|^m} \leq \frac{\left(\frac{1}{A} + \frac{2m}{A}\right)^q}{q!} \frac{1}{|\lambda|^m} \leq \exp\left(\frac{1}{A}\right) \delta^m,$$

for every $q \in \mathbb{N}_0$, $m \in \mathbb{N}$. Therefore, by Lemma 1.4.11, we conclude that for every $\mu > 0$ there is $D_\mu > 0$ such that

$$\sup_{x \in I_{2,m}, \, j \in \mathbb{N}_0, \, q \in \mathbb{N}_0} \frac{(1 + |x|)^q \; |f^{(j)}(x + m)| \, \exp(-\mu \varphi_\omega^*(\frac{j+q}{\mu}))}{|\lambda|^m} \leq D_\mu \, \delta^m$$

for every $m \in \mathbb{N}$.

Since $I_{1,m} \cup I_{2,m} = \mathbb{R}$ for every $m \in \mathbb{N}$, we conclude that for every $\mu > 0$ it holds that

$$\sum_m \frac{\|C_{\psi_m} f\|_{\mu,\omega}}{|\lambda|^m} = \sum_m \sup_{x \in \mathbb{R}, \, q \in \mathbb{N}_0, \, j \in \mathbb{N}_0} \frac{(1 + |x|)^q |f^{(j)}(x + m)| \exp(-\mu \varphi_\omega^*(\frac{j+q}{\mu}))}{|\lambda|^m} < \infty$$

for every $f \in \mathcal{S}_\omega(\mathbb{R})$ and hence, $\lambda \notin \sigma(C_\psi)$. We now consider the case $|\lambda| < 1$. We observe that $C_\psi^{-1} = C_\Phi$ where $\Phi(x) = x - 1$. The same arguments used in the previous case show that

$$0 < |\lambda| < 1 \Rightarrow \frac{1}{\lambda} \notin \sigma(C_\Phi) \Rightarrow \lambda \notin \sigma(C_\psi).$$

Obviously, $0 \notin \sigma(C_\psi)$ and we are done. □

For the spectral theory of linear operators not necessarily everywhere defined on non-normable spaces we follow [4]. We will always work with Fréchet spaces, hence we will give the definitions in this setting. We will write $\lambda - T$ as a shorthand for $\lambda I - T$, where I denotes the identity operator. In the following definition we consider a linear operator T whose domain, denoted $D(T)$, is a linear subspace of the Fréchet space E.

Definition 3.4.7. *Let E be a Fréchet space and $T : D(T) \subset E \to E$ a linear operator. The resolvent set of T is*

$$\rho(T) = \left\{\lambda \in \mathbb{C} : \ \lambda - T : D(T) \to E \text{ is bijective and } (\lambda - T)^{-1} \in L(E)\right\},$$

and the spectrum of T is defined by $\sigma(T) = \mathbb{C} \setminus \rho(T)$.

$\rho^*(T)$ consists of those $\lambda \in \rho(T)$ for which there exists an open neighbourhood $V(\lambda)$ of λ, $V(\lambda) \subset \rho(T)$, such that $\{(\mu - T)^{-1} : \mu \in V(\lambda)\}$ is equicontinuous in $L(E)$.

We put $\sigma^*(T) = \mathbb{C} \setminus \rho^*(T)$, and we call it the Waelbroeck spectrum by similarity with the case in which $T \in L(E)$.

According to [4, Remark 3.1], if $\rho(T) \neq \emptyset$ then T is a closed operator, that is, if $(x_n)_n \subset D(T)$ converges to x in E and $(Tx_n)_n$ converges to y, then $x \in D(T)$ and $y = Tx$. Then, when $\rho^*(T) \neq \emptyset$, [4, Proposition 3.4] can be applied and the map $\rho^*(T) \to L_b(E), \lambda \mapsto (\lambda - T)^{-1}$, is holomorphic.

Proposition 3.4.8. *Let E and F be Fréchet spaces with E continuously included in F. Given $T \in L(F)$, we put $D(A) := \{x \in E : Tx \in E\}$, and $A = T_{|_{D(A)}} : D(A) \to E$. We assume $D(A)$ dense in E. If $\sigma^*(A)$ is compact and there is $R > 0$ such that the series $\sum_{m=0}^{\infty} \frac{T^m x}{\mu^{m+1}}$ converges in F for every $x \in F$ and $|\mu| > R$, then $D(A) = E$.*

Proof. The hypotheses imply that $(\rho^m T^m x)_m$ is bounded in F for every $x \in F$ and $|\rho| < \frac{1}{R}$. Hence, $\displaystyle\sum_{m=0}^{\infty} \frac{T^m x}{\mu^{m+1}}$ is absolutely convergent in F for every $x \in F$ and $|\mu| > R$. From the compactness of $\sigma^*(A)$ we can find $r > R$ such that $\{\mu \in \mathbb{C} : \ |\mu| = r\} \subset \rho^*(A)$. Then the map

$$\{|\mu| = r\} \to L(E),\ \mu \mapsto (\mu - A)^{-1}$$

is continuous and, for each $y \in E$, the integral $\displaystyle\frac{1}{2\pi i}\int_{|\mu|=r} \mu(\mu - A)^{-1} y \ d\mu$ defines an element of E. On the other hand, with convergence in F,

$$\mu(\mu - A)^{-1} y = \sum_{m=0}^{\infty} \frac{T^m y}{\mu^m},$$

therefore

$$\frac{1}{2\pi i}\int_{|\mu|=r} \mu(\mu - A)^{-1} y \ d\mu = \sum_{m=0}^{\infty} \frac{T^m y}{2\pi i}\int_{|\mu|=r} \frac{1}{\mu^m} \ d\mu = Ty.$$

This finishes the proof.

□

By [54], given a polynomial ψ of degree greater than one and without fixed points, $\sigma_{\mathcal{S}(\mathbb{R})}(C_\psi) = \{0\}$, that is, for $\mu \neq 0$, $(\mu - C_\psi)$ is a topological isomorphism onto $\mathcal{S}(\mathbb{R})$. Besides,

$$(\mu - C_\psi)^{-1} f = \sum_{m=0}^{\infty} \frac{C_{\psi_m} f}{\mu^{m+1}},$$

with convergence in $\mathcal{S}(\mathbb{R})$. Using the estimates in Proposition 3.2.4, it is easy to check that $\rho(C_\psi) = \rho^*(C_\psi) = \mathbb{C} \setminus \{0\}$ and, consequently, its Waelbroeck spectrum reduces to $\{0\}$. Now, our next result follows immediately from Proposition 3.4.8 and means that, regarding the Waelbroeck spectrum, the behaviour of C_ψ in the Gelfand-Shilov classes compared with the Schwartz class might be very different.

The condition $d > 2$ in the theorem that follows guarantees that $\Sigma_{\frac{d}{2}}(\mathbb{R})$ is not trivial.

Theorem 3.4.9. *Let ψ be a polynomial of degree greater than one and without fixed points. For $d > 2$, take $E = \Sigma_d(\mathbb{R})$, $F = \mathcal{S}(\mathbb{R})$ and $T = C_\psi$. With the notation of Proposition 3.4.8, $\sigma^*(A)$ is not compact.*

Proof. According to Theorem 2.4.7 we have $\Sigma_{\frac{d}{2}}(\mathbb{R}) \subset D(A)$, hence $D(A)$ is a dense subspace of $E = \Sigma_d(\mathbb{R})$. On the other hand, from Corollary 2.3.8 we have $D(A) \neq E$. The conclusion follows from Proposition 3.4.8. □

Let us write $R_\mu = (\mu - C_\psi)^{-1}$. Given a subadditive weight ω, by Theorem 3.3.14, $(C_{\psi_m})_m$ is equicontinuous in $\mathcal{L}(\mathcal{S}_\omega(\mathbb{R}), \mathcal{S}_\sigma(\mathbb{R}))$, where $\sigma(t) = \omega(t^{\frac{1}{a}})$ for $a > 2$. Hence, it is natural to investigate whether $R_\mu \in \mathcal{L}(\mathcal{S}_\omega(\mathbb{R}), \mathcal{S}_\sigma(\mathbb{R}))$, that is, if the regularity of the solution g of the equation $\mu g - C_\psi g = f$ depends on the regularity of the datum f. This is the content of the next result.

Proposition 3.4.10. *Let ω be any subadditive weight and $\sigma(t) = \omega(t^{\frac{1}{a}})$ for $a > 2$. Given a polynomial ψ of degree greater than one and without fixed points and $\mu \neq 0$, we have that $R_\mu \in \mathcal{L}(\mathcal{S}_\omega(\mathbb{R}), \mathcal{S}_\sigma(\mathbb{R}))$.*

Proof. Given $f \in \mathcal{S}_\omega(\mathbb{R})$ and $\mu \neq 0$, and $\lambda > 0$ we will show that the series $\displaystyle\sum_{m=0}^{\infty} \frac{\|C_{\psi_m} f\|_{\sigma,\lambda}}{\mu^m}$ converges. Then, the result will follow by the Banach-Steinhaus theorem and the fact that $\mu R_\mu f = \sum_{m=0}^{\infty} \frac{C_{\psi_m} f}{\mu^m}$.

Let $b > 1$. By Proposition 3.2.4, there is $m_0 \in \mathbb{N}$ such that

$$|\psi_{m_0+k}(x)| \geq b^{2^k}$$

for all $x \in \mathbb{R}$ and for all $k \in \mathbb{N}$.

Proceeding as in the proof of (3)⇒(1) of Theorem 3.3.14 to check that condition 2 in Proposition 3.3.11 holds, we have that for all $\rho > 0$ there is $C_\rho > 0$ such that

$$|\psi_m^{(n)}(x)| \leq C_\rho \exp\left(\rho \varphi_\sigma^*(\frac{n}{\rho})\right) (1 + |\psi_m(x)|)^p$$

for all $n \in \mathbb{N}$, $x \in \mathbb{R}$, $m \in \mathbb{N}$, where $p = a - 1$. Proceeding as in the proof of Proposition 4.1 in [11] we obtain that given $\rho > 0$ there exist positive numbers ρ' and C such that

$$(1 + |\psi_m(x)|)^q \left|(f \circ \psi_m)^{(n)}(x)\right| \leq C \|f\|_{\omega,\rho'} \exp\left(\rho \varphi_\sigma^*(\frac{n+q}{\rho})\right),$$

for all $x \in \mathbb{R}$, $n \in \mathbb{N}$, $m \in \mathbb{N}$, $q \in \mathbb{N}_0$. Moreover, there is $C_0 > 0$ such that $1 + |x| \leq C_0(1 + |\psi_m(x)|)$ for all $x \in \mathbb{R}$, $m \in \mathbb{N}$. Hence, if $m = m_0 + k$,

$$\begin{aligned} b^{2^k}(1+|x|)^q \left|(f \circ \psi_m)^{(n)}(x)\right| &\leq C_0^q(1+|\psi_m(x)|)^{q+1} \left|(f \circ \psi_m)^{(n)}(x)\right| \\ &\leq C_0^q C \|f\|_{\omega,\rho'} \exp\left(\rho\varphi_\sigma^*(\frac{n+q+1}{\rho})\right). \end{aligned}$$

Given $\lambda > 0$ we choose $\rho > 0$ and $C' > 0$ such that

$$C_0^q C \exp\left(\rho\varphi_\sigma^*(\frac{n+q+1}{\rho})\right) \leq C' \exp\left(\lambda\varphi_\sigma^*(\frac{n+q}{\lambda})\right), \forall x \in \mathbb{R}, \forall n, q \in \mathbb{N}_0.$$

This is possible by the convexity of φ_σ^* and Lemma 3.3 [11]. Then, for $m = m_0 + k$,

$$\|C_{\psi_m}(f)\|_{\sigma,\lambda} \leq \frac{1}{b^{2^{m-m_0}}} C' \|f\|_{\omega,\rho'},$$

from where the convergence of $\displaystyle\sum_{m=0}^{\infty} \frac{\|C_{\psi_m} f\|_{\sigma,\lambda}}{\mu^m}$ follows.

□

For polynomials of degree greater than one and having at least one fixed point, arguing as in [54, Theorem 2.8, Theorem 2.10] we have

Proposition 3.4.11. *Let ω be a subadditive weight and ψ a polynomial of degree greater than one and having at least one fixed point. Then, for every $0 < |\mu| \leq 1$, there is $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$ such that no $g \in \mathcal{S}(\mathbb{R})$ satisfies $\mu g - C_\psi g = f$. If in addition ψ has a fixed point a such that $\psi'(a) > 1$ and $\psi^{(n)}(a) \geq 0$ for all $n \geq 2$, then, for every $\mu \neq 0$ there is $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$ such that the equation $\mu g - C_\psi g = f$ has no solution in $\mathcal{S}(\mathbb{R})$.*

Now we focus on polynomials ψ of degree 2. In this case, the polynomial is linearly equivalent to $x^2 + c$. The parameter c depends on the number of fixed points of the polynomial. In fact, ψ has two different fixed points if and only if $c < \frac{1}{4}$, has no fixed points when $c > \frac{1}{4}$ and has a unique fixed point for $c = \frac{1}{4}$, which occurs when $x = \frac{1}{2}$. According to this, and by our previous results, when $c > \frac{1}{4}$, $R_\mu \in \mathcal{L}(\mathcal{S}_\omega(\mathbb{R}), \mathcal{S}_\sigma(\mathbb{R}))$, for every $\mu \neq 0$ and for every subadditive weight ω and $\sigma(t) = \omega(t^{\frac{1}{a}})$ with $a > 2$, whereas for $c < \frac{1}{4}$

and $\mu \neq 0$ we may find $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$ such that the equation $\mu g - C_\psi g = f$ has no solution in $\mathcal{S}(\mathbb{R})$. The case $c = \frac{1}{4}$ is more cumbersome. On the one hand, for every subadditive weight ω and each $0 < |\mu| \leq 1$ there is $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$ such that $f \notin (\mu - C_\psi)(\mathcal{S}(\mathbb{R}))$. On the other hand, for $|\mu| > 1$, $(\mu - C_\psi)^{-1} \in \mathcal{L}(\mathcal{S}(\mathbb{R}))$ and the series $\displaystyle\sum_{m=0}^{\infty} \frac{C_{\psi_m} f}{\mu^{m+1}}$ converges in $\mathcal{S}(\mathbb{R})$ for every $f \in \mathcal{S}(\mathbb{R})$ [54, Theorem 3.3]. Next we analyze, for $|\mu| > 1$, the range of $R_\mu := (\mu - C_\psi)^{-1}$ restricted to spaces $\mathcal{S}_{(\omega)}(\mathbb{R})$.

Lemma 3.4.12. *Let* $y_0 \geq 4$ *and* $y_{n+1} = \sqrt{2y_n - 1}$. *Then*

$$y_n \geq \left(\frac{n+2}{n+1}\right)^2.$$

Proof. We proceed by induction. By hypotheses $y_0 \geq 4$. Suppose that for some $n \geq 1$, $y_{n-1} \geq \left(\frac{n+1}{n}\right)^2$. We want to show that

$$y_n^2 = 2y_{n-1} - 1 \geq \left(\frac{n+2}{n+1}\right)^4.$$

To this end, it suffices to see that

$$2\left(\frac{n+1}{n}\right)^2 - 1 \geq \left(\frac{n+2}{n+1}\right)^4,$$

which happens if and only if

$$2(n+1)^6 - n^2(n+1)^4 \geq n^2(n+2)^4.$$

The left hand side is

$$n^6 + 8n^5 + 24n^4 + 36n^3 + 29n^2 + 12n + 2$$

whereas the right hand side coincides with

$$n^6 + 8n^5 + 24n^4 + 32n^3 + 16n^2.$$

This finishes the proof.

$\square$

Theorem 3.4.13. *Let* $\psi(x) = x^2 + \frac{1}{4}$, $|\mu| > 1$ *and* $R_\mu = \sum_{m=0}^{\infty} \frac{C_{\psi_m}}{\mu^{m+1}}$. *Then*

1. *For every strong weight* ω *there is* $f \in \mathcal{S}_\omega(\mathbb{R})$ *such that* $R_\mu f \notin \Sigma_2(\mathbb{R})$.
2. *Let* $d, d' > 1$ *be given such that* $d' < d + 2$. *There is* $f \in \Sigma_d(\mathbb{R})$ *such that* $R_\mu f \notin \Sigma_{d'}(\mathbb{R})$.

Proof. (1) Take $x_0 \geq 2$ and define $(x_n)_n$, $x_n \geq 0$, by the recurrence rule $x_{n+1}^2 + \frac{1}{4} = x_n$. Then $(x_n)_n$ is a decreasing sequence converging to $\frac{1}{2}$.

For each $n \in \mathbb{N}_0$ let a_n be the sequence given by $a_n = (\delta_{n,j})_j$, $j \in \mathbb{N}_0$. As the Borel map $B : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{E}_\omega(\{0\})$ is surjective and $(a_n)_n$ is a bounded sequence in the Fréchet nuclear space $\mathcal{E}_\omega(\{0\})$ (see [11, Lemma 3.4]), we may find a bounded sequence $(f_n)_n$ in $\mathcal{S}_{(\omega)}(\mathbb{R})$ such that $(f_n^{(j)}(x_0))_j = a_n$ for all n. Without loss of generality we may assume that all functions f_n have compact support contained in $(x_1, \psi(x_0))$, which implies that f_n and all their derivatives vanish at x_j as well as at $\psi_j(x_0)$ for $j \geq 1$.

Assume that $R_\mu(\mathcal{S}_\omega(\mathbb{R})) \subset \Sigma_2(\mathbb{R})$. By the closed graph theorem, the map $R_\mu : \mathcal{S}_\omega(\mathbb{R}) \to \Sigma_2(\mathbb{R})$ is continuous, thus $(R_\mu(f_n))_n$ is a bounded sequence in $\Sigma_2(\mathbb{R})$. In particular for every $h > 0$ there is $C_h > 0$ such that

$$|(R_\mu f_n)^{(n)}(x_n)| \leq C_h h^n n!^2 \text{ for each } n \geq 1.$$

Observe that $(R_\mu f_n)^{(n)}(x_n) = \displaystyle\sum_{j=0}^{\infty} \frac{1}{\mu^{j+1}} (f_n \circ \psi_j)^{(n)}(x_n)$. We denote $y_k := 2x_k$, so that $y_{n+1}^2 + 1 = 2y_n$. Since $\psi_j(x_n) = x_{n-j}$ for $j \leq n$, using Faà di Bruno's formula and identity (9) in [54], we conclude that

$$\begin{aligned} \left|(R_\mu f_n)^{(n)}(x_n)\right| &= \frac{1}{|\mu|^{n+1}} f_n^{(n)}(x_0) \left(\psi_n'(x_n)\right)^n = \frac{1}{|\mu|^{n+1}} \left(\prod_{j=0}^{n-1} 2\psi_j(x_n) \right)^n \\ &= \frac{1}{|\mu|^{n+1}} \left(\prod_{k=1}^{n} y_k \right)^n \geq \left(\frac{(n+2)^2}{4|\mu|} \right)^n \frac{1}{|\mu|} \end{aligned}$$

by Lemma 3.4.12. Then

$$\frac{1}{|\mu|} \left(\frac{(n+2)^2}{4|\mu|} \right)^n \leq C_h h^n n!^2$$

for all $n \in \mathbb{N}$. Choosing $h > 0$ such that $4|\mu|h < e^2$ we get a contradiction by an application of Stirling's formula.

(2) We put $\omega(t) = t^{\frac{1}{d}}$ and $\sigma(t) = t^{\frac{1}{d'}}$. We take $(x_n)_n$ and $(a_n)_n$ as in (1) and consider $b_n := \exp\left(\lambda_n \varphi^*_\omega(\frac{n}{\lambda_n})\right) a_n$, where $\lambda_n = \log n$. Then $(b_n)_n$ is a bounded sequence in $\mathcal{E}_\omega(\{0\})$ and, again by the surjectivity of the Borel map, we may find a bounded sequence $(f_n)_n$ in $\mathcal{S}_{(\omega)}(\mathbb{R})$ such that $(f_n^{(j)}(x_0))_j = b_n$ for all n. Without loss of generality we may assume that all functions f_n have compact support contained in $(x_1, \psi(x_0))$, which implies that f_n and all their derivatives vanish at x_j as well as at $\psi_j(x_0)$ for $j \geq 1$.

