Group invariant mappings on Banach spaces and applications

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I, Daniel Isert Sales, with ID 53793897W, declare this dissertation, entitled *Group invariant mappings on Banach spaces and applications*, and the work presented in it are my own. I confirm that:

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- I have acknowledged all main sources of help.

Daniel Isert Sales

I declare that this dissertation presented by **Daniel Isert Sales** entitled *Group invariant mappings on Banach spaces and applicaitons* has been done under my supervision at Valencia University. I also state that this work corresponds to the thesis project approved by this institution and it satisfies all the requisites to obtain the degree of Doctor in Mathematics.

Francisco Javier Falcó Benavent

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Summary (English)

In this dissertation we study group invariant mappings obtaining generalizations of classical results in the theory of Banach spaces. We will focus our study on the case of compact topological groups.

The main contents of this document are organized in 4 chapters, where we cover the contents of the accepted paper [32], the published paper [33], and the submitted papers [34, 35]. In this section, we will summarize in English the contents of each chapter.

The first Chapter of this dissertation is just a translation to Catalan of Chapter 2.

Summary of Chapter 2

The contents of this chapter can be found in the article to be published in the Proceedings of XII Congreso del Máster en Investigación Matemática

[32] J. Falco, and D. Isert, Basic properties of the infinite dimensional group invariant points, sets and mappings, *Proceedings of XII Congreso del Máster en Investigación Matemática*, to appear.

The purpose of Chapter 2 is twofold. Firstly it serves us as an introductory chapter. We explain when the study of group invariant mappings started, to the extent of our knowledge, and establish the notation, adhering to the standard notation commonly used in the field. And secondly, it also serves for presenting some of the most basic properties that will provide better understanding of group invariant mappings. This will help us adopt our point of view from the classical setting to the new one. The three notions of group invariance that we are going to work with are the following.

Definition 1. Let X, Y be two normed spaces:

1. A point $x \in X$ is G-invariant, or invariant under the action of G if $g(x) = x$ for all $g \in G$.

2. A set $K \subset X$ is G-invariant if for every $g \in G$, $g(K) = K$.

3. A mapping $f\colon X \to Y$ is G-invariant if for every $x \in X$ and every $g \in G$ we have that
$$f(g(x)) = f(x).$$

We denote the set of all G-invariant points by X_G, and the set of all G-invariant functionals by X_G^*.

It is important to emphasize here the fact that we will consider the norm of the Banach spaces in which we are working with to be group invariant.

After these definitions, a selection of results concerning group invariant sets and mappings are presented. We also present some counterexamples to motivate and highlight these results. The most relevant result that we present in this chapter beyond any doubt is the following proposition that relates two notions of group invariance.

Proposition 2. *Let X be a normed space, and let G be a group acting on X. Then*

1. *$f\colon X \to Y$ is G-invariant if, and only if, $Gr(f)$ is (G, Id)-invariant.*

2. $f\colon X \to Y$ is G-invariant if, and only if, $Epi(f)$ is (G, Id)-invariant.

Summary of Chapter 3

The contents of this Chapter appeared as part of the published work

[33] J. Falco, and D. Isert, Group invariant variational principles, *Revista de la real academia de ciencias exactas, físicas y naturales. Serie A. Matemáticas* **10**(3) (2024), pp. 443–474.

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[34] J. Falco, and D. Isert, G-strong subdifferentiability and applications to norm attaining subspaces, *arXiv preprint arXiv:2403.18467* (2024).

[35] J. Falco, and D. Isert, Variational principles and applications to symmetric PDEs, *arXiv preprint arXiv:2403.18467* (2024).

This Chapter is of vital relevance for the rest of the dissertation since it contains all the necessary definitions, and previous results needed in the rest of the work. We will proceed now to do a study section by section of this chapter, and in each section we will point out when will the contents be relevant.

We start Section 3.1 recalling a few classical results related to the Haar measure and the Bochner integral. We will need this results to introduce one of the most important notions of this thesis which will be present in every chapter and section, the notion of symmetrized point and functional. This definition is a generalization of the Reynolds operator defined in [16, Chapter 7.3].