As before, the closed graph theorem implies that $R_\mu : \mathcal{S}_\omega(\mathbb{R}) \to \mathcal{S}_\sigma(\mathbb{R})$ is continuous provided that $R_\mu(\mathcal{S}_\omega(\mathbb{R})) \subset \mathcal{S}_\sigma(\mathbb{R})$. Therefore $(R_\mu(f_n))_n$ is a bounded sequence in $\mathcal{S}_\sigma(\mathbb{R})$. In particular, there is $C > 0$ such that

$$\left|(R_\mu f_n)^{(n)}(x_n)\right| \leq C \exp\left(\varphi^*_\sigma(n)\right) \quad \forall\, n \in \mathbb{N}_0.$$

On the other hand, proceeding as in the proof of (1) we obtain

$$(R_\mu f_n)^{(n)}(x_n) \geq \frac{(n+2)^{2n}}{(4|\mu|)^{n+1}} \exp\left(\lambda_n \varphi^*_\omega(\frac{n}{\lambda_n})\right).$$

Using that $\varphi^*_\omega(x) = xd \log\left(\frac{xd}{e}\right)$ and $\varphi^*_\sigma(x) = xd' \log\left(\frac{xd'}{e}\right)$ we finally obtain

$$\left(\frac{nd}{\lambda_n e}\right)^{nd} \frac{(n+2)^{2n}}{(4|\mu|)^{n+1}} \leq C \left(\frac{nd'}{e}\right)^{nd'}$$

for all $n \in \mathbb{N}_0$, which is a contradiction. □

Corollary 3.4.14. *Let* $\omega(t) = (\max\{0, \log(t)\})^p$ *with* $p > 1$. *Given a polynomial* ψ *of degree greater than one, the following statements hold:*

1. $\sigma_{\mathcal{S}_\omega(\mathbb{R})}(C_\psi) = \{0\}$ *whenever* ψ *lacks fixed points.*
2. $\overline{\mathbb{D}} \setminus \{0\} \subset \sigma_{\mathcal{S}_\omega(\mathbb{R})}(C_\psi)$ *provided that* ψ *has fixed points.*
3. $\sigma_{\mathcal{S}_\omega(\mathbb{R})}(C_\psi) = \mathbb{C} \setminus \{0\}$ *if* ψ *is of second degree and has two different fixed points.*

We recall that if $\psi(x) = \sqrt{x^2+1}$, the spectrum of $C_\psi : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is $\sigma_p(C_\psi) = \sigma(C_\psi) = \mathbb{D}$ (see [53, Example 3.]). The proof of [53, Example 3.] only works with little or none change for $C_\psi : \mathcal{S}_\omega(\mathbb{R}) \to \mathcal{S}_\omega(\mathbb{R})$, where on the weight function ω a further condition is imposed. Indeed,

Proposition 3.4.15. *If* $\psi(x) = \sqrt{x^2+1}$ *and* ω *is a sub-additive weight such that* $\omega_1(t) = \omega(t^3)$ *is a weight function then, the spectrum of* $C_\psi : \mathcal{S}_\omega(\mathbb{R}) \to \mathcal{S}_\omega(\mathbb{R})$ *is* $\sigma_p(C_\psi) = \mathbb{D}$

Proof. Since $\psi(x) \geq 1$ for all $x \in \mathbb{R}$, C_ψ is not injective in $\mathcal{S}_\omega(\mathbb{R})$ and hence, $0 \in \sigma_p(C_\psi)$. We proceed as in [53, Example 3]. Take $I =]x_0, y_0[= \left]\frac{1}{4}, \frac{1}{2}\right[$ and observe that

$$\psi_n(I) =]x_n, y_n[\quad \text{where} \quad 0 < x_0 < y_0 < x_1 < y_1 < x_2 < y_2 < \dots$$

We now fix $\lambda \in \mathbb{C}$ with $0 < |\lambda| < 1$ and take a function $h \in \mathcal{S}_{\omega_1}(\mathbb{R})$ with compact support contained in I. For $x \geq 0$ we define

$$f(x) = \lambda^n \left(h \circ (\psi_n)^{-1}\right)(x) \quad \text{if} \quad x \in \psi_n(I) \quad (n = 0, 1, 2, \dots) \tag{3.15}$$

and

$$f(x) = 0 \quad \text{in the case that} \quad x \geq 0, x \notin \bigcup_{n=0}^{\infty} \psi_n(I).$$

We observe that $f(x) = 0$ for every x in a neighborhood of the points $\{x_n, y_n\}_{n=0}^{\infty}$. Finally we extend f to the negative real numbers by $f(-x) = f(x)$. Then f is an smooth function that verifies that $C_\psi f = \lambda f$ (see [53, Example 3]). We need to show that $f \in \mathcal{S}_\omega(\mathbb{R}) \setminus \{0\}$. Fix $\mu > 0$. Take $A > 0$ so that $\exp(\frac{1}{A})|\lambda| < 1$. Since $(1+x^2)^q \leq \left(n + \frac{5}{4}\right)^q$ for every $x \in \psi_n(I), n \in \mathbb{N}$ and $q \in \mathbb{N}$ and

$$|\lambda|^n \frac{\left(n + \frac{5}{4}\right)^q}{A^q q!} \leq \exp\left(\frac{5}{4A}\right) \left(|\lambda| \exp(\frac{1}{A})\right)^n \to 0$$

as $n \to \infty$, we conclude that

$$\sup_{x \in \mathbb{R}, q \in \mathbb{N}} \left(1 + x^2\right)^q |f(x)| \exp\left(-\mu \varphi_\omega^*(\frac{q}{\mu})\right) < \infty.$$

In order to control the derivatives of f, we observe that

$$(\psi_n)^{-1}(x) = g(x^2 - n), \quad x \in \psi_n(I),$$

where $g(t) = \sqrt{t}$, for $t \geq 0$. According to Faà di Bruno's formula (1.4),

$$\left((\psi_n)^{-1}\right)^{(j)}(x) = \sum \frac{j!}{j_1! \, j_2!} g^{(j_1+j_2)}(x^2 - n) \, (2x)^{j_1},$$

where the sum extends over pairs $(j_1, j_2) \in \mathbb{N}_0^2$ such that $j_1 + 2j_2 = j$. After some computations, we see that

$$g^{(\ell)}(x) = (-1)^{\ell-1} \frac{1 \cdot 3 \cdots (2\ell-5) \cdot (2\ell-3)}{2^\ell} \, x^{\frac{1}{2}-\ell},$$

for all $\ell \in \mathbb{N}$, $x > 0$. Notice also that $x^2 - n \in]\frac{1}{16}, \frac{1}{4}[$ whenever $x \in \psi_n(I)$, $n \in \mathbb{N}$. So, we conclude that there is $Q_0 > 0$ such that

$$\left| \left((\psi_n)^{-1} \right)^{(j)} (x) \right| \leq j! \, Q_0^j \, |x|^j, \tag{3.16}$$

for all $j \in \mathbb{N}$, $x \in \psi_n(I)$, $n \in \mathbb{N}$. Now, using the same argument that in [52, Proposition 2.1.] with $\omega_1 = \omega(\bullet^3)$ we get that for every $\mu_0 > 0$ there is $B_{\mu_0} \geq 1$ so that

$$\prod_{\ell=1}^{n} \left(\frac{| \left((\psi_n)^{-1} \right)^{(\ell)} (x)|}{\ell!} \right)^{k_\ell} \leq B_{\mu_0}^k \frac{\exp\left(\mu_0 \varphi_{\omega_1}^*(\frac{j-k}{\mu_0}) \right)}{(j-k)!} |x|^j \tag{3.17}$$

for all $x \in \psi_n(I)$, $n \in \mathbb{N}$, $j \in \mathbb{N}$ and $k_1, ..., k_n \in \mathbb{N}_0$ so that $k_1 + ... + nk_n = n$, where $k = k_1 + ... + k_n$. We also have that

$$\frac{(1+x^2)^m}{(1 + [(\psi_n)^{-1}(x)]^2)^m} \leq (n + \frac{5}{4})^m \tag{3.18}$$

for every $m, n \in \mathbb{N}$. Therefore, as before, we conclude that

$$\frac{|\lambda|^n \, (1+x^2)^m}{A^m \, m! \, (1 + [(\psi_n)^{-1}(x)]^2)^m} \leq \exp\left(\frac{5}{4A} \right) \left(|\lambda| \exp(\frac{1}{A}) \right)^n \to 0 \tag{3.19}$$

as $n \to \infty$. Using again Faà di Bruno's formula (1.4) and taking a suitable $\mu_0 > 0$ in (3.17), we obtain that there is $M_{\mu,\lambda} > 0$ such that

$$\sup_{x \in \mathbb{R}, q \in \mathbb{N}_0, j \in \mathbb{N}} \left| (1+x^2)^q f^{(j)}(x) \right| \exp\left(-\mu \varphi_\omega^*(\frac{j+q}{\mu}) \right) \leq M_{\mu,\lambda}. \tag{3.20}$$

□

Let us now present a shorter alternative, which does not need any extra hypothesis. The key ingredient is the following lemma:

Lemma 3.4.16. *The function $f : \mathbb{R} \to \mathbb{R}^2$, defined by $f(z) = \exp(-az^2)$, with real part $\mathcal{R}e(a) > 0$, for all $z \in \mathbb{C}$, belongs to $\mathcal{S}_{(\omega)}(\mathbb{R})$ whenever f does and verifies that*

$$(f \circ \psi)(z) - \exp(-a)f(z) = 0$$

for all $z \in \mathbb{R}$, where $\psi(x) = \sqrt{1+x^2}$ for all $x \in \mathbb{R}$.

Proof. It is well-known that $f \in \mathcal{S}(\mathbb{R})$. By [56, Pag. 208 and 220] and Lemma 1.4.11, the function g_1 defined by $g_1(z) = \exp(-\mathcal{I}m(a)iz^2)$, for all $z \in \mathbb{C}$, is a multiplier of $\mathcal{S}_{(\omega)}(\mathbb{R})$. By [56, Pag. 220] and again Lemma 1.4.11, the function $g_2(z) = \exp(-\mathcal{R}e(a)z^2)$ belongs to $\mathcal{S}_{(\omega)}(\mathbb{R})$, because g_2 is a (real) dilatation of the Gaussian function $h(x) = \exp(-x^2)$ for every $x \in \mathbb{R}$. Since it clearly holds that

$$f(z) = \exp(-\mathcal{R}e(a)z^2)\exp(-\mathcal{I}m(a)iz^2)$$

for all $z \in \mathbb{R}$, we deduce that $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$ as we wanted. Now we observe that

$$(f \circ \psi)(z) = \exp\left(-a(z^2+1)\right) = f(z)\exp(-a)$$

for all $z \in \mathbb{R}$, as we claimed. □

We also need the following.

Lemma 3.4.17. *Given any sub-additive weight function ω, it holds that $C_{\psi_m} f \to 0$ in $\mathcal{S}_{(\omega)}(\mathbb{R})$ as $m \to \infty$, where $\psi(x) = \sqrt{1+x^2}$ for all $x \in \mathbb{R}$.*

Proof. Fix $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$. From the proof above, $\{C_{\psi_m} f : m \in \mathbb{N}\}$ is bounded in $\mathcal{S}_{(\omega)}(\mathbb{R})$. Since $\mathcal{S}_{(\omega)}(\mathbb{R})$ is Montel, the set $\{C_{\psi_m} f : m \in \mathbb{N}\}$ has an accumulation point $h \in \mathcal{S}_{(\omega)}(\mathbb{R})$. We have that $\lim_{m\to\infty} \psi_m(x) = \lim_{m\to\infty} \sqrt{x^2+m} = \infty$ and hence, $h(x) = \lim_{m\to\infty} f(\psi_m(x)) = 0$, for every $x \in \mathbb{R}$. Therefore, exists $\lim_m f \circ \psi_m = 0$ in $\mathcal{S}_{(\omega)}(\mathbb{R})$. □

As we promised above, Proposition 3.4.15 can be improved to exactly be as it was in $\mathcal{S}(\mathbb{R})$, i.e. without further hypotheses on the weight function ω:

Proposition 3.4.18. *If $\psi(x) = \sqrt{x^2+1}$ for all $x \in \mathbb{R}$ and ω is a sub-additive weight then, the spectrum of $C_\psi : \mathcal{S}_\omega(\mathbb{R}) \to \mathcal{S}_\omega(\mathbb{R})$ is $\sigma_p(C_\psi) = \sigma(C_\psi) = \mathbb{D}$.*

Proof. It holds that $\psi_n(x) = \sqrt{x^2+n}$ for all $n \in \mathbb{N}$, $x \in \mathbb{R}$. From Lemma 3.4.17 we have that $C_{\psi_m} f \to 0$ in $\mathcal{S}_\omega(\mathbb{R})$ as $m \to \infty$ for every $f \in \mathcal{S}_\omega(\mathbb{R})$. Therefore, by Corollary 3.4.4 and Proposition 3.4.1 we have $\sigma_p(C_\psi) \subset \mathbb{D}$ and

$\sigma(C_\psi) \subset \overline{\mathbb{D}}$. On the one hand, since $\exp(\{x+iy : x<0\}) = \mathbb{D}$, by Lemma 3.4.16, we deduce that the following inclusion holds:

$$\mathbb{D} \subset \sigma_p(C_\psi) \subset \sigma(C_\psi).$$

On the other hand, by Corollary 3.4.5, $\sigma(C_\psi) \subset \mathbb{D}$. The hypotheses of Corollary 3.4.5 are satisfied by ψ due to the estimates involved in the proof of Proposition 3.3.18 and the following inequality:

$$\sum_{n=1}^{\infty} \frac{1}{(1+|\psi_n(x)|)^p} \leq \sum_{n=1}^{\infty} \frac{1}{m^{\frac{p}{2}}} < +\infty,$$

for all $x \in \mathbb{R}$ and each $p > 2$. Putting both inclusions together, we deduce that $\sigma(C_\psi) = \sigma_p(C_\psi) = \mathbb{D}$ as we wanted to prove. $\square$

Notice that the above approach would also work for the Schwartz space $\mathcal{S}(\mathbb{R})$.

3.5 Power boundedness of composition operators on the Schwartz space and Gelfand-Shilov classes in $\mathbb{R}^N$ for linear maps.

We will consider composition operators with a linear map acting on the Schwartz class $\mathcal{S}(\mathbb{R}^N)$. All the results below hold also true for

$$C_T : \mathcal{S}_{(\omega)}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N)$$

with little changes.

Given a linear map $T : \mathbb{R}^N \to \mathbb{R}^N$, the first task is to study when does it hold that $C_T : \mathcal{S}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N)$, defined by $C_T f = f \circ T$. Let us prove the following initial result on this:

Proposition 3.5.1. *Given a linear map $T : \mathbb{R}^N \to \mathbb{R}^N$, $f \circ T \in \mathcal{S}(\mathbb{R}^N)$ whenever $f \in \mathcal{S}(\mathbb{R}^N)$ if and only if T is biyective. In this case, the composition operator $C_T : \mathcal{S}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N)$, defined by $C_T f = f \circ T$ is continuous.*

Proof. Suppose that $T^{-1}(\{0\})$ is a nontrivial subspace of $\mathbb{R}^N$, then we can find $(x_n)_n$ such that $\|x_n\| \to \infty$ and $Tx_n = 0$ for all $n \in \mathbb{N}$. Take $f \in \mathcal{S}(\mathbb{R}^N)$

so that $f(0) \neq 0$. We have that $\lim_n f(T(x_n)) = f(0)$, but if $f \circ T \in \mathcal{S}(\mathbb{R}^N)$, it should hold that $\lim_n f \circ T(x_n) = 0$. So far we have shown that the linear map $T : \mathbb{R}^N \to \mathbb{R}^N$ has to be bijective if the condition $f \circ T \in \mathcal{S}(\mathbb{R}^N)$ whenever $f \in \mathcal{S}(\mathbb{R}^N)$ holds. Now let us assume that T is bijective. Using the continuity of T^{-1} we deduce that there is $c > 0$ such that

$$\|Tx\| \geq c\|x\|$$

for all $x \in \mathbb{R}^N$. This in turn implies that there is $k > 1$ large enough to ensure that

$$\|Tx\| \geq \|x\|^{\frac{1}{k}},$$

for every $\|x\| \geq k$. So condition (ii) in Theorem 1.4.17 is satisfied. Condition (i) in Theorem 1.4.17 is automatically satisfied since $\partial^\alpha T \equiv 0$, for each $\alpha = (\alpha_1, ..., \alpha_N) \in \mathbb{N}_0^N$ with $|\alpha| := \alpha_1 + ... + \alpha_N > 1$ and $\frac{\partial T}{\partial x_j}$ is a constant vector, for every $j \in \{1, ..., N\}$. Therefore, by Theorem 1.4.17, we obtain that $f \circ T \in \mathcal{S}(\mathbb{R}^N)$ whenever $f \in \mathcal{S}(\mathbb{R}^N)$, as we wanted. □

$\mathcal{S}_*(\mathbb{R}^N)$ stands for either $\mathcal{S}(\mathbb{R}^N)$ or $\mathcal{S}_{(\omega)}(\mathbb{R}^N)$. We are interested in necessary and sufficient conditions for which the composition operator $C_T : \mathcal{S}_*(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N)$ is power bounded (i.e. the set $\{C_T^n f : n \in \mathbb{N}\}$ is bounded in $\mathcal{S}(\mathbb{R}^N)$ for each $f \in \mathcal{S}_*(\mathbb{R}^N)$ is $\mathcal{S}(\mathbb{R}^N)$ or $\mathcal{S}_{(\omega)}(\mathbb{R}^N)$. Some results regarding this question can be established by applying [13, Corollary 2.3] and [13, Corollary 2.4]. However, in an independent way we will relate the spectra structure of T with the condition of $C_T : \mathcal{S}_*(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N)$ being power bounded or mean ergodic.

Let us start with the easiest case where T is diagonalizable over $\mathbb{R}$. In this case we have the following result:

Theorem 3.5.2. *If T is diagonalizable over $\mathbb{R}$ then, the following conditions are equivalent:*

1. *$C_T : \mathcal{S}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N)$ is power bounded.*
2. *$C_T : \mathcal{S}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N)$ is mean ergodic.*
3. *The eigenvalues of T belong to $\{-1, 1\}$.*

Proof. Since $\mathcal{S}(\mathbb{R}^N)$ is a reflexive Fréchet space, 1. $\Longrightarrow$ 2. clearly holds by [3].

Without loss of generality, we may assume that T is of the form

$$T(x_1, x_2, ..., x_N) = (\lambda_1 x_1, \lambda_2 x_2, ..., \lambda_N x_N) \tag{3.21}$$

for each $x = (x_1, ..., x_N) \in \mathbb{R}^N$.

3. $\implies$ 1. Indeed, if $\lambda_i \in \{-1, 1\}$ for all $i \in \{1, 2, ..., N\}$ then, by Proposition 3.5.1, the set $\{C_T^n f : n \in \mathbb{N}\}$ is a finite subset of $\mathcal{S}(\mathbb{R}^N)$ and hence, $C_T : \mathcal{S}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N)$ is indeed power bounded.

2. $\implies$ 3. We first assume that for some $i \in \{1, 2, ..., N\}$ it holds that $0 < |\lambda_i| < 1$. Fix $f \in \mathcal{S}(\mathbb{R}^N)$ so that $f(0, ..., 0, \pm 1, 0, ..., 0) \neq 0$. We take the vector $x^{(n)} := \frac{1}{|\lambda_i|^n} e_i \in \mathbb{R}^N$, for each $n \in \mathbb{N}$, where e_i is the $i-$th element of the usual canonical basis for $\mathbb{R}^N$. Since $n\,|\lambda_i|^n \to \infty$, as $n \to \infty$, we have that

$$\frac{\|x^{(n)}\|}{n} = \frac{1}{n\,|\lambda_i|^n} \to \infty$$

as $n \to \infty$. By (3.21), we get that the following limit holds:

$$\frac{1 + \|x^{(n)}\|}{n} \,(f \circ T^n)\,(x^{(n)}) = \frac{1 + \|x^{(n)}\|}{n} \,|f(0, ..., 0, \pm 1, 0, ..., 0)| \to \infty,$$

as $n \to \infty$ and hence, the set $\{\frac{1}{n} C_T^n f : n \in \mathbb{N}\}$ is not bounded in $\mathcal{S}(\mathbb{R}^N)$.

Now we assume that for some $j \in \{1, 2, ..., N\}$ it holds that $|\lambda_j| > 1$. Take $f \in \mathcal{S}(\mathbb{R}^N)$ such that $\frac{\partial f}{\partial x_j}(0, ..., 0, \pm 1, 0, ..., 0) \neq 0$ and $x^{(n)} := |\lambda_j|^{-n} e_j \in \mathbb{R}^N$, for each $n \in \mathbb{N}$, where e_j is the $j-$th element of the usual canonical basis for $\mathbb{R}^N$. We have the following equality:

$$\frac{\partial (f \circ T^n)}{\partial x_j}(x_1, ..., x_N) = \lambda_j^n \,\frac{\partial f}{\partial x_j}(\lambda_1^n x_1, ..., \lambda_N^n x_N) \tag{3.22}$$

for all $n \in \mathbb{N}$, $x = (x_1, ..., x_N) \in \mathbb{R}^N$. In particular, from (3.22) with $x = x^{(n)}$ we get that

$$\frac{1}{n} \left| \frac{\partial (f \circ T^n)}{\partial x_j}(x^{(n)}) \right| = \frac{|\lambda_j|^n}{n} \left| \frac{\partial f}{\partial x_j}(0, ..., 0, \pm 1, 0, ..., 0) \right| \to \infty,$$

as $n \to \infty$, and hence, the set $\{\frac{1}{n} C_T^n f : n \in \mathbb{N}\}$ is not bounded in $\mathcal{S}(\mathbb{R}^N)$. Using Proposition 3.5.1 again, we conclude that if $C_T : \mathcal{S}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N)$ is mean ergodic then, $\lambda_i \in \{-1, 1\}$ for all $i \in \{1, 2, ..., N\}$. $\square$

Let us recall that two linear maps $T_1, T_2 : \mathbb{R}^N \to \mathbb{R}^N$ are similar if there is an invertible linear map $P : \mathbb{R}^N \to \mathbb{R}^N$ such that $T_1 = P^{-1}T_2P$. So the corresponding composition operators $C_{T_1}, C_{T_2} : \mathcal{S}(\mathbb{R}^N) \to (\mathbb{R}^N)$ have the same spectrum and one is power bounded (or mean ergodic) if and only if the other is also power bounded (or also mean ergodic resp.).

The following result is well-known in linear algebra (see for instance [111, Theorem 2.5.5.]):

Theorem 3.5.3 (Jordan decomposition theorem)**.** *Let V be a finite-dimensional vector space and let $T : V \to V$ be a linear transformation. Then V has a basis B in which $A = [T]_B$ is a matrix in Jordan Canonical Form.*

We now turn our attention to the case where T is not diagonalizable over $\mathbb{R}$ and its N eigenvalues $\lambda_1, ..., \lambda_N$ (counted with multiplicity) are real numbers. In this case, we have the following result:

Theorem 3.5.4. *Let $T : \mathbb{R}^N \to \mathbb{R}^N$ be a linear bijective map such that its N eigenvalues $\lambda_1, ..., \lambda_N$ (counted with multiplicity) are real numbers. If the composition operator $C_T : \mathcal{S}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N)$ is power bounded then, T is diagonalizable over $\mathbb{R}$.*

Proof. Let us start by analyzing what happens when T is a Jordan block of size greater than 1. So we suppose first that $\lambda \in \mathbb{R}$, $N > 1$ and

$$Tx = \begin{bmatrix} \lambda & 1 & 0 & ... & ... & ... & 0 \\ 0 & \lambda & 1 & 0 & ... & ... & 0 \\ 0 & 0 & \lambda & 1 & 0 & ... & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & ... & ... & 0 & \lambda & 1 \\ 0 & 0 & ... & ... & ... & 0 & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N-1} \\ x_N \end{bmatrix}$$

for all $x = (x_1, ..., x_N) \in \mathbb{R}^N$. It can be checked that for any $n \geq N-1$ it holds that (see for instance [64, p. 55])

$$T^n \equiv \begin{bmatrix} \lambda^n & n\lambda^{n-1} & \binom{n}{2}\lambda^{n-2} & ... & ... & ... & \binom{n}{N-1}\lambda^{n-N+1} \\ 0 & \lambda^n & n\lambda^{n-1} & ... & ... & ... & \binom{n}{N-2}\lambda^{n-N+2} \\ 0 & 0 & \lambda^n & ... & ... & ... & \binom{n}{N-3}\lambda^{n-N+3} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & ... & ... & 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & ... & ... & 0 & 0 & \lambda^n \end{bmatrix}$$

for all $x = (x_1, ..., x_N) \in \mathbb{R}^N$, i.e. it holds for every $n \geq N-1$ that

$$T^n x = \left(\sum_{q=0}^{N-1} \binom{n}{q} \lambda^{n-q} x_{q+1}, ..., \sum_{q=0}^{N-k} \binom{n}{q} \lambda^{n-q} x_{q+k}, ..., \lambda^n x_N \right), \qquad (3.23)$$

for all $x = (x_1, ..., x_N) \in \mathbb{R}^N$. Notice that if put $x = (0, ..., 0, x_N)$ in (3.23) we get that

$$T^n x = \left(\binom{n}{N-1} \lambda^{n-N+1}, ..., \binom{n}{N-k} \lambda^{n-N+k}, ..., \lambda^n \right) x_N.$$

For every $n \in \mathbb{N}$, we set $y_n = \left(\binom{n}{N-1} \lambda^{n-N+1}, ..., \binom{n}{N-k} \lambda^{n-N+k}, ..., \lambda^n \right)$. If $|\lambda| < 1$ then, $n\, y_n \to 0$ as $n \to \infty$. This is so because

$$n \left| \binom{n}{N-k} \lambda^n \right| \leq \frac{n^N}{(N-k)!} |\lambda|^n \to 0,$$

as $n \to \infty$, for every $k \in \{1, ..., N\}$. Since T is injective, $|\lambda| > 0$. For every $n \in \mathbb{N}$, take $x^{(n)} = \left(0, ..., 0, \frac{1}{\|y_n\|}\right) \in \mathbb{R}^N$. Obviously, $x_N^{(n)}\, y_n \in \overline{B}(0,1)$ for all $n \in \mathbb{N}$ and $\frac{\|x^{(n)}\|}{n} \to \infty$ as $n \to \infty$. Fix $f \in \mathcal{S}(\mathbb{R}^N)$ such that $f(x) \neq 0$ for all $x \in \overline{B}(0,1)$. Clearly, there is $\ell > 0$ such that $|f(y)| \geq \ell$ for all $y \in \overline{B}(0,1)$. We get that

$$\frac{1 + \|x^{(n)}\|}{n} \left| (f \circ T^n)(x^{(n)}) \right| \geq \frac{1 + \|x^{(n)}\|}{n} \ell \to \infty,$$

as $n \to \infty$. In this case, $C_T : \mathcal{S}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N)$ is not mean ergodic.

Now we suppose that $|\lambda| > 1$. From (3.23) we get that

$$\frac{\partial (f \circ T^n)}{\partial x_1}(x) = \lambda^n \frac{\partial f}{\partial x_1}(T^n x) = \lambda^n \frac{\partial f}{\partial x_1}(\lambda^n x_1, 0, ..., 0)$$

for all $x = (x_1, 0..., 0) \in \mathbb{R}^N$ and $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, this time we take $x^{(n)} = (\frac{1}{\lambda^n}, 0, ..., 0) \in \mathbb{R}^N$ and $f \in \mathcal{S}(\mathbb{R}^N)$ so that $\frac{\partial f}{\partial x_1}(1, 0, ..., 0) \neq 0$. Then,

$$\frac{1}{n} \left| \frac{\partial (f \circ T^n)}{\partial x_1}(x^{(n)}) \right| = \frac{|\lambda|^n}{n} \left| \frac{\partial f}{\partial x_1}(1, 0, ..., 0) \right| \to \infty$$

as $n \to \infty$. So again, $C_T : \mathcal{S}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N)$ is not mean ergodic in this case.

From what we have proven so far and Proposition 3.5.1 we obtain that if $C_T : \mathcal{S}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N)$ is power bounded then, $\lambda \in \{-1, 1\}$.

Let us suppose that $\lambda = 1$. Then, by (3.23), we have that

$$T^n x = \left(\sum_{q=0}^{N-1} \binom{n}{q} x_{q+1}, ..., \sum_{q=0}^{N-k} \binom{n}{q} x_{q+k}, ..., x_N \right)$$

for all $x = (x_1, ..., x_N) \in \mathbb{R}^N$. By the Chain rule, we get that

$$\frac{\partial (f \circ T^n)}{\partial x_j}(x) = \sum_{q=1}^{j} \binom{n}{j-q} \frac{\partial f}{\partial x_q}(T^n x)$$

for all $x \in \mathbb{R}^N$ and $n \geq N-1$. If $x = (0, ..., 0, x_N)$ then, $T^n x = \left(\binom{n}{N-1}, ..., \binom{n}{0}\right) x_N$ for all $n \geq N-1$. Set $\alpha_n = \left(\binom{n}{N-1}, ..., \binom{n}{0}\right)$ for all $n \geq N-1$. Since $N \geq 2$ we have that $\|\alpha_n\| \to \infty$, as $n \to \infty$. Now, take $f \in \mathcal{S}(\mathbb{R}^N)$ such that $f(x_1, ..., x_n) = x_2$ for all $(x_1, ..., x_N) \in \overline{B}(0, 1)$ and $x^{(n)} = \frac{1}{\|\alpha_n\|} e_N$, for all $n \geq N-1$, where e_N is the $N-$th element of the usual canonical basis for $\mathbb{R}^N$. We have that

$$\frac{1}{n} \left| \frac{\partial (f \circ T^n)}{\partial x_3}(x^{(n)}) \right| = \frac{1}{n} \binom{n}{1} \left| \frac{\partial f}{\partial x_2}(T^n x^{(n)}) \right| = 1$$

for all $n \in \mathbb{N}$ and hence, $\frac{1}{n} C_T^n f$ does not tend to 0 in $\mathcal{S}(\mathbb{R}^N)$ as $n \to \infty$. So in this case, $C_T : \mathcal{S}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N)$ is not mean ergodic.

Now we suppose that $\lambda = -1$. Then, by (3.23), we have that

$$T^n x = \left(\sum_{q=0}^{N-1} \binom{n}{q} (-1)^{n-q} x_{q+1}, ..., \sum_{q=0}^{N-k} \binom{n}{q} (-1)^{n-q} x_{q+k}, ..., (-1)^n x_N \right)$$

for all $x = (x_1, ..., x_N) \in \mathbb{R}^N$. By the Chain rule, we get that

$$\frac{\partial (f \circ T^n)}{\partial x_j}(x) = \sum_{q=1}^{j} \binom{n}{j-q} (-1)^{n-j+q} \frac{\partial f}{\partial x_q}(T^n x)$$

for all $x \in \mathbb{R}^N$ and $n \geq N-1$. If $x = (0, ..., 0, x_N)$ then,

$$T^n x = \left(\binom{n}{N-1} (-1)^{n-N+1}, ..., \binom{n}{0} (-1)^n \right) x_N$$

for all $n \geq N-1$. Set $\alpha_n = \left(\binom{n}{N-1}(-1)^{n-N+1}, ..., \binom{n}{0}(-1)^n\right)$ for all $n \geq N-1$. Since $N \geq 2$ we have that $\|\alpha_n\| \to \infty$ as $n \to \infty$. Now, take $f \in \mathcal{S}(\mathbb{R}^N)$ such that $f(x_1, ..., x_n) = x_2$ for all $(x_1, ..., x_N) \in \overline{B}(0,1)$ and $x^{(n)} = \frac{1}{\|\alpha_n\|} e_N$ for all $n \geq N-1$, where e_N is the $N-$th element of the usual canonical basis for $\mathbb{R}^N$. We have that

$$\frac{1}{n}\left|\frac{\partial(f \circ T^n)}{\partial x_3}(x^{(n)})\right| = \frac{1}{n}\binom{n}{1}\left|\frac{\partial f}{\partial x_2}(T^n x^{(n)})\right| = 1,$$

for all $n \in \mathbb{N}$ and hence, $\frac{1}{n} C_T^n f$ does not tend to 0 in $\mathcal{S}(\mathbb{R}^N)$ as $n \to \infty$. So in this case, $C_T : \mathcal{S}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N)$ is not mean ergodic neither. So $C_T : \mathcal{S}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N)$ cannot be mean ergodic, let alone power bounded.

We are ready to proceed with the general case. By Theorem 3.5.3 and Proposition 3.5.1, we can assume that T is in Jordan block form. Let us assume that $C_T : \mathcal{S}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N)$ is mean ergodic and hence, for every $f \in \mathcal{S}(\mathbb{R}^N)$ it holds that $\frac{1}{n} C_T^n f \to 0$ in $\mathcal{S}(\mathbb{R}^N)$ as $n \to \infty$. By Proposition 3.5.1 we have that $C_{T_j} : \mathcal{S}(\mathbb{R}^{d_j}) \to \mathcal{S}(\mathbb{R}^{d_j})$, where $T_j : \mathbb{R}^{d_j} \to \mathbb{R}^{d_j}$ is a Jordan block of T and $d_j \in \mathbb{N}$ its size. Let p be the number of Jordan blocks T is made of. It is not hard to prove that for every $g \in \mathcal{S}(\mathbb{R}^{d_j})$, $\frac{1}{n} C_{T_j}^n g \to 0$ in $\mathcal{S}(\mathbb{R}^{d_j})$ as $n \to \infty$. A way to do this consists in that given $g \in \mathcal{S}(\mathbb{R}^{d_j})$ we define the function f as

$$f(w_1, ..., w_{j-1}, x, y_{j+1}, ..., y_p) = e^{-w_1^T w_1} ... e^{-w_{j-1}^T w_{j-1}} g(x)\, e^{-y_{j+1}^T y_{j+1}} ... e^{-y_p^T y_p}$$

for every $w_1 \in \mathbb{R}^{d_1}$,..., $w_{j-1} \in \mathbb{R}^{d_{j-1}}$, $x \in \mathbb{R}^{d_j}$, $y_{j+1} \in \mathbb{R}^{d_{j+1}}$, ..., $y_p \in \mathbb{R}^{d_p}$, where v^T denotes the transpose of the matrix v. Clearly, it holds that

$$g(x) = f(0, ..., 0, x, 0, ..., 0),$$

for all $x \in \mathbb{R}^{d_j}$. Now observe that since $f \in \mathcal{S}(\mathbb{R}^N)$ we have that $\frac{1}{n} C_T^n f \to 0$ in $\mathcal{S}(\mathbb{R}^N)$ as $n \to \infty$. After some computations and using [64, Proposition 2.57.], we deduce that $\frac{1}{n} C_{T_j}^n g \to 0$ in $\mathcal{S}(\mathbb{R}^{d_j})$ as $n \to \infty$. As we have already proved before, this is a contradiction unless $d_j = 1$. Since $j \in \{1, ..., p\}$ was arbitrary, we deduce that $d_j = 1$ for all $j \in \{1, ..., p\}$. This last condition automatically implies that T is in fact diagonalizable over $\mathbb{R}$, which is again a contradiction. □

Notice that we used critically that $N > 1$, otherwise both α_n above would be equal to 1, for each $n \in \mathbb{N}$, and hence $1 = \lim_{n\to\infty} \alpha_n \neq \infty$, as required during the proof.

As we already mentioned at the beginning of the section, all the results above hold true for $C_T : \mathcal{S}_{(\omega)}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N)$ with little changes. One starts by proving Proposition 3.5.1 in this setting by means of Faà Di Bruno's formula (1.4) on several variables and then, notice that all the functions used above can be taken to be in $\mathcal{S}_{(\omega)}(\mathbb{R})$.

It remains to investigate what happens when T has some eigenvalue λ such that $\lambda \in \mathbb{C} \setminus \mathbb{R}$ (Open Problem 9).

Chapter 4

Topologizability and related properties of composition operators of polynomials acting on Gelfand-Shilov classes

4.1 Introduction

The aim of this chapter is to continue the research of the previous chapters concerning composition operators acting on Gelfand-Shilov spaces $\mathcal{S}_{(\omega)}(\mathbb{R})$ when ψ is a polynomial. We study what happens with the intermediate properties of topologizability and $m-$topologizability of iterates of composition operators $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\sigma)}(\mathbb{R})$, where ψ is a polynomial. In Section 4.3, the case of polynomials of degree greater than one and $\sigma(t) = \omega(t^{\frac{1}{a}})$, with $a > 2$, is analyzed. Unlike in the Schwartz space setting, composition operators C_ψ associated with polynomials ψ are not always $m-$topologizable. In Section 4.4 the case of polynomial ψ of degre one and $a = 1$ (i.e. $\sigma = \omega$) is studied and a complete characterization of topologizability and $m-$topologizability of $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\omega)}(\mathbb{R})$ is given.

The results presented are contained in the paper, under peer review and accepted by the journal Integral Equations and Operator Theory:

-Albanese, A., Ariza, H.: *Topologizability and related properties of composition operators of polynomials acting on Gelfand-Shilov classes.*. Arxiv

(2025).

4.2 Topologizabilty and m-topologizability

Let E be a locally convex Hausdorff space. Let $cs(E)$ be a fundamental system of semi-norms of the lcHs E.

The concept of topologizability and $m-$topologizability was defined originally and studied by Zelazko in [112].

Definition 4.2.1. *An operator $T \in L(E)$ on a locally convex Hausdorff space E is called topologizable if for $p \in cs(E)$ there is $q \in cs(E)$ such that for every $k \in \mathbb{N}$ there is $M_k > 0$ such that*

$$p(T^k(x)) \leq M_k q(x)$$

for each $x \in E$.

Definition 4.2.2. *An operator $T \in L(E)$ on a locally convex Hausdorff space E is called $m-$topologizable if for $p \in cs(E)$ there are $q \in cs(E)$ and $C \geq 1$ such that*

$$p(T^k(x)) \leq C^k q(x)$$

for each $k \in \mathbb{N}$ and for each $x \in E$.

Observe that in the definitions above it is essential that the seminorm q only depends on the seminorm p and not on the iteration k. Clearly, we have that power boundedness of an operator $T \in L(E)$ on a locally convex Hausdorff space E (i.e., equicontinuity of the sequence $(T^k)_{k\in\mathbb{N}}$) implies $m-$topologizability, which in turn implies topologizability.

As it was done with the concept of power boundedness and mean ergodicity in previous chapters, we extend the concepts of topologizability and of $m-$topologizability to an arbitrary family of operators $\{T_m : E \to F : m \in \mathbb{N}\}$, where E and F are locally convex Hausdorff spaces:

Definition 4.2.3. *Let E, F be locally convex Hausdorff spaces. A family of operators $\{T_m : E \to F : m \in \mathbb{N}\}$ is called topologizable if for every $p \in cs(F)$ there is $q \in cs(E)$ such that for every $k \in \mathbb{N}$ there is $M_k > 0$ such that*

$$p(T_k(x)) \leq M_k q(x)$$

for each $x \in E$.

Definition 4.2.4. *Let E, F be locally convex Hausdorff spaces. A family of operators $\{T_m : E \to F : m \in \mathbb{N}\}$ is called $m-$topologizable if for every $p \in cs(F)$ there are $q \in cs(E)$ and $C \geq 1$ such that*

$$p(T_k(x)) \leq C^k q(x)$$

for each $k \in \mathbb{N}$ and for each $x \in E$.

In the above definitions, it is essential yet again that the semi-norm q only depends on the semi-norm p and not on the iteration k; otherwise, all families of continuous linear operators $\{T_m : E \to F : m \in \mathbb{N}\}$ would fulfill them automatically. Also in this case, equicontinuity of the family of operators $\{T_m : E \to F : m \in \mathbb{N}\}$ implies $m-$topologizability of $\{T_m : E \to F : m \in \mathbb{N}\}$, which in turn implies topologizability of $\{T_m : E \to F : m \in \mathbb{N}\}$ and hence $T_m : E \to F$ is continuous for every $m \in \mathbb{N}$.

4.3 Topologizability and $m-$opologizability of the composition operators associated to a polynomial ψf degree greater than one on Gelfand-Shilov classes.