Definition 3. Let X be a Banach space and let G be a compact topological group acting on X. For $x \in X$ we define the symmetrization point of x with respect to G, or the G-symmetrization point, to be

$$\overline{x} = \int_G g(x)\, d\mu(g),$$

where the integral is the Bochner integral, and the measure is the Haar measure associated to the compact group G. Similarly, for a functional $F\colon X \to \mathbb{R}$, we define the symmetric functional of F with respect to G, or the G-symmetric functional, to be

$$\overline{F}(x) = \int_G F(g(x)) d\mu(g),$$

where again the integral is the Bochner integral and the measure is the Haar measure associated to the compact group G.

We also present the notions of linearity and convexity with respect to the group, and give two examples in which we observe that these notions are a bit more general to those of linearity and convexity. This two notions arose naturally to us while studying the generalization of Ekeland's variational principle.

Definition 4. Let X be a Banach space and G a compact topological group acting on X. Let $\varphi\colon X \to \mathbb{R}$ be a function. We say that φ is convex with respect to G given that

$$\varphi\left(\int_G g(x) d\mu(g)\right) \leqslant \int_G \varphi(g(x)) d\mu(g) \quad \forall\, x \in X.$$

And we say that φ is linear with respect to G if

$$\varphi\left(\int_G g(x) d\mu(g)\right) = \int_G \varphi(g(x)) d\mu(g) \quad \forall\, x \in X.$$

Let us show two examples to illustrate the previous definitions.

Example 5. Let $X = \mathbb{R}^2$ with the Euclidean topology and $G = \{Id, \sigma\}$, where

$$\begin{array}{rccc} \sigma: & \mathbb{R}^2 & \rightarrow & \mathbb{R}^2 \\ & (x,y) & \mapsto & (-x,y), \end{array}$$

and Id is the identity mapping. Now, define $f(x,y) = y^2$. This function is linear with respect to the group G but not linear

Example 6. Let $X = \mathbb{R}$ with the Euclidean topology and $G = \{Id, -Id\}$. Define

$$h(x) = \begin{cases} -x-1 & \text{if } x \leqslant -1, \\ x+1 & \text{if } -1 < x < 0, \\ -1 & \text{if } x = 0, \\ -x+1 & \text{if } 0 < x < 1, \\ x-1 & \text{if } x \geqslant 1. \end{cases}$$

This function is convex with respect to the group but not convex.

Section 3.2 is part of a current work in progress. We have two main aspirations in this study. First, to give a characterization of the well known result that ensures that for a space having the Bishop-Phelps-Bollobas property is equivalent to the space having the Radon-Nikódym property. And also, to give a negative example for the converse implication of [19, Theorem 4.21]. So, in this line of work, we present and adaptation to the G-invariant context of many definitions such as exposed, extremal and strongly exposed points.

Definition 7. Let X be a Banach space, G a compact topological group acting on X, and $C \subseteq X$ a closed, convex, and G-invariant set. We say that a point $x_0 \in C$ is G-exposed if there exists a functional $x^* \in (X^* \backslash \{0\})_G$ such that

1. $\langle x^*, x_0\rangle = \sup_{x\in C}\langle x^*, x\rangle$.

2. $\langle x^*, x\rangle < \sup_{y\in C}\langle x^*, y\rangle$ for all $x \in C\setminus\{x_0\}$.

We say that a point $x_0 \in C$ is G-strongly exposed if there exists a functional $x^* \in (X^*\setminus\{0\})_G$ such that for every sequence $\{x_n\}_{n=1}^{+\infty} \subseteq C$ with $\langle x^*, x_n\rangle \to \sup_{x\in C}\langle x^*, x\rangle$, then $\|\overline{x_n} - \overline{x_0}\| \to 0$.

Definition 8. Let V be a vectorial space, G a compact topological group acting on V, and C a convex and G-invariant subset. A point $x_0 \in C$ is G-extreme if for $x_1, x_2 \in C$ and $0 < \lambda < 1$ such that $\overline{x_0} = \lambda\overline{x_1} + (1-\lambda)\overline{x_2}$, then $\overline{x_0} = \overline{x_1} = \overline{x_2}$. We will denote the set of G-extremal points in C as $\text{Ext}_G(C)$.