In this section we prove that even when the polynomial ψ has fixed points we still have that the family of iterates $\{C_{\psi_m} : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\sigma)}(\mathbb{R}) : m \in \mathbb{N}\}$ is topologizable in the setting considered in Chapter 3, i.e. where $\sigma(t) = \omega(t^{\frac{1}{a}})$, with $a > 2$:

Theorem 4.3.1. *Let ω be a sub-additive weight function and ψ be a polynomial of degree strictly greater than one. Then the family of iterates $\{C_{\psi_m} : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\sigma)}(\mathbb{R}) : m \in \mathbb{N}\}$ is topologizable, whenever $\sigma(t) = \omega(t^{\frac{1}{a}})$ and $a > 2$.*

Proof. We recall that $\varphi^*_\sigma(x) = \varphi^*_\omega(ax)$ for all $x \geq 0$. Let ρ_m be the degree of the polynomial ψ_m and $I_j = \{(k_1, ..., k_j) \in \mathbb{N}_0^j : k_1 + 2k_2 + \ldots + jk_j = j\}$, for every $m, j \in \mathbb{N}$. The fact that ψ is a polynomial of degree greater than one clearly implies that there are $\alpha \in]1, 2[$ and $b > 1$ such that $|\psi(x)| \geq |x|^\alpha$ for all $|x| \geq b$. So, it easily follows that there is $C_0 \geq 1$ such that $1 + |x| \leq C_0(1 + |\psi_m(x)|)$ for all $x \in \mathbb{R}$ and $m \in \mathbb{N}$. Since $\psi_m^{(j)} = 0$ for every $m \in \mathbb{N}$

and $j \in \mathbb{N}$ with $j > \rho_m$, we also have that for each $m \in \mathbb{N}$ there is $D_m > 0$ such that

$$|\psi_m^{(\ell)}(x)| \le D_m \, (1 + |\psi_m(x)|)^{\delta_m}$$

for all $x \in \mathbb{R}$, $\ell \in \mathbb{N}$, where $0 < \delta_m = \frac{\rho_m - 1}{\rho_m} < 1$. Therefore, for every $j, m \in \mathbb{N}$ and $(k_1, \ldots, k_j) \in I_j$ we obtain that

$$\prod_{\ell=1}^{j} \left| \frac{\psi_m^{(\ell)}(x)}{\ell!} \right|^{k_\ell} \le D_m^k (1 + |\psi_m(x)|)^{\delta_m k} \prod_{l=1}^{j} \frac{1}{(l!)^{k_\ell}}$$

for all $x \in \mathbb{R}$, where $k = k_1 + k_2 + \ldots k_j$, and hence, for every $\mu > 0$ and $x \in \mathbb{R}$ that

$$\prod_{\ell=1}^{j} \left| \frac{\psi_m^{(\ell)}(x)}{\ell!} \right|^{k_\ell} \le D_m^k (1 + |\psi_m(x)|)^k \prod_{l=1}^{j} \frac{\exp\left(2k_\ell \mu \varphi_\omega^*(\frac{\ell}{2\mu})\right)}{(l!)^{k_\ell}} \cdot \frac{1}{\exp\left(2k_\ell \mu \varphi_\omega^*(\frac{\ell}{2\mu})\right)}.$$

Since ω is a sub-additive weight, we can now argue as in the proof of [52, Proposition 2.1] to show that for every $\mu > 0$ and $m \in \mathbb{N}$ there is $B_{m,\mu} \ge 1$ such that

$$\begin{aligned} \prod_{\ell=1}^{j} \left| \frac{\psi_m^{(\ell)}(x)}{\ell!} \right|^{k_\ell} &\le B_{m,\mu}^k \frac{\exp\left(\mu \varphi_\omega^*(\frac{j-k}{\mu})\right)}{(j-k)!} (1 + |\psi_m(x)|)^k \prod_{\ell=1}^{j} \frac{1}{\exp\left(2k_\ell \mu \varphi_\omega^*(\frac{\ell}{2k_\ell \mu})\right)} \\ &\le B_{m,\mu}^k \frac{\exp\left(\mu \varphi_\omega^*(\frac{j-k}{\mu})\right)}{(j-k)!} (1 + |\psi_m(x)|)^k \end{aligned}$$

for all $(k_1, ..., k_j) \in I_j$, $j \in \mathbb{N}$ and $x \in \mathbb{R}$, where $k = k_1 + ... + k_j$.

Fix $\lambda > 0$ and $\delta > 0$. Then there exist $L_\lambda > 0$ and $\mu = \mu(\lambda) > 0$ such that

$$4^j \, C_0^q \exp\left(\mu \varphi_\omega^*\left(\frac{(2+\delta)(q+j)}{\mu}\right)\right) \le L_\lambda \exp\left(\lambda \varphi_\omega^*\left(\frac{(2+\delta)(q+j)}{\lambda}\right)\right)$$

for every $j, q \in \mathbb{N}$ (Lemma 1.4.11). On the other hand, for each $m \in \mathbb{N}$ there is $\square_{m,\mu,\delta} > 0$ such that

$$B_{m,\mu}^k \le \square_{m,\mu,\delta} \exp\left(\mu \varphi_\omega^*\left(\frac{\delta k}{\mu}\right)\right)$$

for all $k \in \mathbb{N}$ (Lemma 1.4.11).

By the Faá Di Bruno's formula and the fact that

$$\sum_{(k_1,\ldots,k_j)\in I_j} \frac{(k_1+\ldots+k_j)!}{k_1!\ldots k_j!} = 2^{j-1}$$

for all $j \in \mathbb{N}$ (see, for instance, [11] for a proof of this identity), it follows that

$$\begin{aligned}
&|(1+|x|)^q (f\circ\psi_m)^{(j)}(x)| \\
&\leq C_0^q (1+|\psi_m(x)|)^q \sum_{(k_1,\ldots,k_j)\in I_j} \frac{j!}{k_1!\ldots k_j!} |f^{(k)}(\psi_m(x))| \prod_{\ell=1}^{j} \left| \frac{\psi_m^{(\ell)}(x)}{\ell!} \right|^{k_\ell} \\
&\leq C_0^q \sum_{(k_1,\ldots,k_j)\in I_j} \frac{j!}{k_1!\ldots k_j!} |f^{(k)}(\psi_m(x))| B_{m,\mu}^k \frac{\exp\left(\mu\varphi_\omega^*\left(\frac{j-k}{\mu}\right)\right)}{(j-k)!} (1+|\psi_m(x)|)^{k+q} \\
&\leq 2^j C_0^q p_{\omega,\mu}(f) \sum_{(k_1,\ldots,k_j)\in I_j} \frac{k!}{k_1!\ldots k_j!} \exp\left(\mu\varphi_\omega^*\left(\frac{2k+q+j-k}{\mu}\right)\right) B_{m,\mu}^k \\
&\leq \square_{m,\mu,\delta} 2^j C_0^q p_{\omega,\mu}(f) \sum_{(k_1,\ldots,k_j)\in I_j} \frac{k!}{k_1!\ldots k_j!} \exp\left(\mu\varphi_\omega^*\left(\frac{(1+\delta)k+q+j}{\mu}\right)\right) \\
&\leq \square_{m,\mu,\delta}\, 4^j C_0^q p_{\omega,\mu}(f) \exp\left(\mu\varphi_\omega^*\left(\frac{(2+\delta)(q+j)}{\mu}\right)\right) \\
&\leq \square_{m,\mu,\delta} L_\lambda \exp\left(\lambda\varphi_\omega^*\left(\frac{(2+\delta)(q+j)}{\lambda}\right)\right) p_{\omega,\mu}(f)
\end{aligned}$$

for all $j \in \mathbb{N}, q \in \mathbb{N}_0$, $x \in \mathbb{R}$, $m \in \mathbb{N}$ and $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$. Since $\lambda > 0$ and $\delta > 0$ were arbitrary, the proof is complete. □

As in Corollary 3.3.16, we can deduce the following immediate consequence in the case where the weight ω satisfies the additional rather technical condition (4.1), which power of logarithms are particular cases of:

Corollary 4.3.2. *Let ω be a sub-additive weight such that the following condition is satisfied:*

$$\exists \gamma > 1 \; \exists C \geq 1 \; \forall t \geq 0: \; \omega(t^\gamma) \leq C\omega(t) + C. \tag{4.1}$$

If ψ is a polynomial of degree strictly greater than one, then the family of iterates $\{C_{\psi_m} : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\omega)}(\mathbb{R}) : m \in \mathbb{N}\}$ is topologizable.

In the following result we show that Theorem 4.3.1 cannot be improved to obtain the m-topologizability of the composition operator C_ψ in the same setting, unlike what happens in the classical Schwartz space setting (see, for instance, [13, Example 4.14.]). In particular, we establish that there is at least one polynomial ψ of degree greater than one and a sub-additive weight ω for which $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{\omega(\bullet^{\frac{1}{2}})}(\mathbb{R})$ is not $m-$topologizable.

Proposition 4.3.3. *If $\psi(x) = x^2$ for all $x \in \mathbb{R}$ and $\omega(t) = |t|^{\frac{1}{s}}$, with $s > 1$, then $\{C_{\psi_m} : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{\omega(\bullet^{\frac{1}{2}})}(\mathbb{R}) : m \in \mathbb{N}\}$ is not $m-$topologizable.*

Proof. We denote $\sigma(t) := \omega(t^{\frac{1}{2}})$. We observe that

$$\exp\left(-\lambda\varphi^*_\sigma\left(\frac{m}{\lambda}\right)\right) = \left(\frac{\lambda e}{2sm}\right)^{2sm}$$

for all $m \in \mathbb{N}$ and $\lambda > 0$, as an easy computation shows. We also note that $\psi_m(x) = x^{2^m}$, for all $x \in \mathbb{R}$ and $m \in \mathbb{N}$.

We suppose that the family of iterates $\{C_{\psi_m} : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_\sigma(\mathbb{R}) : m \in \mathbb{N}\}$ is $m-$topologizable, i.e. for every $\lambda > 0$ there are $\mu > 0$ and $C > 0$ such that $p_{\sigma,\lambda}(C_{\psi_m} f) \leq C^m p_{\omega,\mu}(f)$ for all $m \in \mathbb{N}$ and $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$. Accordingly,

$$\sup_{x\in\mathbb{R}, j,q\in\mathbb{N}_0} (1+|x|)^q |(f\circ\psi_m)^{(j)}(x)| \exp\left(-\lambda\varphi^*_\sigma\left(\frac{j+q}{\lambda}\right)\right) \leq C^m p_{\omega,\mu}(f) \quad (4.2)$$

for all $m \in \mathbb{N}$ and $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$. Since ω is a strong weight, we can select $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$ so that $f'(1) = 1$ and $f^{(h)}(1) = 0$ for all $h \geq 2$. By Faá Di Bruno's formula, it follows that

$$|(f\circ\psi_m)^{(j)}(1)| = 2^m(2^m-1)\dots(2^m-j+1)$$

for all $m \in \mathbb{N}$, $j \leq 2^m$. Observe that the following inequality is satisfied

$$2^m(2^m-1)\dots(2^m-m+1) \geq (2^m-m+1)^m \geq 2^{\frac{m^2}{2}}$$

for all $m \in \mathbb{N}$ large enough.

If we put $q = 0, x = 1, j = m (\leq 2^m)$ in (4.2) and use the inequality above, we get the following estimate:

$$p_{\omega,\mu}(f)C^m \geq [2^m(2^m-1)\dots(2^m-m+1)]\exp\left(-\lambda\varphi^*_\sigma\left(\frac{m}{\lambda}\right)\right) \geq 2^{\frac{m^2}{2}}\left(\frac{\lambda e}{2sm}\right)^{2sm}$$

for all $m \in \mathbb{N}$ large enough. This would imply that there is $Q = Q(f, \lambda, s) > 0$ such that

$$Q \geq \frac{2^{\frac{m}{2}}}{m^{2s}}$$

for all $m \in \mathbb{N}$, which is a contradiction with the obvious fact that

$$\frac{2^{\frac{m}{2}}}{m^{2s}} \to \infty$$

as $m \to \infty$. $\square$

Notice that $\psi'(1) = 2 > 1$ and $\psi(1) = 1$, i.e. 1 is a repelling fixed point of ψ. Let us show the following rather more general result:

Theorem 4.3.4. *Let ψ be a polynomial of degree greater than one that possesses at least one repelling fixed point, i.e. there is $x_0 \in \mathbb{R}$ such that $\psi(x_0) = x_0$ and $|\psi'(x_0)| > 1$. Then, the family of iterates $\{C_{\psi_m} : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{\omega(\bullet^{\frac{1}{2}})}(\mathbb{R}) : m \in \mathbb{N}\}$ is not $m-$topologizable, where $\omega = |\bullet|^{\frac{1}{d}}$, with $d > 1$.*

Proof. Set $\sigma(t) := \omega(t^{\frac{1}{2}})$ and $\alpha := |\psi'(x_0)| > 1$. Proceeding by contradiction, we assume that for every $\lambda > 0$ there are $\mu > 0$, $C > 0$ such that

$$p_{\sigma,\lambda}(f \circ \psi_m) \leq C^m p_{\omega,\mu}(f) \tag{4.3}$$

for all $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$, $m \in \mathbb{N}$. Fix $\lambda > 0$ and $\mu > 0$ so that (4.3) holds. We also recall that

$$\varphi_\omega^*(x) = xd \log\left(\frac{xd}{e}\right), \varphi_\sigma^*(x) = 2xd \log\left(\frac{2xd}{e}\right)$$

for all $x \geq 0$, and hence, we have that

$$\exp(-\lambda \varphi_\sigma^*(\frac{j}{\lambda})) = \left(\frac{\lambda e}{2dj}\right)^{2dj} = A_{\lambda,d}^j \frac{1}{j^{2dj}},$$

for all $j \in \mathbb{N}$, where $A_{\lambda,d} = (\frac{\lambda e}{2d})^{2s} > 0$. Without loss of generality, we may assume $A_{\lambda,d} > 1$; otherwise we take $\lambda > 0$ larger in order to ensure so.

Proceeding as in the proof of [11, Theorem 3.5.], since

$$a_j = \exp\left(\log(j)\varphi_\omega^*(\frac{j}{\log(j)})\right) = B^j \left(\frac{j}{\log(j)}\right)^{jd},$$

for all $j \in \mathbb{N}$, where $B \equiv B_d = (\frac{d}{e})^d > 0$, we can construct a sequence of functions $(f_\ell)_\ell \subset \mathcal{S}_{(\omega)}(\mathbb{R})$ that is bounded in $\mathcal{S}_{(\omega)}(\mathbb{R})$ and that verifies:

$$f_\ell^{(j)}(x_0) = \delta_\ell^j B^j \left(\frac{j}{\log(j)}\right)^{jd},$$

for all $\ell \in \mathbb{N}, j \in \mathbb{N}$, where $\delta_\ell^j = 0$ whenever $\ell \neq j$ and $\delta_\ell^j = 1$ otherwise.

Obviously, $\psi_m(x_0) = x_0$ and $\psi'_m(x_0) = \psi'(x_0)^m$ for all $m \in \mathbb{N}$. So, after applying Faa Di Bruno's formula (see, for instance, [11, Lemma 4.1.]), we obtain that

$$\begin{aligned}|(f_j \circ \psi_m)^{(j)}(x_0)|\, A_{\lambda,d}^j \frac{1}{j^{2dj}} &= |f_j^{(j)}(x_0)\psi'_m(x_0)^j|\, A_{\lambda,d}^j \frac{1}{j^{2dj}} \\ &= (A_{\lambda,d}B)^j \left(\frac{j}{\log(j)}\right)^{jd} \alpha^{jm} \frac{1}{j^{2dj}} \\ &= (A_{\lambda,d}B)^j \left(\frac{1}{j\log(j)}\right)^{jd} \alpha^{jm}\end{aligned}$$

for all $j \in \mathbb{N}$, $m \in \mathbb{N}$. Now, putting $j = m$ in the above equality, we get

$$|(f_m \circ \psi_m)^{(m)}(x_0)|\, A_{\lambda,d}^m \frac{1}{m^{2dm}} \geq \left(\frac{1}{m\log(m)}\right)^{md} \alpha^{m^2} (A_{\lambda,d}B)^m \tag{4.4}$$

for all $m \in \mathbb{N}$. Using the definition of $p_{\sigma,\lambda}(f \circ \psi_m)$ (with $q = 0, x = x_0, j = m$), the fact that there is $D_\mu > 0$ so that $p_{\omega,\mu}(f_m) \leq D_\mu$, for all $m \in \mathbb{N}$ and combining (4.3) with (4.4), we obtain:

$$D_\mu C^m \geq \left(\frac{1}{m\log(m)}\right)^{md} \alpha^{m^2} (A_{\lambda,d}B)^m$$

for all $m \in \mathbb{N}$. In turn, this implies that there is a constant $H_{\lambda,\mu,d} > 0$ such that

$$\left(\frac{\alpha^{\frac{m}{d}}}{m\log(m)}\right)^d \leq H_{\lambda,\mu,d}$$

for all $m \in \mathbb{N}$, which is a contradiction with the fact that

$$\left(\frac{\alpha^{\frac{m}{d}}}{m\log(m)}\right) \to \infty$$

as $m \to \infty$. $\square$

Recall that if ψ is a polynomial of degree greater than or equal to two, then the equicontinuity of the sequence of the iterates of the composition operator $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_\sigma(\mathbb{R})$, with $\sigma(t) = \omega(t^{\frac{1}{a}})$ for $a > 2$, is equivalent to the lack of fixed points of the polynomial ψ (Proposition 3.3.15). On the other hand, equicontinuity implies always m-topologizability. Let us observe that if $\psi(x) = x^2 + \frac{1}{4}$ one has that $\psi(x) - x = (x - \frac{1}{2})^2$ and $\psi'(\frac{1}{2}) = 1$. So we could not apply Theorem 4.3.4 and hence, we do not know yet if the associated composition operator $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{\omega(\bullet^{\frac{1}{2}})}(\mathbb{R})$ is $m-$topologizable or not. More generally, the following problem arises: Open Problem 7.

4.4 Topologizability (and m-topologizability) of the composition operator associated with a polynomial ψf degree one on Gelfand-Shilov classes

The aim of this section is to study the topologizability of the composition operator C_ψ acting on Gelfand-Shilov classes, when the polynomial ψ is of degree one. In this case, we know that the composition operator $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\omega)}(\mathbb{R})$ is continuous and hence, this is the usual and proper setting to work in. In Proposition 3.3.10, it was proved that if $\psi(x) = ax + b$, with $a \neq 0$, for $x \in \mathbb{R}$ then, $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\omega)}(\mathbb{R})$ is power bounded if and only if $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\omega)}(\mathbb{R})$ is mean ergodic if and only if $\psi(x) = x$ for $x \in \mathbb{R}$ or $\psi(x) = -x$ for $x \in \mathbb{R}$.

It is useful to notice that if two polynomials ψ, ϕ are linearly equivalent, then the composition operator C_ψ is clearly topologizable ($m-$topologizable, respectively) if and only if the composition operator C_ϕ is topologizable ($m-$topologizable, respectively). The proof is almost immediate when one observes that if $\psi = \ell \circ \phi \circ \ell^{-1}$, with ℓ being a non-constant affine function, then $C_{\psi_m} = C_\ell^{-1} \circ C_{\phi_m} \circ C_\ell$ for all $m \in \mathbb{N}$, where the composition operator $C_\ell : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\omega)}(\mathbb{R})$ is clearly an isomorphism onto. Therefore, we may assume without loss of generality that either $\psi(x) = ax$, for all $x \in \mathbb{R}$, with $a \neq 0, \pm 1$, or $\psi(x) = x + 1$, for all $x \in \mathbb{R}$.

First, we deal with the case where ψ is a translation. We know that composition operators acting on Gelfand-Shilov classes associated with translations are not power bounded (see Propositon 3.3.10). However, as in the Schwartz class (see [13, Example 4.14.]), we still have $m-$topologizability.

Proposition 4.4.1. *Let* ω *be a weight function and* $\psi(x) = x+1$, *for* $x \in \mathbb{R}$. *Then the composition operator* $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\omega)}(\mathbb{R})$ *is m-topologizable.*

Proof. Fix $\lambda > 0, \mu > 0$ and observe that that

$$q_{\omega,\lambda,\mu}(C_{\psi_m} f) = \sup_{j\in\mathbb{N}_0} \sup_{y\in\mathbb{R}} |f^{(j)}(y)| e^{-\lambda\varphi^*_\omega(\frac{j}{\lambda})+\mu\omega(y-m)},$$

for every $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$ and $m \in \mathbb{N}$.

Since $\mu\omega(y-m) \leq \mu L\,(1+\omega(y)+\omega(m))$, for all $y \in \mathbb{R}$ and $m \in \mathbb{N}$, it follows, for every $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$, $\lambda > 0$ and $\mu > 0$, that

$$q_{\omega,\lambda,\mu}(C_{\psi_m} f) \leq \exp\left(\mu L(1+\omega(m))\right) q_{\omega,\lambda,\mu L}(f),$$

for all $m \in \mathbb{N}$.

It is well-known that condition (β) in Definition 1.4.7 together with the fact that ω is increasing imply that $\frac{\omega(t)}{t} \to 0$ as $t \to \infty$. Therefore, there is $Q > 0$ such that $\omega(t) \leq Qt$ for all $t \geq 0$. It follows for every $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$ that

$$q_{\omega,\lambda,\mu}(C_{\psi_m} f) \leq \exp\left(\mu L\right) \exp\left(\mu LQ\,m\right) q_{\omega,\lambda,\mu L}(f),$$

for all $m \in \mathbb{N}$. This clearly completes the proof. □

Remark 4.4.2. *This result is still true for weaker classes of Braun-Meise-Taylor weight functions* ω, *provided that* ω *verifies the condition that* $\omega(t) \leq Qt$, *for all* $t > 0$ *large enough and for some* $Q > 0$ *(see, for instance, [92] and the references therein for a survey of such classes).*

In the setting of the Gelfand-Shilov spaces $\Sigma_s(\mathbb{R})$, $s > 1$, an interesting and non-obvious consequence from Proposition 4.4.1 can be derived by using the equivalent fundamental system of semi-norms $(p_\lambda)_{\lambda>0}$ for $\Sigma_s(\mathbb{R})$ given after Definiton 1.5.3 rather than $(q_{\omega,\lambda,\mu})_{\lambda>0,\mu>0}$.

Corollary 4.4.3. *Let* $s > 1$. *Then there is* $\lambda > 0$ *such that for every* $A > 0$ *there is no* $f \in \Sigma_s(\mathbb{R}) \setminus \{0\}$ *such that the conditions* $\mathrm{supp} f \subset [-A, A]$ *and* $p_\lambda(f)$ *is attained in* $j, q \in \mathbb{N}_0$ *with* $q - j$ *arbitrarily large are simultaneously satisfied.*

Proof. If $\psi(x) = x + 1$, for $x \in \mathbb{R}$, by Proposition 4.4.1 the composition operator $C_\psi \colon \Sigma_s(\mathbb{R}) \to \Sigma_s(\mathbb{R})$ is m-topologizable. Therefore, there exist $\mu \geq 1$ and $D \geq 1$ such that for each $m \in \mathbb{N}$

$$\begin{aligned} p_1(f(\bullet + m)) &= \sup_{j,q \in \mathbb{N}_0} \sup_{x \in \mathbb{R}} \left(\frac{|x|}{|x+m|} \right)^q \frac{|x+m|^q \, |f^{(j)}(x+m)|}{j!^s q!^s} \\ &\leq D^m \, p_\mu(f) \end{aligned} \tag{4.5}$$

for all $f \in \Sigma_s(\mathbb{R})$. Without loss of generality, we may assume that $\log_\mu(D) \geq 1$ and hence, $\log_\mu(D^m) \geq 1$ for all $m \in \mathbb{N}$.

Proceeding by contradiction, we suppose that there exist $A > 0$ and sequences $(f_m)_m \subset \Sigma_s(\mathbb{R}) \setminus \{0\}$ with $supp f_m \subset [-A, A]$, $(q_m)_m \subset \mathbb{N}_0$ and $(j_m)_m \subset \mathbb{N}_0$ such that $p_\mu(f_m) = \sup_{x \in \mathbb{R}} |x|^{q_m} |f_m^{(j_m)}(x)| \frac{\mu^{j_m + q_m}}{j_m!^s q_m!^s}$ and $q_m - j_m \geq \log_\mu(D^m) > 0$, for all $m \in \mathbb{N}$. Since $\frac{m-A}{A} \to \infty$ as $m \to \infty$, we have that $\frac{m-A}{A} > \mu^2$ for all $m \in \mathbb{N}$ large enough. So, by inequality (4.5) we obtain that

$$\begin{aligned} D^m p_\mu(f_m) &\geq \left(\frac{m-A}{A} \right)^{q_m} \left(\frac{1}{\mu} \right)^{j_m + q_m} p_\mu(f_m) \\ &> \mu^{q_m - j_m} \, p_\mu(f_m) > D^m \, p_\mu(f_m) \end{aligned}$$

for all $m \in \mathbb{N}$ large enough. This is clearly a contradiction. □

We now investigate the topologizability of C_ψ when ψ is a dilatation, i.e., $\psi(x) = ax$, for $x \in \mathbb{R}$, with $a \neq 0$. To this end, some results are needed.

Lemma 4.4.4. *Let ω be a weight function, $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$ and $\lambda > 0$. Then for every $\varepsilon > 0$ there is $M = M(\varepsilon) > 0$ such that for all $j, q \in \mathbb{N}_0$ verifying $j + q \geq M$ one has that* $\sup_{x \in \mathbb{R}} |x|^q |f^{(j)}(x)| \exp\left(-\lambda \varphi^*_\omega(\frac{j+q}{\lambda})\right) \leq \varepsilon$.

Proof. By Lemma 1.4.11, for the given $\lambda > 0$ there are $\mu > 0$, $A > 1$ and $D > 0$ such that

$$\exp\left(-\lambda \varphi^*_\omega\left(\frac{j+q}{\lambda}\right)\right) \leq D \left(\frac{1}{A}\right)^{j+q} \exp\left(-\mu \varphi^*_\omega\left(\frac{j+q}{\mu}\right)\right)$$

for all $j, q \in \mathbb{N}_0$. Since $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$, we have that

$$\sup_{x \in \mathbb{R}} |x|^q |f^{(j)}(x)| \exp\left(-\lambda \varphi^*_\omega\left(\frac{j+q}{\lambda}\right)\right) \leq D \left(\frac{1}{A}\right)^{j+q} p_{\omega,\mu}(f)$$

for all $j, q \in \mathbb{N}_0$, with $p_{\omega,\mu}(f) < \infty$. The result easily follows. □

Remark 4.4.5. *Let ω be a weight function. For $\lambda > 0$ and $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$ given, note that Lemma 4.4.4 implies that there is a finite number of pairs $(j,q) \in \mathbb{N}_0^2$ for which the supremum in the norm*

$$p_\lambda(f) := \sup_{j,q\in\mathbb{N}_0} \sup_{x\in\mathbb{R}} |x|^q |f^{(j)}(x)| \exp\left(-\lambda\varphi_\omega^*\left(\frac{j+q}{\lambda}\right)\right)$$

is attained (the system $\{p_\lambda : \lambda > 0\}$ of norms also generates the topology of $\mathcal{S}_{(\omega)}(\mathbb{R})$).

Proposition 4.4.6. *For every $\lambda > 0$ and $m \in \mathbb{N}$, there is $g \in \mathcal{S}_{(\omega)}(\mathbb{R})$ for which the supremum in $p_\lambda(g)$ is only attained in $j, q \in \mathbb{N}_0$ verifying $q - j \geq m$. In particular, $q \geq m$.*

Proof. Fix $f \in \mathcal{S}_{(\omega)}(\mathbb{R}) \setminus \{0\}$, $\lambda > 0$, $m \in \mathbb{N}$. For the sake of brevity, denote $a_{j,q} = \max_{x\in\mathbb{R}} |x|^q\, |f^{(j)}(x)| \exp\left(-\lambda\varphi_\omega^*(\frac{j+q}{\lambda})\right)$, for all $j, q \in \mathbb{N}_0$. Observe that $a_{j,q} > 0$ for all $j, q \in \mathbb{N}_0$ because p_λ is a continuous norm over $\mathcal{S}_{(\omega)}(\mathbb{R})$. Consider the function $g(x) = f(\rho x)$, for all $x \in \mathbb{R}$, where $0 < \rho < 1$ to be chosen later on. Clearly, $g \in \mathcal{S}_{(\omega)}(\mathbb{R}) \setminus \{0\}$ and also,

$$\max_{x\in\mathbb{R}} |x|^q\, |g^{(j)}(x)| \exp\left(-\lambda\varphi_\omega^*(\frac{j+q}{\lambda})\right) = \rho^{j-q}\, a_{j,q} \tag{4.6}$$

for all $j, q \in \mathbb{N}_0$. Fix $m_0 > m$. By Lemma 4.4.4, there is $M > 0$ such that for all $j, q \in \mathbb{N}_0$ verifying that $j + q \geq M$ one has that $a_{j,q} \leq \frac{\min\{a_{r,\ell} : r \leq m_0, \ell \leq m_0\}}{2}$. Now we choose $0 < \rho < 1$ small enough so that the sequence $\{\rho^{-k}\, a_{j,j+k} : k \geq -j\}$ is strictly increasing up to m, for each $0 \leq j \leq M$. This choice is possible because the condition:

$$Q^{k+1}\, a_{j,j+k+1} > Q^k\, a_{j,j+k}$$

for some $Q > 0$, is equivalent to $Q > \frac{a_{j,j+k}}{a_{j,j+k+1}}$, and hence we only need to take $\rho^{-1} > \max\{\frac{a_{j,j+k}}{a_{j,j+k+1}} : j \in \{0, 1, ..., M\}, k \in \{-j, -j+1, ..., 0, ..., m\}\}$. This choice of $\rho > 0$ implies that the maximum of the sequence $\{\rho^{-k} a_{j,j+k} : k \geq -j\}$ is only achieved in $k \geq m$. Denote $L_0 := \max\{\rho^{-k}\, a_{j,j+k} : j \in \mathbb{N}_0, k \geq -j\}$. Since $a_{0,1} > 0$, we can also assume, without loss of generality, that

$$\rho < \frac{a_{0,1}}{\max\{a_{r,\ell} : r, \ell \in \mathbb{N}_0\}}.$$

By doing so, we guarantee that the following inequality holds:

$$L_0 \geq (\frac{1}{\rho})^1 a_{0,1} > \max\{a_{r,\ell} : r, \ell \in \mathbb{N}_0\}.$$

Now we observe that for all $j \geq M$, it is not possible neither to have that $\rho^{-k} a_{j,j+k} = L_0$ with $k \leq 0$. Otherwise, since $\rho < 1$ we would have for $k \leq 0$ that

$$L_0 > \max\{a_{j,j+k} : j \in \mathbb{N}_0,\, k \geq -j\} \geq \rho^{-k} a_{j,j+k} = L_0,$$

which obviously is a contradiction. Finally, we will see that it is not possible to have that $\rho^{-k} a_{j,j+k} = L_0$, with $j \geq M$ and $k \leq m$. Indeed, if we suppose that $\rho^{-k} a_{j,j+k} = L_0$, with $j \geq M$ and $k \leq m$, then we get

$$L_0 = (\frac{1}{\rho})^k a_{j,j+k} \leq (\frac{1}{\rho})^m a_{j,j+k} < (\frac{1}{\rho})^m a_{0,m} \leq L_0$$

since $\frac{1}{\rho} > 1$, which obviously is a contradiction and we are done. $\square$

Remark 4.4.7. *Switching the roles of j and q and choosing $\rho > 1$ large enough in the proof of Proposition 4.4.6, we obtain that for every $\lambda > 0$ and $m \in \mathbb{N}$, there is $g \in \mathcal{S}_{(\omega)}(\mathbb{R})$ for which the supremum in $p_\lambda(g)$ is only attained in $j, q \in \mathbb{N}_0$ verifying $j - q \geq m$. In particular, $j \geq m$.*

As far as we know, there is no literature available about whether all $3-$tuples $(j, q, \lambda) \in \mathbb{N}_0 \times \mathbb{N}_0 \times \mathbb{R}_+$ verify that there is $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$ such that $p_\lambda(f) = \sup_{x\in\mathbb{R}} |x|^q |f^{(j)}(x)| \exp\left(-\lambda\varphi_\omega^*\left(\frac{j+q}{\lambda}\right)\right)$, not even when the space $\mathcal{S}_{(\omega)}(\mathbb{R})$ is a classical Gelfand-Shilov class. We state the problem explicitly in the last chapter (see Open Problem 8), for which we only have a partial answer.

Other useful fact is the following result:

Proposition 4.4.8. *Let ω be a weight function. Let $a \in \mathbb{R}\backslash\{\pm 1, 0\}$, $\psi(x) := ax$ and $\Phi(x) = \frac{x}{a}$, for $x \in \mathbb{R}$. The following conditions are equivalent.*

1. *The composition operator $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\omega)}(\mathbb{R})$ is topologizable.*
2. *The composition operator $C_\Phi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\omega)}(\mathbb{R})$ is topologizable.*

.

Proof. The result follows by observing that for all $b \neq 0, \eta \in \mathbb{R}$ one has the following equalities:

$$\begin{aligned}(\mathcal{F}(f(b\bullet)))(\eta) &= \int_{\mathbb{R}} e^{-i\eta} f(bx)dx = \frac{1}{b}\int_{\mathbb{R}} e^{-i\eta\frac{y}{b}} f(y)dy \\ &= \frac{1}{b}(\mathcal{F}f)(\frac{\eta}{b}),\end{aligned}$$

where the Fourier transform $\mathcal{F} : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\omega)}(\mathbb{R})$ is an isomorphism onto. Accordingly, it follows that $\mathcal{F} \circ C_{\psi_m} = (\frac{1}{a^m} C_{\Phi_m}) \circ \mathcal{F}$ for all $m \in \mathbb{N}$. Putting these facts together, we see that the family of iterates $\{C_{\psi_m} : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\omega)}(\mathbb{R}) : m \in \mathbb{N}\}$ is topologizable if and only if $\{\frac{1}{a^m} C_{\Phi_m} : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\omega)}(\mathbb{R}) : m \in \mathbb{N}\}$ is topologizable if and only if the family of iterates $\{C_{\Phi_m} : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\omega)}(\mathbb{R}) : m \in \mathbb{N}\}$ is topologizable, as we wanted to show. □

We know that composition operators of non-trivial dilatations are not power bounded (see Proposition 3.3.10). Contrary to what was expected from the case of translations worked out above, composition operators of non-trivial dilatations are not topologizable on Gelfand-Shilov classes. Surprisingly enough, it depends on the possibility of finding $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$ whose seminorm $p_\lambda(f)$ is attained in $j, q \in \mathbb{N}_0$ verifying conditions like the one appearing in Proposition 4.4.6. The following result, whose proof is very different from the techniques used in previous chapters, is also unexpected from the classical Schwartz class $\mathcal{S}(\mathbb{R})$ ([13, Example 4.15.])), where $C_\psi : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is always $m-$topologizable when $\psi(x) = ax$, for $x \in \mathbb{R}$.

Theorem 4.4.9. *Let ω be a weight function and $\psi(x) := ax$, for all $x \in \mathbb{R}$, with $a \neq 0$. If the weight function ω satisfies condition* (1.14), *i.e. there exists $H \geq 1$ such that*

$$2\omega(t) \leq \omega(Ht) + H$$

for all $t \geq 0$. Then the following conditions are equivalent.

1. *$C_\psi \colon \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\omega)}(\mathbb{R})$ is power bounded.*
2. *$C_\psi \colon \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\omega)}(\mathbb{R})$ is m-topologizable.*
3. *$C_\psi \colon \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\omega)}(\mathbb{R})$ is topologizable.*
4. *$\{C_{\psi_m} : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}(\mathbb{R}) : m \in \mathbb{N}\}$ is equicontinuous.*
5. *$a = \pm 1$.*

Proof. Clearly, 1. $\Rightarrow$ 2. $\Rightarrow$ 3. holds. By [12, Proposition 3.1], we have that 1. $\Leftrightarrow$ 4. $\Leftrightarrow$ 5.. So, to conclude the proof, it suffices to show that if $\psi(x) = ax$, for $x \in \mathbb{R}$, with $a \neq \pm 1$, then the composition operator $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\omega)}(\mathbb{R})$ is not topologizable. To this end, we first observe that $\psi_m(x) = a^m x$, for all $x \in \mathbb{R}$ and $m \in \mathbb{R}$. Moreover, via the properties of φ^*_ω (see Lemma 1.4.11), we can assume that

$$p_\lambda(f) = \sup_{j,q\in\mathbb{N}_0} \sup_{x\in\mathbb{R}} |x|^q |f^{(j)}(x)| \exp\left(-\lambda\varphi^*_\omega(\frac{j}{\lambda})\right) \exp\left(-\lambda\varphi^*_\omega(\frac{q}{\lambda})\right),$$

for every $\lambda > 0$ and $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$. On the other hand, the sequence $\{p_k\}_{k\in\mathbb{N}}$ of norms generates the topology of $\mathcal{S}_{(\omega)}(\mathbb{R})$.

Now, we assume that the composition operator $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\omega)}(\mathbb{R})$ is topologizable. By Proposition 4.4.8, we may also assume that $|a| > 1$ and hence, $\frac{1}{|a|^{mq}} < 1$ for all $m, q \in \mathbb{N}$. Therefore, for every $k \in \mathbb{N}$ there exists $h \in \mathbb{N}$ with $h \geq k$ and a sequence of positive constants $\{C_m\}_{m\in\mathbb{N}} \subset \mathbb{R}_+$ such that

$$p_k(C_{\psi_m} f) \leq C_m \, p_h(f)$$

for every $f \in \mathcal{S}_\omega(\mathbb{N})$ and $m \in \mathbb{R}$, i.e., applying Lemma 4.4.4 (cf. Remark 4.4.5),

$$\begin{aligned} & p_k(C_{\psi_m} f) \\ & = \sup_{j,q\in\mathbb{N}_0} \sup_{x\in\mathbb{R}} |x|^q |f^{(j)}(a^m x)| |a|^{mj} \exp\left(-k\varphi^*_\omega\left(\frac{q}{k}\right)\right) \exp\left(-k\varphi^*_\omega\left(\frac{j}{k}\right)\right) \\ & \leq C_m p_h(f) = C_m \sup_{y\in\mathbb{R}} |y|^{\overline{q}} |f^{(\overline{j})}(y)| \exp\left(-h\varphi^*_\omega\left(\frac{\overline{q}}{h}\right)\right) \exp\left(-h\varphi^*_\omega\left(\frac{\overline{j}}{h}\right)\right) \end{aligned}$$

where $\overline{j}, \overline{q} \in \mathbb{N}_0$ depend only on f and $h \in \mathbb{N}$. On the other hand, we also have that

$$\begin{aligned} & p_k(C_{\psi_m} f) \\ & = \sup_{j,q\in\mathbb{N}_0} \sup_{y\in\mathbb{R}} |y|^q |a|^{m(j-q)} |f^{(j)}(y)| \exp\left(-k\varphi^*_\omega\left(\frac{q}{k}\right)\right) \exp\left(-k\varphi^*_\omega\left(\frac{j}{k}\right)\right) \\ & \geq |a|^{m(\overline{j}-\overline{q})} \frac{\exp(-k\varphi^*_\omega(\frac{\overline{q}}{k})) \exp(-k\varphi^*_\omega(\frac{\overline{j}}{k}))}{\exp(-h\varphi^*_\omega(\frac{\overline{q}}{h})) \exp(-h\varphi^*_\omega(\frac{\overline{j}}{h}))} p_h(f), \end{aligned}$$

for every $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$ and $m \in \mathbb{N}$. Combining the inequalities above, it follows that

$$|a|^{m(\overline{j}-\overline{q})} \frac{\exp(-k\varphi^*_\omega(\frac{\overline{q}}{k}))\exp(-k\varphi^*_\omega(\frac{\overline{j}}{k}))}{\exp(-h\varphi^*_\omega(\frac{\overline{q}}{h}))\exp(-h\varphi^*_\omega(\frac{\overline{j}}{h}))} \leq C_m \tag{4.7}$$

for every $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$ and $m \in \mathbb{N}$, where $\overline{j}, \overline{q} \in \mathbb{N}_0$ depend only on f and $h \in \mathbb{N}$. We observe that (4.7) is equivalent in turn to

$$m(\overline{j}-\overline{q})\log(|a|) + h\varphi^*_\omega(\frac{\overline{q}}{h}) - k\varphi^*_\omega(\frac{\overline{q}}{k}) + h\varphi^*_\omega(\frac{\overline{j}}{h}) - k\varphi^*_\omega(\frac{\overline{j}}{k}) \leq \log(C_m),$$

for all $m \in \mathbb{N}$, where $\overline{j}, \overline{q} \in \mathbb{N}_0$ depend only on f and $h \in \mathbb{N}$, as it easy to prove.

Since ω is a weight function satisfying condition (1.14), we have that $\mathcal{S}_{(\omega)}(\mathbb{R}) = \mathcal{S}_M(\mathbb{R})$, [29, Corollary 16], for some weight sequence $M = (M_p)_{p\in\mathbb{N}_0}$. Accordingly, using the system $\{p_{M,h}\}_{h>0}$ of norms generating the topology of $\mathcal{S}_M(\mathbb{R})$, the inequality (4.7) is equivalent to

$$|a|^{m(\overline{j}-\overline{q})} \frac{A^{\overline{j}+\overline{q}}}{B^{\overline{j}+\overline{q}}} \leq C_m$$

for all $m \in \mathbb{N}$, and for some $A, B > 0$. This is in turn equivalent to

$$\left(m\log(|a|) + \log\left(\frac{A}{B}\right)\right)\overline{j} - \overline{q}\left(m\log(|a|) - \log\left(\frac{A}{B}\right)\right) \leq \log(C_m) \tag{4.8}$$

for all $m \in \mathbb{N}$. Observe that $m\log(|a|) - \log\left(\frac{A}{B}\right) > 0$ and $m\log(|a|) + \log\left(\frac{A}{B}\right) > 0$ for $m \in \mathbb{N}$ large enough.

By Proposition (4.4.6) (see also Remark 4.4.7) we can construct sequences $(f_\ell)_\ell \subset \mathcal{S}_{(\omega)}(\mathbb{R}) = \mathcal{S}_M(\mathbb{R})$, $(j_\ell)_\ell \subset \mathbb{N}_0, (q_\ell)_\ell \subset \mathbb{N}$ such that the supremum involving $p_{M,h}(f_\ell)$ is attained for j_ℓ and q_ℓ satisfying $j_\ell - q_\ell \geq \ell$ for all $\ell \in \mathbb{N}$. Therefore, $j_l \to \infty$ as $l \to \infty$. Moroever, two different cases can occur:

1. $\limsup_{\ell\to\infty} \frac{j_\ell}{q_\ell} = +\infty$.
2. $\limsup_{\ell\to\infty} \frac{j_\ell}{q_\ell} < +\infty$.