We also talk briefly about G-slices which we define in the following way.

Definition 9. Let X be a Banach space, G a compact topological group acting on X, and A a G-invariant subset. We define the G-slice of A as

$$S_G(x^*, A, \alpha) = \left\{x \in A_G \mid \langle x^*, x\rangle > \sup_{x\in A}\langle x^*, x\rangle - \alpha\right\},$$

where $x^* \in X^*_G$ and $\alpha > 0$.

After this, we show the following well known relation between this three definitions remains unchanged in the group invariant case.

Proposition 10. *Let X be a Banach space, G a compact topological group acting on X, and C a closed convex and G-invariant set. If $x_0 \in C$ is G-strongly exposed, then x_0 is G-exposed. If x_0 is G-exposed, then x_0 is G-extreme.*

Finally we present a counterexample of the Krein-Milman theorem for G-invariant sets and we obtain the best alternative that we can in this context.

Example 11. Take in $\mathbb{R}^2$ the group $G = \{Id, \sigma\}$ where $\sigma\colon \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $\sigma(x,y) = (y,x)$. Consider the convex hull of the sets $B_{\|\cdot\|_2}((1,-1),1)$, $B_{\|\cdot\|_2}((-1,1),1)$, and $B_{\|\cdot\|_1}((0,0),1)$, and call the resultant set K. Then K is a G-invariant compact set which does not coincide with the closure of the convex hull of its G-extreme points.

Proposition 12. *Let X be a Banach space, G a compact topological group acting on X, and K a compact, convex and G-invariant subset. Then:*

$$K_G \subseteq \overline{conv}(Ext_G(K)) \subseteq K.$$

In [19] the authors explore a group invariant Hahn-Banach separation theorem. Section 3.3 keeps exploring more geometric forms of the Hahn-Banach theorem in our context of group invariant mappings. The results presented in this section will be of use in Chapter 4. We resume this results in the following theorem.

Theorem 13. *Let X be a Banach space and G a compact topological group acting on X.*

1. *If A, B are two convex and G-invariant sets, with A open and $A \cap B = \varnothing$, then there exists a G-invariant hyperplane that separates A and B.*

2. *If $A, B \subseteq X$ are nonempty, convex and G-invariant sets such that A is closed, B is compact and $A \cap B = \varnothing$, then there exists a G-invariant hyperplane that strictly separates A and B.*

The results studied in Section 3.4 will be used in Chapter 5. To continue, we generalize the concepts of weak and weak-star topologies, which we define in the following way.

Definition 14. Let X be a normed space and G a compact topological group acting on X. We define the weak group invariant topology on X

as the topology generated by the sets

$$\{x \in X \mid \langle f_i, x - x_0 \rangle < \epsilon, \text{ for } 1 \leqslant i \leqslant n\},$$

for all choices of $x_0 \in X$, $f_1, \ldots, f_n \in X_G^*$ and $\epsilon > 0$. We denote this topology by w_G or $\sigma_G(X, X^*)$.

We define the weak-star group invariant topology on X^* as the topology generated by the sets

$$\{f \in X^* \mid \langle f - f_0, x_i \rangle < \epsilon, \text{ for } 1 \leqslant i \leqslant n\},$$

for all choices of $f_0 \in X^*$, $x_1, \ldots, x_n \in X_G$ and $\epsilon > 0$. We denote this topology by w_G^* or $\sigma_G(X^*, X)$.

One of the properties that makes the weak and weak-star topologies so helpful is that they are Hausdorff. In this direction, we obtain the opposite result for the topologies that we just defined.

Proposition 15. *Let X be a normed space and G a compact topological group acting on X. If $X_G^* \subsetneq X^*$, then*

1. *The weak group invariant topology on X, w_G, is strictly weaker than the weak topology of X, w.*
2. *The topologies w_G and w_G^* are not Hausdorff.*

Finally we focus on some classic results concerning the reflexivity of a Banach space and the geometry of the ball. For this purpose we need to define the notion of G-reflexivity.

Definition 16. Let X be a Banach space and G a compact topological group acting on X. We say that X is G-reflexive if the canonical injection $\pi\colon X \to X^{**}$ is G-surjective, and by this we mean that, $\pi(X_G) = X_{G^{**}}^{**}$.

We resume now the results that we generalized.

Theorem 17. *Let X be a Banach space and G a compact topological group acting on X. Then*

1. *B_{X^*} is w_G^*-compact.*
2. *$\overline{B_X}^{w_G^*} = B_{X_{G^{**}}^{**}}$.*
3. *X is G-reflexive if, and only if, B_X is compact in the $\sigma_G(X, X^*)$ topology.*

To conclude this chapter, in section 3.5 we recall the notions of Fréchet and Gâteaux differentiability, and we give the definition of G-subdifferential. The results presented in this chapter will be required for Chapter 4 and Chapter 5.

Definition 18. Let X be a Banach space, G a group acting on X, and let $f\colon X \to]-\infty, +\infty]$ be a proper function. For $x_0 \in \mathrm{Dom} f$ we define the G-subdifferential of f at x_0 as

$$\partial_G f(x_0) = \{h \in X_G^* \mid \langle h, x - x_0\rangle \leqslant f(x) - f(x_0), \forall\, x \in X\}.$$

Taking a closer look to the definition of G-subdifferential we observe the three following equivalent ways of defining it:

$$\begin{aligned}\partial_G f(x_0) &= \{x^* \in X_G^* \mid \langle x^*, x - x_0\rangle \leqslant f(x) - f(x_0) \ \ \forall\, x \in X\}\\ &= \{x^* \in X_G^* \mid \langle x^*, \overline{x} - x_0\rangle \leqslant f(\overline{x}) - f(x_0) \ \ \forall\, x \in X\}\\ &= \{x^* \in X_G^* \mid \langle x^*, x - x_0\rangle \leqslant f(x) - f(x_0) \ \ \forall\, x \in X_G\}\,.\end{aligned}$$

Summary of Chapter 4

The first half of this Chapter appears in the published work

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And the second half in the submitted work

[35] J. Falco, and D. Isert, Variational principles and applications to symmetric PDEs, *arXiv preprint arXiv:2403.18467* (2024).

We start this chapter by presenting the G-invariant Ekeland's result, G-EVP. The most remarkable fact here is that when generalizing this result to G-invariant mappings, we cannot obtain in general a positive result, we need to add the condition of convexity with respect to the group on the mapping for the result to be true.

Theorem 19. *Let X be a Banach space and G a compact topological group acting on X. Let $\varphi\colon X \to]0, +\infty]$ be proper, lower semicontinuous, bounded below, G-invariant, and convex with respect to the group G. Then, given $\epsilon > 0$ and $\delta > 0$, there exists a G-invariant point $\tilde{x} \in X$ such that*

$$\varphi(\tilde{x}) < \varphi(x) + \epsilon\|\overline{x} - \tilde{x}\| \quad \forall\, x \in X,\ x \neq \tilde{x}.$$

Moreover, if $x_0 \in X$ satisfies that $\varphi(x_0) < \inf\{\varphi(x) \mid x \in X\} + \delta$, then we can choose $\tilde{x}$ to be such that

$$\|\overline{x_0} - \tilde{x}\| < \frac{\delta}{\epsilon}.$$

We also present an example of a function that is not convex with respect to the group and does not satisfies the result. To continue, in Section 4.2 we present many consequences of the G-EVP. One of the most remarkable consequences is the one that we present here. This result allows us to completely describe the dual space of G-invariant functionals.

Corollary 20. *Let X be a Banach space and G a compact topological group acting on X. Let φ be a continuous function, Gâteaux differentiable, bump, G-invariant and linear with respect to G. Then*

$$X_G^* = \overline{Span}\{\partial\varphi(x) \mid x \in X_G\}.$$

The two last consequences of the G-EVP that we study are the Bishop-Phelps theorem and the Brønsted-Rockafellar theorem, both of them in their respective G-invariant versions. We present both results here.