First, assume that $\limsup_{\ell\to\infty} \frac{j_\ell}{q_\ell} = +\infty$. Accordingly, there exists a subsequence $\left(\frac{j_{\ell_h}}{q_{\ell_h}}\right)_h$ such that $\frac{j_{\ell_h}}{q_{\ell_h}} \to \infty$ as $h \to \infty$, and hence, $\frac{q_{\ell_h}}{j_{\ell_h}} \to 0$ as $h \to \infty$. From (4.8), it follows that

$$j_{\ell_h}\left[\left(m\log(|a|) + \log\left(\frac{A}{B}\right)\right) - \frac{q_{\ell_h}}{j_{\ell_h}}\left(\log(|a|) - \log\left(\frac{A}{B}\right)\right)\right] \le \log(C_m), \tag{4.9}$$

for all $h \in \mathbb{N}$ and for $m \in \mathbb{N}$ large enough. For any fixed $m \in \mathbb{N}$ large enough, letting $h \to \infty$ we obtain a contradiction because the left-hand side of (4.9) obviously tends to $+\infty$ as $h \to \infty$ and the right-hand side of (4.9) continues to be equal to $\log(C_m) \in \mathbb{R}$.

Next, assume that $\limsup_{l\to\infty} \frac{j_\ell}{q_\ell} < L$ for some $L > 0$. Of course, $L \ge 1$. Then, by definition of $\limsup_{l\to\infty}$, there is $h_0 > 0$ so that

$$\sup_{\ell \ge h_0} \frac{j_\ell}{q_\ell} < L,$$

and hence, $j_\ell < Lq_\ell$ for all $\ell \ge h_0$. So, we have that

$$q_\ell \le j_\ell < Lq_\ell \tag{4.10}$$

for all $\ell \ge h_0$. Combining (4.8) with (4.10), we obtain that

$$\left((m-1)\log(|a|) + 2\log\left(\frac{A}{B}\right)\right) j_\ell \le \log(C_m), \tag{4.11}$$

for all $\ell \ge h_0$ and $m \ge 1$. For a fixed $m \in \mathbb{N}$ large enough to ensure that the condition $(m-1)\log(|a|) + 2\log\left(\frac{A}{B}\right) > 0$ is satisfied, we have that the left-hand side of (4.11) tends to $+\infty$ as $\ell \to \infty$. This is a contradiction because the right-hand side of (4.11) continues to be equal to $\log(C_m)$. $\square$

In the following result we will show that the composition operator $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_\sigma(\mathbb{R})$, associated with a dilation, is topologizable for certain weight function σ such that $\mathcal{S}_{(\omega)}(\mathbb{R}) \subset \mathcal{S}_\sigma(\mathbb{R})$.

Proposition 4.4.10. *Let* ω *be a weight function and* $\psi(x) = ax$, *for* $x \in \mathbb{R}$, *with* $a \neq 0$. *Then* $\{C_{\psi_m} : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{\omega(\bullet^{\frac{1}{1+\delta}})}(\mathbb{R}) : m \in \mathbb{N}\}$ *is topologizable, for every* $\delta > 0$. *In particular, for the weights* $\omega(t) = (\max(0, \log t)^p$, $p > 1$, *we have that* $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\omega)}(\mathbb{R})$ *is topologizable.*

Proof. For $a = \pm 1$ the result easily follows from the fact that $\{\psi_m : m \in \mathbb{N}\}$ is a finite set.

First, suppose that $|a| > 1$ and fix $\lambda > 0, \delta > 0$. Since $\lim_{j\to\infty} \frac{|a|^{mj}}{j!^{\delta}} = 0$ for every $m \in \mathbb{N}$ and that for all $D > 0, \mu > 0$ there is $B > 0$ such that

$$D^j j! \leq B \exp\left(\mu\varphi^*_\omega\left(\frac{j}{\mu}\right)\right)$$

for all $j \in \mathbb{N}_0$, we obtain that for each $m \in \mathbb{N}$ there is $D_{\lambda,\delta,m} > 0$ such that

$$|a|^{mj} \leq D_{\lambda,\delta,m} \exp\left(\lambda\varphi^*_\omega\left(\frac{\delta j}{\lambda}\right)\right)$$

for all $j \in \mathbb{N}_0$.

In view of the inequality above, we obtain for every $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$ that

$$\begin{aligned}|x|^q|(f\circ\psi_m)^{(j)}(x)| &= |a^m x|^q|f^{(j)}(a^m x)||a|^{m(j-q)}\\ &\leq p_{\omega,\lambda}(f)\exp\left(\lambda\varphi^*_\omega\left(\frac{j+q}{\lambda}\right)\right)|a|^{mj}\\ &\leq D_{\lambda,\delta,m}\exp\left(\lambda\varphi^*_\omega\left(\frac{(1+\delta)(j+q)}{\lambda}\right)\right)p_{\omega,\lambda}(f)\end{aligned}$$

for all $x \in \mathbb{R}$, $q \in \mathbb{N}_0$, $j \in \mathbb{N}$ and $m \in \mathbb{N}$. Since $\lambda > 0$ and $\delta > 0$ are arbitrary, we have done with the case $|a| > 1$. We similarly deal with the case $0 < |a| < 1$. $\square$

Remark 4.4.11. *We can conclude that given a weight function σ, the composition operator having some dynamic property (such as topologizability) for every Gelfand-Shilov class $\mathcal{S}_{\sigma_\delta}(\mathbb{R})$, with $\delta > 1$ and $\sigma_\delta(t) = \omega(t^{\frac{1}{\delta}})$, is not enough to state that such a dynamic property is also valid in the strictly smaller Gelfand-Shilov class $\mathcal{S}_\sigma(\mathbb{R})$.*

Remark 4.4.12. *Let $p > 1$ and $\omega(t) = (\max\{0, \log(t)\})^p$, for $t \geq 0$. By Proposition 1.4.15, we know that the weight ω does not verify condition (1.14). Since $\mathcal{S}_{\sigma_\delta}(\mathbb{R}) = \mathcal{S}_\omega(\mathbb{R})$, for every $\delta > 1$, where $\sigma_\delta(t) = \omega(t^{\frac{1}{\delta}})$, if $\psi(x) = ax$, for $x \in \mathbb{R}$, with $a \neq 0$, by Proposition 4.4.10, the corresponding $C_\psi : \mathcal{S}_\omega(\mathbb{R}) \to \mathcal{S}_\omega(\mathbb{R})$ is topologizable.*

Chapter 5

Some composition operators on modulation spaces

5.1 Introduction

Given $f \in \mathcal{S}(\mathbb{R}^d)$ and $\psi : \mathbb{R}^d \to \mathbb{R}^d$ we may write

$$\begin{aligned} f\left(\psi(x)\right) &= \int_{\mathbb{R}^d} \widehat{f}(y) e^{2\pi i y \psi(x)}\, dy \\ &= \int_{\mathbb{R}^d} \sigma(x,y)\widehat{f}(y) e^{2\pi i x y}\, dy = \sigma(x,D)f(x), \end{aligned}$$

where

$$\sigma(x,y) = \exp\left(i\phi(x)y\right), \ \phi(x) = 2\pi(\psi(x) - x).$$

That is, at least formally, the composition operator can be written as a Kohn-Nirenberg pseudodifferential operator with symbol σ defined as above. Pseudodifferential operators arise as a generalization of variable coefficient differential operators

$$\sum_{|\alpha|\leq m} c_\alpha(x)\partial^\alpha.$$

Pseudodifferential operators were introduced independently by Hörmander [68] and Kohn and Nirenberg [77] and have been studied in several clases of functions and (ultra-)distributions

Methods from time-frequently analysis have proved to be very useful in the investigation of pseudodifferential operators. For instance, the classical

Calderon-Vaillancourt theorem states that any bounded smooth symbol having bounded derivatives defines a bounded pseudodifferential operator on L^2. In 1994-95, Sjöstrand extended this result by considering a new symbol class whose elements need not be smooth. This new symbol class was recognized to be the modulation space $M^{\infty,1}$ earlier introduced by Feichtinger [51] and it was proved that symbols in $M^{\infty,1}$ produce bounded operators on the so-called modulation spaces $M^{p,q}$ [63]. We refer to [44] and the references therein for the background on psedodifferential operators on modulation spaces.

In what follows we will study the action of certain composition operators on some (weighted) modulation spaces by representing them as Kohn-Nirenberg pseudodifferential operators. This is an ongoing research.

5.2 Composition operators on modulation spaces of tempered distributions

Let $f \in \mathcal{S}(\mathbb{R}^d)$ and $\psi : \mathbb{R}^d \to \mathbb{R}^d$ be given. Recall that we can represent the composition operator C_ψ as a Kohn-Nirenberg pseudodifferential operator:

$$f\left(\psi(x)\right) = \sigma(x, D) f(x),$$

where

$$\sigma(x, y) = \exp\left(i\phi(x)y\right), \; \phi(x) = 2\pi(\psi(x) - x).$$

Definition 5.2.1. *The Rihaczek distribution is defined by*

$$W_0(f, g)(x, \xi) = R(f, g)(x, \xi) := e^{-2\pi i x\xi} f(x)\overline{\widehat{g}(\xi)}, \quad f, g \in \mathcal{S}(\mathbb{R}^d).$$

For any $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ we have $\sigma(x, D) : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ and it satisfies (see [44, page 192 and Proposition 4.3.1])

$$\langle \sigma(x, D) f, g\rangle = \langle \sigma, R(g, f)\rangle, \quad f, g \in \mathcal{S}(\mathbb{R}^d).$$

In order to study the continuity of $\sigma(x, D)$ between appropriate modulation spaces we need information about the short-time Fourier transform of σ and $R(g, f)$. This is so because using the orthogonality relations (1.17), fixing $\Phi \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$, we have

$$\langle \sigma(x, D) f, g\rangle = \langle \sigma, R(g, f)\rangle = \langle V_\Phi \sigma, V_\Phi R(g, f)\rangle \|\Phi\|^2 \quad f, g \in \mathcal{S}(\mathbb{R}^d).$$

With respect to $R(g,f)$, according to [44, Lemma 1.3.39], for $f,g,\varphi_1,\varphi_2 \in \mathcal{S}(\mathbb{R}^d)$ and $\Phi_0 = R(\varphi_1,\varphi_2) \in \mathcal{S}(\mathbb{R}^{2d})$ one has

$$V_{\Phi_0}\left(R(g,f)\right)(z,\zeta) = e^{-2\pi i z_2\zeta_2} V_{\varphi_1} g(z_1, z_2+\zeta_1)\overline{V_{\varphi_2} f(z_1+\zeta_2, z_2)},$$

where $z=(z_1,z_2), \zeta=(\zeta_1,\zeta_2) \in \mathbb{R}^{2d}$.

All relations above, with the appropriate changes, extend to the ultradistibutional setting, therefore we will use them in the next sections.

The information we need regarding σ is collected in the following result.

Lemma 5.2.2. *Let us assume* $\phi \in \mathcal{B}_{L^\infty}(\mathbb{R}^d,\mathbb{R}^d)$ *and* $\sigma(x,y) = \exp(i\phi(x)y)$. *Then, for any* $s \geq 0$ *and* $N \in \mathbb{N}$, *we have*

$$\sigma \in M^\infty_{m_1}(\mathbb{R}^{2d}) \cap M^\infty_{m_2}(\mathbb{R}^{2d})$$

where

$$m_1(z,\zeta) = (1+|\zeta_2|)^s\,, \quad m_2(z,\zeta) = (1+|\zeta_1|)^N\,(1+|z_2|)^{-N}\,.$$

Proof. Since $\phi \in \mathcal{B}_{L^\infty}(\mathbb{R}^d,\mathbb{R}^d)$ then

$$|\partial_y^\alpha \sigma(x,y)| \leq C_\alpha, \quad |\partial_x^\beta \sigma(x,y)| \leq C_\beta(1+|y|)^{|\beta|}.$$

Let us fix a non-zero compactly supported smooth function φ on $\mathbb{R}^d$. Then

$$V_{\varphi\otimes\varphi}\sigma(z,\zeta) = \int_{\mathbb{R}^{2d}} \sigma(x,y)\varphi(x-z_1)\varphi(y-z_2)e^{-2\pi i(x\zeta_1+y\zeta_2)}\;d(x,y).$$

For any multi index $\alpha \in \mathbb{N}_0^d$ we obtain, after integrating by parts, that $(-2\pi i\zeta_2)^\alpha\left(V_{\varphi\otimes\varphi}\sigma\right)(z,\zeta)$ can be written as

$$(-1)^\alpha \int_{\mathbb{R}^{2d}} \partial_y^\alpha\left(\sigma(x,y)\varphi(y-z_2)\right)\varphi(x-z_1)e^{-2\pi i(x\zeta_1+y\zeta_2)}\;d(x,y).$$

Hence

$$\begin{aligned}
|\zeta_2^\alpha|\,|\left(V_{\varphi\otimes\varphi}\sigma\right)(z,\zeta)| \quad &\leq \sum_{\beta\leq\alpha} C_{\alpha,\beta}\int_{\mathbb{R}^{2d}} |\varphi^{(\beta)}(y-z_2)||\varphi(x-z_1)|d(x,y) \\
&= \sum_{\beta\leq\alpha} C_{\alpha,\beta}\|\varphi^{(\beta)}\|_1\|\varphi\|_1 =: D_\alpha.
\end{aligned}$$

For every $s \geq 0$ take $N \in \mathbb{N}$ such that $N \geq s$. Since

$$(1+|\zeta_2|)^s \leq d^s(1+\|\zeta_2\|_\infty)^N \leq d^s \sum_{j\leq N} \binom{N}{j} \sum_{k=1}^{d} |\zeta_2^{je_k}|$$

we conclude that

$$\sup_{(z,\zeta)\in\mathbb{R}^{2d}} (1+|\zeta_2|)^s \left|\left(V_{\varphi\otimes\varphi}\sigma\right)(z,\zeta)\right| < \infty.$$

Also, for any multi index $\beta \in \mathbb{N}_0^d$ we have

$$\begin{aligned}
&|\zeta_1^\beta|\left|\left(V_{\varphi\otimes\varphi}\sigma\right)(z,\zeta)\right| \leq \int_{\mathbb{R}^{2d}} \left|\partial_x^\beta\left(\sigma(x,y)\varphi(x-z_1)\right)\right| \varphi(y-z_2)\ d(x,y) \\
&\leq \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} C_{\beta-\gamma} \int_{\mathbb{R}^{2d}} (1+|y|)^{|\beta-\gamma|} |\varphi^{(\gamma)}(x-z_1)||\varphi(y-z_2)|\ d(x,y) \\
&\lesssim \sum_{\gamma\leq\beta} \int_{\mathbb{R}^{2d}} |\varphi^{(\gamma)}(x-z_1)|\ dx \cdot \int_{\mathbb{R}^{2d}} (1+|y|)^{|\beta-\gamma|} |\varphi(y-z_2)|\ d(x,y) \\
&\leq \sum_{\gamma\leq\beta} \|\varphi^{(\gamma)}\|_1 \cdot \int_{\mathbb{R}^d} (1+|y|+|z_2|)^{|\beta-\gamma|} |\varphi(y)|\ dy \\
&\leq \sum_{\gamma\leq\beta} \|\varphi^{(\gamma)}\|_1 (1+|z_2|)^{|\beta-\gamma|} \int_{\mathbb{R}^d} (1+|y|)^{|\beta-\gamma|} |\varphi(y)|\ dy \\
&\lesssim (1+|z_2|)^{|\beta|}.
\end{aligned}$$

Since $(1+|\zeta_1|)^N \leq C\sum_{|\beta|\leq N} |\zeta_1^\beta|$ we finally conclude

$$(1+|\zeta_1|)^N \left|\left(V_{\varphi\otimes\varphi}\sigma\right)(z,\zeta)\right| \leq C\,(1+|z_2|)^N$$

for some constant $C > 0$. □

Remark 5.2.3. For any $s > 0$ we denote

$$v_s(z,\zeta) = (1+|(z,\zeta)|)^s = \left(1+\sqrt{|z|^2+|\zeta|^2}\right)^s,$$

which is a submultiplicative weight on $\mathbb{R}^{4d}$. Then m_1 is v_s-moderate while m_2 is v_{2N}-moderate.

In what follows we fix a non-zero window $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and take $\Phi_0 = R(\varphi, \varphi)$. All the norms in modulation spaces use one of these windows, depending on the dimension.

We are now in a position to obtain a boundedness result for composition operators acting on modulation spaces.

Proposition 5.2.4. *Let us assume* $\phi \in \mathcal{B}_{L^\infty}(\mathbb{R}^d, \mathbb{R}^d)$ *and* $\sigma(x,y) = \exp(i\phi(x)y)$. *Given* $s > 0$ *we take* $N \in \mathbb{N}$ *such that* $N > 2s + 2d$ *and consider the weights* $v_s(z) = (1 + |z|)^s$ *and* $m(z) = v_s(z)(1 + |z_2|)^N$, $z = (z_1, z_2) \in \mathbb{R}^{2d}$. *Then*

$$\sigma(x, D) : M^\infty_m(\mathbb{R}^d) \to M^\infty_{v_s}(\mathbb{R}^d).$$

Proof. For any $f, g \in \mathcal{S}(\mathbb{R}^d)$ we have

$$\begin{aligned} \langle \sigma(x,D)f, g\rangle &= \int_{\mathbb{R}^{2d}} \sigma(x,y)\overline{R(g,f)}(x,y)\; d(x,y) \\ &= \int_{\mathbb{R}^{4d}} \left(V_{\Phi_0}\sigma\right)(z,\zeta)\overline{V_{\Phi_0}(R(g,f))}(z,\zeta)\; d(z,\zeta). \end{aligned}$$

Hence

$$|\langle \sigma(x,D)f, g\rangle| \lesssim \int_{\mathbb{R}^{4d}} |\left(V_{\Phi_0}\sigma\right)(z,\zeta)|\,|V_\varphi g(z_1, z_2 + \zeta_1)|\,|V_\varphi f(z_1 + \zeta_2, z_2)|\; d(z,\zeta)$$

$$\leq \|f\|_{M^\infty_m} \int_{\mathbb{R}^{4d}} |\left(V_{\Phi_0}\sigma\right)(z,\zeta)|\,(1 + |z_2|)^{-N} v_s^{-1}(z_1 + \zeta_2, z_2)\,|V_\varphi g(z_1, z_2 + \zeta_1)|\; d(z,\zeta)$$

$$\leq \|f\|_{M^\infty_m} \times$$

$$\int_{\mathbb{R}^{4d}} |\left(V_{\Phi_0}\sigma\right)(z,\zeta)|\,(1 + |z_2|)^{-N} v_s^{-1}(z_1 + \zeta_2, z_2) v_s(z_1, z_2 + \zeta_1)|F(z_1, z_2 + \zeta_1)|\; d(z,\zeta)$$

where

$$F(a,b) = v_s^{-1}(a,b)V_\varphi g(a,b).$$

It turns out that

$$\int_{\mathbb{R}^{2d}} |F(z_1, z_2 + \zeta_1)|\, dz = \int_{\mathbb{R}^{2d}} |F(z_1, z_2)|\, dz = \|g\|_{M^1_{1/v_s}}$$

is independent of ζ_1. Moreover, having in mind that v_s and v_s^{-1} are v_s-moderate we have that

$$|(V_{\Phi_0}\sigma)(z,\zeta)|\,(1+|z_2|)^{-N}v_s^{-1}(z_1+\zeta_2,z_2)v_s(z_1,z_2+\zeta_1)$$

is less than or equal to some constant times

$$|(V_{\Phi_0}\sigma)(z,\zeta)|\,(1+|z_2|)^{-N}v_s(\zeta_2,0)v_s(0,\zeta_1).$$

From Lemma 5.2.2, for every constant $\alpha > 0$ we have

$$|(V_{\Phi_0}\sigma)(z,\zeta)|\,(1+|z_2|)^{-N} \lesssim \min\left((1+|\zeta_1|)^{-N},(1+|\zeta_2|)^{-\alpha}\right).$$

Hence

$$|\langle\sigma(x,D)f,g\rangle| \lesssim \|f\|_{M_m^\infty}\|g\|_{M^1_{1/v_s}}\int_{\mathbb{R}^{2d}} G(\zeta)\ d\zeta$$

where

$$G(\zeta)=\min\left((1+|\zeta_1|)^{-N},(1+|\zeta_2|)^{-\alpha}\right)v_s(\zeta_2,0)v_s(0,\zeta_1).$$

Since

$$\int_{|\zeta_1|<|\zeta_2|} d\zeta_1 \lesssim |\zeta_2|^d$$

we get

$$\begin{aligned}\int_{|\zeta_1|<|\zeta_2|} G(\zeta)\ d\zeta &\leq \int_{|\zeta_1|<|\zeta_2|}\frac{v_s(\zeta_2,0)^2}{(1+|\zeta_2|)^\alpha}\ d\zeta\\ &\lesssim \int_{\mathbb{R}^d}\frac{|\zeta_2|^d v_s(\zeta_2,0)^2}{(1+|\zeta_2|)^\alpha}\ d\zeta_2<\infty\end{aligned}$$

for α large enough. On the other hand

$$\int_{|\zeta_2|<|\zeta_1|} G(\zeta)\ d\zeta \lesssim \int_{\mathbb{R}^d}\frac{|\zeta_1|^d v_s(0,\zeta_1)^2}{(1+|\zeta_1|)^N}\ d\zeta_1<\infty$$

since $N > 2s+2d$. Finally we conclude

$$|\langle\sigma(x,D)f,g\rangle| \lesssim \|f\|_{M_m^\infty}\|g\|_{M^1_{1/v_s}}$$

for every $f,g\in\mathcal{S}(\mathbb{R}^d)$. □

According to Beurling-Helson Theorem no modulation space $M^\infty_{v_s}(\mathbb{R}^d)$ is invariant under the action of the composition operator $\sigma(x,D)$ unless ψ is a linear function. However, we have the following result:

Corollary 5.2.5. *Let us assume* $\phi \in \mathcal{B}_{L^\infty}(\mathbb{R}^d, \mathbb{R}^d)$ *and* $\sigma(x,y) = \exp(i\phi(x)y)$. *Given* $s > 0$ *we take* $N = [2s] + s + 2d + 1$. *Then*

$$\sigma(x,D) : M^\infty_{v_N}(\mathbb{R}^d) \to M^\infty_{v_s}(\mathbb{R}^d).$$

Using that

$$\mathcal{S}(\mathbb{R}^d) = \bigcap_{s>0} M^\infty_{v_s}(\mathbb{R}^d),$$

we have that $\sigma(x,D) : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$.

5.3 Composition operators on ultra-modulation spaces of ultradistributions. One variable.

We want to obtain boundedness results in the context of ultramodulation spaces. An important tool will be the ω-ultradifferential operators, whose definition and basic properties we review below.

Let $G(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function such that $\log|G(z)| = O(\omega(z))$ as $|z| \to \infty$. Then there exist $C > 0$ and $m \in \mathbb{N}$ such that

$$|a_n| \leq C e^{-m\varphi^*(\frac{n}{m})} \quad \forall n \in \mathbb{N}_0. \tag{5.1}$$

Moreover there exists $\mu \in \mathcal{E}'_{(\omega)}(\mathbb{R})$ whose support reduces to $\{0\}$ and with Fourier-Laplace transform $\widehat{\mu}(z) = G(-z)$. Then

$$G(D) : \mathcal{D}'_{(\omega)}(\mathbb{R}) \to \mathcal{D}'_{(\omega)}(\mathbb{R}), \nu \mapsto \mu * \nu,$$

is called ultradifferential operator of (ω)-class. It turns out that

$$G(D) : \mathcal{E}_{(\omega)}(\mathbb{R}) \to \mathcal{E}_{(\omega)}(\mathbb{R})$$

is given by

$$G(D)f(x) = \sum_{n=0}^{\infty} i^n a_n f^{(n)}(x).$$

That is, ultradifferential operators are differential operators of infinite order.

- We have
$$G(D)\left(e^{i\xi x}\right) = G(-\xi)e^{i\xi x}.$$
- Also, for any $f, g \in \mathcal{E}_{(\omega)}(\mathbb{R})$, one of them compactly supported, we have
$$\int_{\mathbb{R}} f(x)\left(G(D)g\right)(x)\, dx = \int_{\mathbb{R}} \left(G(-D)f\right)(x) g(x)\, dx.$$
- According to [34, Theorem 1], for every $A > 0$ there exist an entire function G such that $\log|G(z)| = O(\omega(z))$ as $|z| \to \infty$ and
$$\log|G(x)| \geq A\omega(x) \ \ \forall x \in \mathbb{R}. \tag{5.2}$$
  In this case $G(D)$ is said to be strongly elliptic.

Strongly elliptic operators are in fact elliptic operators in the sense that if $f \in \mathcal{D}'_{(\omega)}(\mathbb{R})$ and $G(D)f$ is a real analytic function in a given open set, then f has to be a real analytic function in that set.

Lemma 5.3.1. *There exists $m_0 \in \mathbb{N}$ such that the following holds. For every $g \in \mathcal{B}_{\infty,\omega}(\mathbb{R})$ and $h \in \mathcal{D}_{(\omega)}(\mathbb{R})$ we have*
$$G(-D)\left(gh\right)(x) = \sum_{k=0}^{\infty} a_k(x) h^{(k)}(x)$$
where the functions $\left(a_k(x)\right)_{k\in\mathbb{N}_0}$ depend on g and satisfy
$$\forall \ell \in \mathbb{N}\ \exists C_\ell > 0 : \sup_{x\in\mathbb{R}} |a_k^{(r)}(x)| \leq C_\ell \exp\left(-m_0\varphi^*(\frac{k}{m_0}) + \ell\varphi^*(\frac{r}{\ell})\right) \quad \forall k \in \mathbb{N}$$
and the constant C_ℓ only depends on $\|g\|_{2\ell,B_{\infty,\omega}}$.

Proof. We put $b_n := (-1)^n a_n$. We first observe that
$$\begin{aligned} G(-D)(gh)(x) &= \sum_{n=0}^{\infty} b_n (gh)^{(n)}(x) = \sum_{n=0}^{\infty} b_n \left(\sum_{k=0}^{n} \binom{n}{k} h^{(k)}(x) g^{(n-k)}(x)\right) \\ &= \sum_{k=0}^{\infty} a_k(x) h^{(k)}(x), \end{aligned}$$

where

$$a_k(x) = \sum_{n\geq k} \binom{n}{k} b_n g^{(n-k)}(x) = \sum_{j=0}^{\infty} \binom{k+j}{k} b_{k+j} g^{(j)}(x).$$

The change in the order of summation in the expression for $G(-D)(gh)(x)$ is justified by the estimates below. We now fix $r \in \mathbb{N}$ and observe that, for every $\ell \in \mathbb{N}$,

$$\begin{aligned} \left|g^{(j+r)}(x)\right| &\leq \|g\|_{2\ell,\infty,\omega} \exp\left(2\ell\varphi^*(\frac{j+r}{2\ell})\right) \\ &\leq \|g\|_{2\ell,\infty,\omega} \exp\left(\ell\varphi^*(\frac{j}{\ell}) + \ell\varphi^*(\frac{r}{\ell})\right). \end{aligned}$$

Since

$$\begin{aligned} |b_{k+j}| &\leq C \exp\left(-m\varphi^*(\frac{k+j}{m})\right) \\ &\leq C \exp\left(-m\varphi^*(\frac{j}{m}) - m\varphi^*(\frac{k}{m})\right) \end{aligned}$$

we conclude that

$$\sum_{j=0}^{\infty} \binom{k+j}{k} |b_{k+j}||g^{(r+j)}(x)|$$

is less than or equal to

$$\|g\|_{2\ell,\infty,\omega} 2^k \exp\left(\ell\varphi^*(\frac{r}{\ell}) - m\varphi^*(\frac{k}{m})\right) \sum_{j=0}^{\infty} 2^j \exp\left(-m\varphi^*(\frac{j}{m}) + \ell\varphi^*(\frac{j}{\ell})\right).$$

Take $L \in \mathbb{N}$ such that $\omega(et) \leq L(1+\omega(t))$. For every $\ell \geq Lm$ we have

$$j + \ell\varphi^*(\frac{j}{\ell}) - m\varphi^*(\frac{j}{m}) \leq j + mL\varphi^*(\frac{j}{mL}) - m\varphi^*(\frac{j}{m}) \leq mL.$$

Consequently

$$\begin{aligned} &\sum_{j=0}^{\infty} 2^j \exp\left(-m\varphi^*(\frac{j}{m}) + \ell\varphi^*(\frac{j}{\ell})\right) \\ &= \sum_{j=0}^{\infty} (\frac{2}{e})^j \exp\left(j - m\varphi^*(\frac{j}{m}) + \ell\varphi^*(\frac{j}{\ell})\right) \leq e^{mL} \sum_{j=0}^{\infty} (\frac{2}{e})^j =: D \end{aligned}$$

and we conclude

$$|a_k^{(r)}(x)| \leq 2^k D\|g\|_{2\ell,\infty,\omega} \exp\left(-m\varphi^*(\frac{k}{m}) + \ell\varphi^*(\frac{r}{\ell})\right).$$

Finally we take $m_0 = Lm$. Since

$$k + m_0\varphi^*(\frac{k}{m_0}) \leq m\varphi^*(\frac{k}{m}) + m_0,$$

the conclusion follows. $\square$

Lemma 5.3.2. *Let* $\phi \in \mathcal{B}_{\infty,\omega}(\mathbb{R})$ *and* $\sigma(x,y) = e^{i\phi(x)y}$*. For every* $A > 0$ *we have* $\sigma \in M_m^\infty(\mathbb{R}^2)$*, where* $m(z,\zeta) = e^{A\omega(\zeta_2)}$*.*

Proof. We fix $\varphi \in \mathcal{D}_{(\omega)}(\mathbb{R})$ and consider an ultradifferential operator $G(D)$ as in (5.1) and (5.2). We assume $\operatorname{supp}\varphi \subset [-M,M]$. For every $z = (z_1,z_2)$ we put

$$K_z = (z_1 + [-M,M]) \times (z_2 + [-M,M]).$$

Then

$$G(2\pi i\zeta_2)\left(V_{\varphi\otimes\varphi}\sigma\right)(z,\zeta)$$

is given by

$$\int_{\mathbb{R}^2} \sigma(x,y)\varphi(x-z_1)\varphi(y-z_2)G(D_y)\left(e^{-2\pi i(x\zeta_1+y\zeta_2)}\right)\, d(x,y)$$

$$= \int_{K_z} G(-D_y)\left(\sigma(x,y)\varphi(y-z_2)\right)\varphi(x-z_1)e^{-2\pi i(x\zeta_1+y\zeta_2)}\, d(x,y).$$

From Lemma 5.3.1 with $g(y) = \varphi(y-z_2)$ we have

$$\begin{aligned} G(-D_y)\left(\sigma(x,y)\varphi(y-z_2)\right) &= \sum_{k=0}^{\infty} a_k(y,z_2)D_y^k\left(\sigma(x,y)\right) \\ &= \sigma(x,y)\sum_{k=0}^{\infty} a_k(y,z_2)(i\phi(x))^k \end{aligned}$$

for some functions $a_k(\cdot,z_2)$. Since the family of translates of φ is bounded in $\mathcal{B}_{\infty,\omega}(\mathbb{R})$ it follows that

$$\exists m_0 \in \mathbb{N}, C > 0 : \sup_{y,z_2}|a_k(y,z_2)| \leq Ce^{-m_0\varphi^*(\frac{k}{m_0})}.$$

We put

$$D := \sum_{k=0}^{\infty} e^{-m_0\varphi^*(\frac{k}{m_0})}\|\phi\|_\infty^k < \infty.$$

Condition (5.2) now gives

$$e^{A\omega(\zeta_2)}\left|\left(V_{\varphi\otimes\varphi}\sigma\right)(z,\zeta)\right| \leq CD\int_{K_z}|\varphi(x-z_1)|\,d(x,y) \leq 2M\|\varphi\|_1.$$

□

Lemma 5.3.3. *Let $\sigma(x,y) = e^{i\phi(x)y}$, where $\phi \in \mathcal{B}_{\infty,\omega}(\mathbb{R})$ and ω is a subadditive weight. For every $\ell \in \mathbb{N}$ there exists $B_\ell > 0$ such that*

$$\left|\left(D_x^n\sigma\right)(x,y)\right| \leq 4^n e^{\ell\omega(B_\ell y)}e^{\ell\varphi^*(\frac{n}{\ell})} \quad \forall n \in \mathbb{N}_0.$$

Proof. Since

$$\|\phi\|_{\ell,\infty,\omega} = \sup_{x\in\mathbb{R}}\sup_{k\in\mathbb{N}_0}|\phi^{(k)}(x)|e^{-\ell\varphi^*(\frac{k}{\ell})} < \infty$$

then we get, from Faà di Bruno formula, that there exist a constant $B_\ell > 0$ such that

$$\begin{aligned}\left|\left(D_x^n\sigma\right)(x,y)\right| &\leq \sum\frac{n!}{k_1!\ldots k_n!}|y|^k\prod_{j=1}^{n}\left|\frac{\phi^{(j)}(x)}{j!}\right|^{k_j}\\ &\leq \sum\frac{n!}{k_1!\ldots k_n!}|y|^kB_\ell^k\frac{\exp\left(\ell\varphi^*(\frac{n-k}{\ell})\right)}{(n-k)!}\\ &\leq 2^n\sum\frac{k!}{k_1!\ldots k_n!}\left(B_\ell|y|\right)^k\exp\left(\ell\varphi^*(\frac{n}{\ell})-\ell\varphi^*(\frac{k}{\ell})\right).\end{aligned}$$

Using

$$\left(B_\ell|y|\right)^k\exp\left(-\ell\varphi^*(\frac{k}{\ell})\right) \leq \exp\left(\ell\omega(B_\ell y)\right)$$

we finally conclude

$$\begin{aligned}\left|\left(D_x^n\sigma\right)(x,y)\right| &\leq 2^n\exp\left(\ell\omega(B_\ell y)+\ell\varphi^*(\frac{n}{\ell})\right)\sum\frac{k!}{k_1!\ldots k_n!}\\ &\leq 4^ne^{\ell\omega(B_\ell y)}e^{\ell\varphi^*(\frac{n}{\ell})}.\end{aligned}$$

□

Lemma 5.3.4. *Let* $\sigma(x,y) = e^{i\phi(x)y}$, *where* $\phi \in \mathcal{B}_{\infty,\omega}(\mathbb{R})$ *and* ω *is a subadditive weight. For every* $A > 0$ *there exists* $k \in \mathbb{N}$ *such that we have* $\sigma \in M^{\infty}_{m}(\mathbb{R}^2)$, *where* $m(z,\zeta) = \exp\left(A\omega(\zeta_1) - k\omega(z_2)\right).$

Proof. We fix $\varphi \in \mathcal{D}_{(\omega)}(\mathbb{R})$ and consider an ultradifferential operator $G(D)$ as in (5.1) and (5.2). We assume $\operatorname{supp}\varphi \subset [-M,M]$. For every $z = (z_1, z_2)$ we put

$$K_z = (z_1 + [-M,M]) \times (z_2 + [-M,M]).$$

Then

$$\begin{aligned} &G(2\pi i\zeta_1)\left(V_{\varphi\otimes\varphi}\sigma\right)(z,\zeta) \\ &= \int_{\mathbb{R}^2} G(-D_x)\left(\sigma(x,y)\varphi(x-z_1)\right)\varphi(y-z_2)e^{-2\pi i(x\zeta_1 + y\zeta_2)}\, d(x,y). \end{aligned}$$

From Lemma 5.3.1 with $g(x) = \varphi(x - z_1)$ we have

$$G(-D_x)\left(\sigma(x,y)\varphi(x-z_1)\right) = \sum_{k=0}^{\infty} a_k(x,z_1) D_x^k\left(\sigma(x,y)\right)$$

and there exists $C > 0$ such that

$$|a_k(x,z_1)| \le C e^{-m_0\varphi^*(\frac{k}{m_0})} \quad \forall k \in \mathbb{N}_0, x, z_1 \in \mathbb{R}.$$

From Lemma 5.3.3, for every $\ell \in \mathbb{N}$ we have

$$|G(-D_x)\left(\sigma(x,y)\varphi(x-z_1)\right)| \le C \sum_{k=0}^{\infty} e^{-m_0\varphi^*(\frac{k}{m_0})} 4^k \exp\left(\ell\omega(B_\ell y) + \ell\varphi^*(\frac{k}{\ell})\right).$$

Now take $L > 0$ such that $\omega(et) \le L(1 + \omega(t))$ and fix $\ell \ge m_0 L^2$. From

$$e^{-m_0\varphi^*(\frac{k}{m_0})} \le e^{-2k} e^{-\ell\varphi^*(\frac{k}{\ell})}$$

we get

$$|G(-D_x)\left(\sigma(x,y)\varphi(x-z_1)\right)| \le C \sum_{k=0}^{\infty} \left(\frac{4}{e^2}\right)^k e^{\ell\omega(B_\ell y)}.$$

Consequently,

$$\begin{aligned} e^{A\omega(\zeta_1)}\left|\left(V_{\varphi\otimes\varphi}\sigma\right)(z,\zeta)\right| &\lesssim \int_{K_z} e^{\ell\omega(B_\ell y)} |\varphi(y - z_2)|\, d(x,y) \\ &\lesssim \int_{\mathbb{R}} |\varphi(y)| e^{\ell\omega(B_\ell(y+z_2))}\, dy. \end{aligned}$$

Since

$$\int_{\mathbb{R}} |\varphi(y)| e^{M\omega(My)}\, dy < \infty \ \ \forall M > 0$$

and

$$\ell\omega(B_\ell(y+z_2)) \leq \ell L \left(1 + \omega(B_\ell y) + \omega(B_\ell z_2)\right)$$

we finally conclude

$$e^{A\omega(\zeta_1)} \left|\left(V_{\varphi\otimes\varphi}\sigma\right)(z,\zeta)\right| \lesssim e^{k\omega(z_2)}$$

for some constant k only depending on m_0 and the weight ω. $\square$

Theorem 5.3.5. *Let $\sigma(x,y) = e^{i\phi(x)y}$, where $\phi \in \mathcal{B}_{\infty,\omega}(\mathbb{R})$ and ω is a subadditive weight. For every $s > 0$ and $k > 0$ we denote*

$$v_s(z) = e^{s(\omega(z_1)+\omega(z_2))}, \quad m_{s,k}(z) = v_s(z) e^{k\omega(z_2)}, \quad z = (z_1, z_2).$$

Then for every $s > 0$ there exists $k > 0$ such that

$$\sigma(x,D) : M^\infty_{m_{s,k}}(\mathbb{R}) \to M^\infty_{v_s}(\mathbb{R}).$$

Proof. We fix a non-zero window $\varphi \in \mathcal{S}_\omega(\mathbb{R})$ and take $\Phi_0 = R(\varphi,\varphi)$. For any $f, g \in \mathcal{S}_\omega(\mathbb{R})$ we have

$$\begin{aligned}\langle \sigma(x,D)f, g\rangle &= \int_{\mathbb{R}^2} \sigma(x,y)\overline{R(g,f)}(x,y)\, d(x,y) \\ &= \int_{\mathbb{R}^4} \left(V_{\Phi_0}\sigma\right)(z,\zeta)\overline{V_{\Phi_0}(R(g,f))}(z,\zeta)\, d(z,\zeta).\end{aligned}$$

We fix $A > 2s$ and take k as in Lemma 5.3.4. Then

$$|\langle \sigma(x,D)f, g\rangle| \lesssim \int_{\mathbb{R}^4} \left|\left(V_{\Phi_0}\sigma\right)(z,\zeta)\right| \left|V_\varphi g(z_1, z_2+\zeta_1)\right| \left|V_\varphi f(z_1+\zeta_2, z_2)\right| \ d(z,\zeta)$$

$$\leq \|f\|_{M^\infty_{m_s}} \int_{\mathbb{R}^4} \left|\left(V_{\Phi_0}\sigma\right)(z,\zeta)\right| e^{-k\omega(z_2)} v_s^{-1}(z_1+\zeta_2, z_2) \left|V_\varphi g(z_1, z_2+\zeta_1)\right| \ d(z,\zeta)$$

$$\leq \|f\|_{M^\infty_{m_s}} \int_{\mathbb{R}^4} \left|\left(V_{\Phi_0}\sigma\right)(z,\zeta)\right| e^{-k\omega(z_2)} v_s^{-1}(z_1+\zeta_2, z_2) v_s(z_1, z_2+\zeta_1) |F(z_1, z_2+\zeta_1)| \ d(z,\zeta)$$

where

$$F(a,b) = v_s^{-1}(a,b) V_\varphi g(a,b).$$

It turns out that

$$\int_{\mathbb{R}^2} |F(z_1, z_2 + \zeta_1)|\, dz = \int_{\mathbb{R}^2} |F(z_1, z_2)|\, dz = \|g\|_{M^1_{1/v_s}}$$

is independent of ζ_1. Moreover, having in mind that v_s and v_s^{-1} are v_s-moderate we have that

$$|(V_{\Phi_0}\sigma)(z,\zeta)|\, e^{-k\omega(z_2)} v_s^{-1}(z_1 + \zeta_2, z_2) v_s(z_1, z_2 + \zeta_1)$$

is less than or equal to some constant times

$$|(V_{\Phi_0}\sigma)(z,\zeta)|\, e^{-k\omega(z_2)} v_s(\zeta_2, 0) v_s(0, \zeta_1).$$

Now apply Lemmas 5.3.2 and 5.3.4 to conclude

$$|(V_{\Phi_0}\sigma)(z,\zeta)|\, e^{-k\omega(z_2)} \lesssim \min\left(e^{-A\omega(\zeta_1)}, e^{-A\omega(\zeta_2)}\right).$$

Hence

$$|\langle \sigma(x,D)f, g\rangle| \lesssim \|f\|_{M^\infty_{m_s}} \|g\|_{M^1_{1/v_s}} \int_{\mathbb{R}^2} G(\zeta)\ d\zeta$$

where

$$G(\zeta) = \min\left(e^{-A\omega(\zeta_1)}, e^{-A\omega(\zeta_2)}\right) v_s(\zeta_2, 0) v_s(0, \zeta_1).$$

Finally

$$\int_{|\zeta_1| \le |\zeta_2|} G(\zeta)\ d\zeta \lesssim \int_{\mathbb{R}} |\zeta_2| e^{(2s-A)\omega(\zeta_2)}\ d\zeta_2 < \infty$$

and also

$$\int_{|\zeta_2| \le |\zeta_1|} G(\zeta)\ d\zeta < \infty.$$

Finally we conclude

$$|\langle \sigma(x,D)f, g\rangle| \lesssim \|f\|_{M^\infty_{m_s}} \|g\|_{M^1_{1/v_s}}$$

for every $f, g \in \mathcal{S}_\omega(\mathbb{R})$. □

Remark 5.3.6. For every $s > 0$ take $k = k(s)$ as in Theorem 5.3.5. Then

$$\bigcap_{s>0} M^\infty_{m_{s,k}}(\mathbb{R}) = \bigcap_{s>0} M^\infty_{v_s}(\mathbb{R}) = S_\omega(\mathbb{R}).$$

5.4 Composition operators on ultra-modulation spaces of ultradistributions. Several variables.

For a subadditive weight ω and $\ell > 0$ we recall the following facts, to be used later.