Theorem 21 (G-invariant Bishop-Phelps)**.** *Let X be a real Banach space and G a compact topological group acting on X. If $C \subseteq X$ is a convex, closed, bounded and G-invariant subset, then the norm-attaining functionals in C that are G-invariant are dense in X_G^*.*

Theorem 22 (G-invariant Brønsted-Rockafellar)**.** *Let X be a Banach space and G a compact topological group acting on X. Suppose that $f \colon X \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex, lower semicontinuous and G-invariant functional. Then for any G-invariant point x_0 and any functional x_0^*, with $x_0 \in Dom(f)$ and $x_0^* \in \partial_\epsilon f(x_0)$, and for all $\epsilon, \lambda > 0$, there exists a G-invariant point $z \in Dom(f)$ and a functional $x^* \in X_G^*$ such that*

$$x^* \in \partial_G f(x), \quad \|z - x_0\| \leqslant \frac{\epsilon}{\lambda}, \quad \|x^* - x_0^*\| \leqslant \lambda.$$

The second part of this chapter focuses in exploring many different equivalences of Ekeland's variational principle (EVP). In Section 4.3 we focus on the equivalences between EVP, the drop theorem and the petal theorem all of them in their G-invariant version. We present here the two new results that form the core of this section together with Theorem 19.

Theorem 23 (G-invariant Petal theorem)**.** *Let (M, d) be a metric space, $G \subseteq \mathcal{L}(M)$ be a compact topological group of isometries acting on M, and $C \subset M$ a complete G-invariant subset. Assume that d is G-invariant and convex with respect to G. Let $x_0 \in C_G$, $b \in (M \backslash C)_G$, $r \leqslant d(b, C)$ and $s = d(b, x_0)$. Then, for all $\gamma > 0$, there exists a G-invariant point $a \in C \cap P_\gamma(x_0, b)$ such that $C \cap P_\gamma(a, b) = \{a\}$.*

Theorem 24 (G-invariant Drop theorem)**.** *Let $(X, \|\cdot\|)$ be a normed space, $G \subseteq \mathcal{L}(X)$ be a compact topological group of isometries acting on X, and $C \subset X$ a complete G-invariant subset. Assume that $x_0 \in C_G$, and $B = \overline{B(b, r)}$, where $b \in X_G$ and $r < d(b, C)$. Then there exists a G-invariant point $a \in C \cap D(x_0, B)$ such that $C \cap D(a, B) = \{a\}$.*

Using an alternative approach in the previous two results we can omit the condition of the points x_0 being G-invariant, and use the original result for every translation of x_0 of the group and obtain more general results than the original ones. For this, we first define a new function that we use to see the lowest distance between a point and all of his translations of the group. This function is the following.

Definition 25. Let (M, d) be a metric space and G be a compact topological group of isometries acting on M. For a point $x \in E$ we define

$$s_G(x) := \inf\{d(x, g(x)) \mid g \in G \text{ and } g(x) \neq x\}.$$

The two more generalistic results that we obtained by using recursively the original ones are the following.

Proposition 26 (Flower petal theorem)**.** *Let (M, d) be a metric space, $G \subseteq \mathcal{L}(M)$ be a compact topological group of isometries acting on M, and $C \subset M$ a complete G-invariant subset of M. Let $x_0 \in C_G$, $b \in M \backslash C$. Then, for every $\gamma > 0$, there exists $a \in C \cap P_\gamma(x_0, b)$ such that*

$$C \cap P_\gamma(g(a), g(b)) = \{g(a)\} \text{ for every } g \in G.$$

Furthermore, for every $g, g' \in G$ *with* $d(g(b), g'(b)) > 2d(b, C)$ *we have that*

$$P_\gamma(g(a), g(b)) \cap P_\gamma(g'(a), g'(b)) = \varnothing.$$

Proposition 27 (Generalized drop theorem)**.** *Let* $(E, \|\cdot\|)$ *be a normed space,* $G \subseteq \mathcal{L}(E)$ *be a compact topological group of isometries acting on* E*, and* $C \subset E$ *be a complete* G*-invariant subset of* E*. Let* $x_0 \in C_G$*,* $b \in E \backslash C$*. Then, there exists* $a \in C \cap D(x_0, b)$ *such that*

$$C \cap D(g(a), g(b)) = \{g(a)\} \text{ for every } g \in G.$$

Furthermore, for every $g, g' \in G$ *with* $d(g(b), g'(b)) > 2d(b, C)$ *we have that*

$$D(g(a), g(b)) \cap D(g'(a), g'(b)) = \varnothing$$

for every $g, g' \in G$ *with* $g(b) \neq g'(b)$*.*

Observe that we use the function previously defined for guarantee that all the petals and all the drops that we obtain are disjoint.

To continue, in Section 4.4 we present the equivalence of between Ekeland's variational principle, Caristi-Kirk theorem, and Takahashi's theorem all of them in their G-invariant version. The main result of this section is the following.

Theorem 28. *Let* (M, d) *be a complete metric space and* $G \subseteq \mathcal{L}(M)$ *be a compact topological group of isometries acting on* M*. Then, the following results are equivalent:*

1. *Let* $U \subseteq M$ *be* G*-invariant satisfying that*

   $$\forall\, y \in S_0 \backslash U, \exists\, x \in M_G \text{ with } x \neq \overline{y} \text{ and } f(\overline{y}, x) + d(\overline{y}, x) \leqslant 0.$$

   Then, there exists $\hat{x} \in (S_0 \cap U)_G$*.*

2. *There exists* $\hat{x} \in S_0$ *such that* $\hat{x}$ *is* G*-invariant, and* $f(\hat{x}, x) + d(\hat{x}, x) > 0$ *for all* $x \in M$, $x \neq \hat{x}$.

3. *Suppose* $\forall\, y \in S_0$ *with* $\inf_{x \in M} f(\overline{y}, x) < 0$*, there exists*

$$x \in M_G \text{ with } x \neq \overline{y} \text{ and } f(\overline{y}, x) + d(\overline{y}, x) \leqslant 0.$$

   Then, there exists $\hat{x} \in (S_0)_G$ *such that* $f(\hat{x}, x) \geqslant 0$ *for all* $x \in M_G$.

4. *Let* $T \colon M \to M$ *be a multivalued mapping such that for every* $y \in S_0$ *there exists*

$$x \in (T(y))_G \text{ with } f(\overline{y}, x) + d(\overline{y}, x) \leqslant 0.$$

   Then, there exists $\hat{x} \in (S_0)_G$ *such that* $\hat{x} \in T(\hat{x})$.

Finally in section 4.5 we study some applications of the Ekeland's variational principle. The first one is a geometric application, it is a consequence of Proposition 27 related to the contingent cone of a convex set.

Theorem 29. *Let* X *be a normed space and* $G \subseteq \mathcal{L}(X)$ *be a compact topological group of isometries acting on* X*. Assume that* $C \subseteq X$ *is a complete* G*-invariant set. Let* $x_0 \in C$ *be* G*-invariant, and choose* $y \in X$ *such that the segment* $[x_0, y]$ *is not contained in* C*. Then, for each* $\rho > 0$ *and for each* $g \in G$*, there exists* $g(a) \in C$ *such that*

$$\|x_0 - g(a)\| \leqslant \|x_0 - g(y)\| + \rho, \text{ and } g(y) \notin g(a) + K_C(g(a)).$$

Moreover, if $s_G > 2d(z, C)$*, where* z *is such that* $z \in (E \backslash C)_G$ *and* $y = \lambda z$*, then for every* $g, g' \in G$*, we have that*

$$g(a) + K_C(g(a)) \cap g'(a) + K_C(g'(a)) = \varnothing.$$

In the text we can also find an application of theorem 24 in which we obtain a quite similar result to the previous one. Then, we move to present some applications of G-Ekeland's variational principle in PDE's. The first result we obtain ensures us that the perturbed Plateau problem has a unique G-invariant solution.

Theorem 30. *Suppose Ω, and $v_0 \in W_0^{1,1}(\Omega)$ are G-invariant. Then, there exists in $W^{-1,\infty}(\Omega)$ a neighbourhood of the origin, and a dense subset $\mathcal{T}$ in this neighbourhood, such that, for every G-invariant $T \in \mathcal{T}$, the perturbed minimal hypersurface equation*

$$T = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \frac{\nabla v}{(1+|\nabla v|^2)^{\frac{1}{2}}},$$

$$v - v_0 \in W_0^{1,1}(\Omega),$$

and the perturbed Plateau's problem

$$\inf \left(\int_\Omega 1 + |\nabla v|^2 dx \right)^{\frac{1}{2}} - \langle T, v \rangle,$$

$$v - v_0 \in W_0^{1,1}(\Omega),$$

both have a unique G-invariant solution.