Since φ^* is convex there is $D_\ell > 0$ such that

$$\exp\left(2\ell\varphi^*(\frac{r}{2\ell})\right) \leq D_\ell \exp\left(\ell\varphi^*(\frac{r-1}{\ell})\right)$$

for every $r \geq 1$. Moreover, the sequence $a_j := \frac{1}{j!}\exp\left(\varphi^*(\frac{j}{\ell})\right)$ satisfies $a_j a_k \leq a_{j+k}$. Hence, for $k_1 + \ldots + k_n = k$, $\sum_{j=1}^n jk_j = n$, we have

$$\begin{aligned}\prod_{r=1}^n \left(\frac{e^{2\ell\varphi^*(\frac{r}{2\ell})}}{r!}\right)^{k_r} &\leq D_\ell^k \prod_{r=1}^n \left(\frac{e^{\ell\varphi^*(\frac{r-1}{\ell})}}{(r-1)!}\right)^{k_r} \leq D_\ell^k \prod_{r=1}^n \frac{e^{\ell\varphi^*(\frac{(r-1)k_r}{\ell})}}{((r-1)k_r)!} \\ &\leq D_\ell^k \frac{e^{\ell\varphi^*(\frac{n-k}{\ell})}}{(n-k)!}.\end{aligned}$$

Lemma 5.4.1. *Given* $\phi_1, \ldots, \phi_s \in \mathcal{B}_{L_\infty,\omega}(\mathbb{R})$, $y = (y_1, \ldots, y_s)$ *let us put*

$$\tilde{\sigma}(x,y) := e^{i\sum_{j=1}^s \phi_j(x)y_j}.$$

Then, for every $\ell > 0$,

$$|D_x^n \tilde{\sigma}(x,y)| \leq 4^n e^{\ell\omega(B_\ell\|y\|)} e^{\ell\varphi^*(\frac{n}{\ell})},$$

where B_ℓ *is a constant which only depends on* ℓ, ω *and the norms* $\|\phi_j\|_{2\ell,\infty,\omega}$, $1 \leq j \leq s$.

Proof. From Faà di Bruno formula and Cauchy-Schwarz inequality we have

$$\begin{aligned}|D_x^n \tilde{\sigma}(x,y)| &\leq \sum \frac{n!}{k_1! \ldots k_n!} \prod_{r=1}^n \left|\sum_{j=1}^s \frac{\phi_j^{(r)}(x)}{r!} y_j\right|^{k_r} \\ &\leq \sum \frac{n!}{k_1! \ldots k_n!} \|y\|^k \prod_{r=1}^n \left(\left|\frac{\phi_j^{(r)}(x)}{r!}\right|^2\right)^{\frac{k_r}{2}}.\end{aligned}$$

Since

$$|\phi_j^{(r)}(x)| \leq \|\phi_j\|_{2\ell,\infty,\omega} e^{2\ell\varphi^*(\frac{r}{2\ell})}$$

then

$$\begin{aligned} |D_x^n \tilde{\sigma}(x,y)| &\leq \sum \frac{n!}{k_1! \dots k_n!} \|y\|^k B_\ell^k e^{-\ell\varphi^*(\frac{k}{\ell})} \frac{e^{\ell\varphi^*(\frac{n}{\ell})}}{(n-k)!} \\ &\leq e^{\omega(B_\ell\|y\|)} e^{\ell\varphi^*(\frac{n}{\ell})} \sum \binom{n}{k} \frac{k!}{k_1! \dots k_n!}, \end{aligned}$$

where

$$B_\ell = D_\ell \left(\sum_{j=1}^{s} \|\phi_j\|^2_{2\ell,\infty,\omega} \right)^{\frac{1}{2}}.$$

□

Note that if $A \subset \mathcal{B}_{L_\infty,\omega}(\mathbb{R})$ is a bounded set then the estimates in the previous Lemma hold uniformly when $\phi_1, \dots, \phi_s \in A$.

Corollary 5.4.2. *Let* $\phi_1, \dots, \phi_d \in \mathcal{B}_{L_\infty,\omega}(\mathbb{R}^d)$, $\phi = (\phi_1, \dots, \phi_d)$ *and*

$$\sigma(x,y) = e^{i\phi(x)y}, \quad x, y \in \mathbb{R}^d.$$

Then for every $n, \ell \in \mathbb{N}$ *and* $1 \leq k \leq d$ *we have*

$$\left|D_{x_k}^n \sigma(x,y)\right| \leq 4^n e^{\ell\omega(B_\ell\|y\|)} e^{\ell\varphi^*(\frac{n}{\ell})},$$

where B_ℓ *is a constant which only depends on* ϕ.

Proof. For every $x \in \mathbb{R}^d$ and $1 \leq k \leq d$ we denote

$$\widetilde{x_k} = (x_1, \dots, x_{k-1}, 0, x_{k+1}, \dots, x_d).$$

It turns out that the family of functions

$$t \mapsto \phi_j(\widetilde{x_k} + te_k), \quad x \in \mathbb{R}^d,\ 1 \leq j \leq d,\ 1 \leq k \leq d,$$

is bounded in $\mathcal{B}_{L_\infty,\omega}(\mathbb{R})$. Now it suffices to apply Lemma 5.4.1. □

Lemma 5.4.3. *For every* $\widetilde{x_k} \in \mathbb{R}^{d-1}$ *and* $y \in \mathbb{R}^d$ *we denote*

$$f_{\widetilde{x_k},y}(t) = \sigma\left(\widetilde{x_k} + te_k, y\right).$$

Then, for every $\varphi \in \mathcal{D}_{(\omega)}(\mathbb{R})$ *and* $A > 0$ *there exist* $\ell \in \mathbb{N}$, $B_\ell > 0$ *and* $C > 0$ *such that*

$$\left|\left(V_\varphi f_{\widetilde{x_k},y}\right)(u,v)\right| \leq C e^{-A\omega(v)} e^{\ell\omega(B_\ell\|y\|)} \quad \forall (u,v) \in \mathbb{R}^2.$$

Proof. Let $G(D)$ an ultradifferential operator as in (5.1) and (5.2). Then

$$G(2\pi v)\left(V_\varphi f_{\widetilde{x_k},y}\right)(u,v) = \int_{\mathbb{R}} G(-D_t)\left(\sigma\left(\widetilde{x_k}+te_k,y\right)\varphi(t-u)\right)e^{-2\pi i t v}\,dt.$$

As in Lemma 5.3.4 we have

$$\left|G(-D_t)\left(\sigma\left(\widetilde{x_k}+te_k,y\right)\right)\right| \leq C\sum_{r=0}^{\infty}\left(\frac{4}{e^2}\right)^r e^{\ell\omega(B_\ell\|y\|)}$$

where ℓ depends on the coefficients of $G(D)$. Since $|G(2\pi v)| \geq e^{A\omega(v)}$, the conclusion follows. $\square$

Lemma 5.4.4. *Let* $\phi_1,\ldots,\phi_d \in \mathcal{B}_{L_\infty,\omega}(\mathbb{R}^d)$, $\phi = (\phi_1,\ldots,\phi_d)$ *and*

$$\sigma(x,y) = e^{i\phi(x)y}, \quad x,y\in\mathbb{R}^d.$$

For every $A>0$ *and* $k=1,\ldots,d$ *there is* $p>0$ *such that* $\sigma\in M^\infty_m(\mathbb{R}^{2d})$ *where*

$$m(z,\zeta) = e^{A\omega(\zeta_k)-p\omega(z_{d+1},\ldots,z_{2d})}, \quad z,\zeta\in\mathbb{R}^{2d}.$$

Proof. For every $x\in\mathbb{R}^d$ we denote

$$\widehat{x_k} = \left(x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_d\right).$$

We consider $\psi = \overbrace{\varphi\otimes\ldots\otimes\varphi}^{(2d)}$ and observe that $V_\psi\sigma(z,\zeta)$ is given by

$$\int_{\mathbb{R}^{2d-1}}\prod_{j=1,j\neq k}^{d}\varphi(x_j-z_j)\prod_{j=1}^{d}\varphi(y_j-z_{j+d})\exp\left(-2\pi i\left(\sum_{j=1,j\neq k}^{d}\zeta_j x_j+\sum_{j=1}^{d}\zeta_{j+d}y_j\right)\right)\times$$

$$\times\left(V_\varphi f_{\widetilde{x_k},y}\right)(z_k,\zeta_k)\,d\widehat{x_k}\,dy.$$

Consequently, from Lemma 5.4.3 we conclude

$$\begin{aligned}|V_\psi\sigma(z,\zeta)| &\lesssim \|\varphi\|_1^{d-1}e^{-A\omega(\zeta_k)}\int_{\mathbb{R}^d}\prod_{j=1}^{d}|\varphi(y_j-z_{j+d})|e^{\ell\omega(B_\ell\|y\|)}\,dy\\ &\lesssim e^{-A\omega(\zeta_k)}e^{\ell\omega(B_\ell\|(z_{d+1},\ldots,z_{2d})\|)}\left(\int_{\mathbb{R}}\varphi(t)e^{\ell\omega(B_\ell t)}\,dt\right)^d.\end{aligned}$$

$\square$

Lemma 5.4.5. *Let* $\phi_1,\dots,\phi_d \in \mathcal{B}_{L_\infty,\omega}(\mathbb{R}^d)$, $\phi=(\phi_1,\dots,\phi_d)$ *and*

$$\sigma(x,y)=e^{i\phi(x)y},\quad x,y\in\mathbb{R}^d.$$

For every $A>0$ *and* $k=1,\dots,d$ *we have* $\sigma\in M^\infty_m(\mathbb{R}^{2d})$ *where* $m(z,\zeta)=e^{A\omega(\zeta_{k+d})}$.

Proof. We use the notation of Lemma 5.4.4 and denote

$$\sigma_{\widetilde{x_k}}(t,s)=e^{i\phi_k(\widetilde{x_k}+te_k)s},\quad (t,s)\in\mathbb{R}^2.$$

Then $V_\psi\sigma(z,\zeta)$ is given by

$$\int_{\mathbb{R}^{2d-2}}\prod_{j\neq k}\varphi(x_j-z_j)\varphi(y_j-z_{j+d})\exp\left(-2\pi i\sum_{j\neq k}(\zeta_jx_j+\zeta_{j+d}y_j)\right)\times$$

$$\times\left(V_{\varphi\otimes\varphi}\sigma_{\widetilde{x_k}}\right)(z_k,z_{k+d};\zeta_k,\zeta_{k+d})\,d\widehat{x_k}\,d\widehat{y_k}.$$

Since the familly of functions

$$\left\{t\mapsto\phi_k(\widetilde{x_k}+te_k):\widetilde{x_k}\in\mathbb{R}^{d-1}\right\}$$

is a bounded set in $\mathcal{B}_{L_\infty,\omega}(\mathbb{R})$ we can apply Lemma 5.3.2 to conclude

$$\begin{aligned}e^{-A\omega(\zeta_{k+d})}\left|V_\psi\sigma(z,\zeta)\right| &\lesssim \int_{\mathbb{R}^{2d-2}}\prod_{j\neq k}\left|\varphi(x_j-z_j)\varphi(y_j-z_{j+d})\right|\,d\widehat{x_k}d\widehat{y_k}\\ &=\|\varphi\|_1^{2d-2}.\end{aligned}$$

□

In the next results we put $z=(z_1,z_2)$ and $\zeta=(\zeta_1,\zeta_2)$ where $z_1,z_2,\zeta_1,\zeta_2\in\mathbb{R}^d$.

Corollary 5.4.6. *Let* $\phi_1,\dots,\phi_d \in \mathcal{B}_{L_\infty,\omega}(\mathbb{R}^d)$, $\phi=(\phi_1,\dots,\phi_d)$ *and*

$$\sigma(x,y)=e^{i\phi(x)y},\quad x,y\in\mathbb{R}^d.$$

For every $A>0$ *there exists* $p>0$ *such that* $\sigma\in M^\infty_{m_1}(\mathbb{R}^d)\cap M^\infty_{m_2}(\mathbb{R}^d)$ *where*

$$m_1(z,\zeta)=e^{A\omega(|\zeta_1|)-p\omega(|z_2|)},\quad m_2(z,\zeta)=e^{A\omega(|\zeta_2|)}.$$

Theorem 5.4.7. *Let* $\phi_1, \ldots, \phi_d \in \mathcal{B}_{L_\infty,\omega}(\mathbb{R}^d)$, $\phi = (\phi_1, \ldots, \phi_d)$ *and*

$$\sigma(x, y) = e^{i\phi(x)y}, \quad x, y \in \mathbb{R}^d.$$

For every $s, p > 0$ *we denote*

$$v_s(z) = e^{s(\omega(|z_1|)+\omega(|z_2|))}, \quad m_{s,p}(z) = v_s(z)e^{p\omega(|z_2|)}.$$

Then for every $s \in \mathbb{N}$ *(large enough) there exists* $p > 0$ *such that*

$$\sigma(x, D) : M^\infty_{m_{s,p}}(\mathbb{R}^d) \to M^\infty_{v_s}(\mathbb{R}^d).$$

Chapter 6

Concluding remarks, open problems and further research.

We devote this last chapter to make some concluding remarks, put together the open problems and further avenues of research.

We start by listing the open problems so far.

The first open problem left is to see if Theorem 2.3.11 remains true for ψ'', ψ''' and so on, i.e.

Open problem 1. *Let* $d > 1$ *and* $\psi \in C^\infty(\mathbb{R})$ *be given such that* $C_\psi\left(\Sigma_d(\mathbb{R})\right) \subset \Sigma_d(\mathbb{R})$. *Is* $\psi^{(\ell)}$ *bounded, for each* $\ell \geq 2$*?*

We lack a complete characterization of the symbols for $\mathcal{S}_{(\omega)}(\mathbb{R})$. Is condition (b) of Proposition 2.4.1 also a necessary condition? I.e.,

Open problem 2. *Let* ω *be a subadditive weight,* $a \geq 1$, *and* $\sigma(t) = \omega(t^{\frac{1}{a}})$. *If* $C_\psi\left(\mathcal{S}_{(\omega)}(\mathbb{R})\right) \subset \mathcal{S}_{(\sigma)}(\mathbb{R})$ *then,*

(a) $|x| \leq C_0(1 + |\psi(x)|)^a$ *for every* $x \in \mathbb{R}$.

(b) For every $m \in \mathbb{N}$ *there is* $C_m > 0$ *such that*

$$|\psi^{(j)}(x)| \leq C_m \exp(m\varphi^*_\sigma(\frac{j}{m}))(1 + \psi(x)|)^p \ \forall j \in \mathbb{N}\ \forall x \in \mathbb{R},$$

where $p = a - 1$.

In Theorem 2.2.2, it was shown that composition operators acting on $\mathcal{S}_{(\omega)}(\mathbb{R})$ are never compact, provided that ω is such that the Borel map $B : \mathcal{E}_{(\omega)}(\mathbb{R}) \to \mathcal{E}_{(\omega)}(\{0\})$ is open (this is the case for instance when ω is strong). The following problems remain unsolved:

Open problem 3. 1. *What happens with Theorem 2.2.2 when ω is not an strong weight?*

2. *Assume that $\psi \in C^\infty(\mathbb{R})$ satisfies $C_\psi(\mathcal{S}_{(\omega)}(\mathbb{R})) \subset \mathcal{S}_{(\sigma)}(\mathbb{R})$. Could the composition operator $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\sigma)}(\mathbb{R})$ be compact?*

In case of the weight ω being sub-additive, we proved that Theorem 2.4.7 was optimal (see Theorem 2.3.9). When ω is not sub-additive, we were still able to show Remark 2.4.8. Can we say more when ω is not sub-additive?

Open problem 4. *Is Theorem 2.4.7 true when ω is not sub-additive?*

Another open problem is the following:

Open problem 5. *If $C_\psi : \mathcal{S}_{(\omega)}(\mathbb{R}) \to \mathcal{S}_{(\sigma)}(\mathbb{R})$ then, is ψ a symbol for $\mathcal{S}(\mathbb{R})$?*

From Corollary 3.4.14 case 2., $\overline{\mathbb{D}}\setminus\{0\} \subset \sigma_{\mathcal{S}_\omega(\mathbb{R})}(C_\psi)$, where $\psi(x) = x^2 + \frac{1}{4}$, for $x \in \mathbb{R}$, provided that ω verifies condition (3.13). Clearly, $0 \in \sigma_{p,\mathcal{S}_\omega(\mathbb{R})}(C_\psi)$ since $\psi(x) \geq \frac{1}{4}$, for all $x \in \mathbb{R}$.

Open problem 6. *Let the weight ω verify condition* (3.13). *Is $\sigma_{\mathcal{S}_\omega(\mathbb{R})}(C_\psi) = \overline{\mathbb{D}}$, where $\psi(x) = x^2 + \frac{1}{4}$, for $x \in \mathbb{R}$?*

Open problem 7. What happens with Theorem 4.3.4 if the polynomial ψ appearing therein only have non-repelling fixed points. An example of such polynomial is $\psi(x) = x^2 + \frac{1}{4}$.

The following problem also remains unsolved:

Open problem 8. Let ω be any weight function (for instance, the Gevrey weight). Given $\lambda > 0$, $j, q \in \mathbb{N}_0$, there is $f \in \mathcal{S}_{(\omega)}(\mathbb{R})$ such that

$$p_\lambda(f) = \max_{x\in\mathbb{R}} |x|^q |f^{(j)}(x)| \exp\left(-\lambda\varphi_\omega^*\left(\frac{j+q}{\lambda}\right)\right).$$

Yet another unsolved question is:

Open problem 9. *Given a bijective linear map* $T : \mathbb{R}^N \to \mathbb{R}^N$*, can the composition operator* $C_T : \mathcal{S}_{(\omega)}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N)$ *be power bounded without being diagonalizable over* $\mathbb{R}$*? What are the eigenvalues* $\lambda \in \mathbb{D}$ *of* $C_T : \mathcal{S}_{(\omega)}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N)$*?*

A natural avenue of future research would be to study the most general case of weighted composition operators acting on Gelfald-Shilov classes on several variables in the spirit of [13].

Another avenue of future research is to consider the Roumieu case $\mathcal{S}_{\{\omega\}}(\mathbb{R})$.

Definition 6.0.1. *Let* ω *be a weight function. The Gelfand-Shilov space of Roumieu type* $\mathcal{S}_{\{\omega\}}(\mathbb{R})$ *consists of those functions* $f \in L^1(\mathbb{R})$ *with the property that* $f, \widehat{f} \in C^\infty(\mathbb{R})$ *and*

$$\max\Big\{\sup_{x\in\mathbb{R}} |f^{(j)}(x)|\, e^{\lambda\omega(x)},\ \sup_{\xi\in\mathbb{R}} |\widehat{f}^{(j)}(\xi)|\, e^{\lambda\omega(\xi)}\Big\} < +\infty$$

for some $\lambda > 0$*, and all* $j \in \mathbb{N}_0$.

When $\omega(t) = |t|^{\frac{1}{d}}$, $d > 1$, we recover the space described in Remark 2.3.10.

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