We conclude this chapter by presenting an application of G-Ekeland's variational principle to control theory.

Theorem 31. *Suppose f satisfies the previous assumptions, and let $h\colon \mathbb{R}^n \to \mathbb{R}$ be a differentiable function. Then, for all $\epsilon > 0$, there exists a G-invariant measurable control v, whose trajectory is y, such that*

$$\begin{cases} h(y(T)) \leqslant \inf h(x(T)) + \epsilon, \\ \langle f(t, y(t), v(t)), p(t) \rangle \leqslant \min_{u \in K} \langle f(t, y(t), u(t)), p(t) \rangle + \epsilon, \end{cases}$$

where p is the solution of the differential system

$$\begin{cases} \frac{dp_i}{dt}(t) = -\sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i}(t, y(t), v(t))p_j(t) \quad \forall 1 \leqslant i \leqslant n, \\ p(T) = h'(y(T)). \end{cases}$$

Summary of Chapter 5

The contents of this Chapter appeared in the submitted work

[34] J. Falco, and D. Isert, G-strong subdifferentiability and applications to norm attaining subspaces, *arXiv preprint arXiv:2403.18467* (2024).

All of the work done in Chapter 3 and in Chapter 5 Section 5.1 is to give all the necessary prerequisites to show Theorem 39. In Section 5.1 the first result that we study is a Hahn-Banach extension theorem but on the predual. The result is the following.

Theorem 32. *Let X be a Banach space, G a compact topological group acting on X, and let A be w^*_G-closed, convex and G^*-invariant in X^*. If $f \in X^* \backslash A$ is G-invariant, then there exists a G-invariant point, x, such that* $\sup_{h \in A}\langle h, x\rangle < \langle f, x\rangle$.

Then, we move to study G-James theorem. Before presenting the result, we obtained this previous one which is quite natural. This result tells us that for a spaces being G-reflexive is an equivalent condition for the space X_G being reflexive.

Theorem 33. *Let X be a Banach space and G a compact topological group acting on X. Then, X is G-reflexive if, and only if, X_G is reflexive.*

With this result in mind, and [30, Theorem 6], we can easily obtain James G-invariant.

Theorem 34 (James G-invariant)**.** *Let X be a Banach space and G a compact topological group acting on X. Then X is G-reflexive if, and only if, every G-invariant functional is norm-attaining.*

Then, we present the G-invariant version of Moreau theorem. This theorem is just a restriction to the space of G-invariant point of the classical result. In this direction we could not obtain a better result than this one.

Proposition 35 (Moreau G-invariant)**.** *Let X be a Banach space and G a compact topological group acting on X. Let $f\colon X \to \mathbb{R}\cup\{+\infty\}$ be a proper, convex, G-invariant function that is continuous at $x_0 \in Dom(f) \cap X_G$. Then,*

$$d^+ f(x_0)(x) = \sup\{\langle x^*, x\rangle \ \mid \ x^* \in \partial_G f(x_0)\} \quad \forall\, x \in X_G.$$

Moreover, this supremum is attained at some point $x^* \in \partial_G f(x_0)$.

We also give an example of why this result cannot be improved by taking the condition of x being G-invariant. The example is the following.

Example 36. Let $X = \mathbb{R}$ and $G = \{Id, -Id\} \subseteq \mathbb{R}$. Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x|$. Then, if $x_0 = 0$ which is the only G-invariant point, and $x = 1$ which is not G-invariant, observe that

$$d^+ f(0)(1) = 1, \text{ and } \partial_G|\cdot|(0) = \{0\}.$$

Henche, in general, G-invariant Moreau's maximum formula is not true if we drop the condition of $x \in X_G$.

Finally, we quickly present the notion of G-polar set and move to present Theorem 38.

Definition 37. Let X be a Banach space, G a compact topological group acting on X, and A a subset of X. We define the G-invariant polar set as follows

$$A^{\circ_G} = \{x^* \in X_G^* \mid |\langle x^*, x\rangle| \leqslant 1, \ \forall\, x \in A\}\,.$$

Theorem 38 (Krein-Smulian G-invariant)**.** *Let X be a Banach space, G a compact topological group acting on X, and $C \subseteq X^*$ a G^*-invariant convex set. If $\delta B_{X^*} \cap C$ is w_G^*-closed for every $\delta > 0$, then C is w_G^*-closed in X^*.*

Finally, in Section 5.2 we present a few definitions such as G-SSD, G-dissipative set, G-duality mapping, and study some relations between them. Then, we obtain the necessary condition for a space to be G-reflexive by means of the G-SSD, which is the following.

Theorem 39 (Main result)**.** *Let X be a Banach space and G a compact topological group acting on X. If X^* is G-SSD, then X is G-reflexive.*

We close this chapter with two applications of this result to the existence of large vectorials spaces inside the set of norm-attaining functionals, and a relation of having vectorial spaces of norm-attaining functionals and the property of being G-reflexive.

Corollary 40. *Let X be a Banach space and G a compact topological group acting on X. If X^* is G-SSD, then the set of norm-attaining operators on X contains, at least, the vectorial space X_G^*.*

Proposition 41. *Let X be a Banach space. If the set of norm-attaining functionals on X contains a finite dimensional Banach space E, then there exists a compact topological group G acting on X such that $E = X_G^*$. In particular, X is G-reflexive.*

We conclude the present dissertation with a Conclusions chapter, which includes some remarks and open questions. At the end of the document, there is an extensive list of references used in this work.

Resumen (Castellano)

En esta tesis estudiamos aplicaciones grupo invariantes, obteniendo generalizaciones de resultados clásicos en la teoría de espacios de Banach. Centraremos nuestro estudio en el caso de grupos topológicos compactos.

Los principales contenidos de este documento están organizados en cuatro capítulos, en los cuáles cubrimos los contenidos del artículo aceptado [32], el artículo publicado [33], y los artículos enviados a revisión por pares [34, 35]. En esta sección resumiremos en Castellano los contenidos de cada capítulo.

El primer Capítulo de esta tesis es una traducción al Valenciano del Capítulo 2.

Resumen del Capítulo 2

Los contenidos de este capítulo se pueden encontrar en el artículo pendiente de publicación en

> [32] J. Falco, and D. Isert, Basic properties of the infinite dimensional group invariant points, sets and mappings, *Proceedings of XII Congreso del Máster en Investigación Matemática*, to appear.

El Capítulo 2 tiene doble propósito, en primer lugar nos sirve como un capítulo introductorio. Explicamos cuándo empieza a aparecer la investigación de aplicaciones grupo invariantes, en la medida de nuestro conocimiento, y fijamos notación, adheriéndonos a las notaciones estándar usadas habitualmente en el área. En segundo lugar, nos sirve para presentar algunas de las propiedades más elementales que nos aportará un mejor conocimiento de las aplicaciones grupo invariantes. También nos ayudará a adaptar nuestro punto de vista del escenario clásico al nuevo escenario. Las tres nociones de grupo invarianza con las que vamos a trabajar son las siguientes.

Definición 1. Sean X, Y dos espacios normados:

1. Un punto $x \in X$ es G-invariante, o invariante sobre la acción de G, si $g(x) = x$ para todo $g \in G$.

2. Un conjunto $K \subset X$ es G-invariante si para todo $g \in G$, $g(K) = K$.

3. Una aplicación $f\colon X \to Y$ es G-invariante si para todo $x \in X$ y todo $g \in G$ tenemos
$$f(g(x)) = f(x).$$

Denotamos el conjunto de todos los puntos G-invariantes como X_G, y el conjunto de todos los funcionales G-invariantes como X^*_G.

Es importante denotar aquí el hecho de que vamos a considerar que la norma de los espacios de Banach con los que trabajemos, será grupo invariante.

Después de estas definiciones, presentamos una selección de resultados relativos a aplicaciones y conjuntos grupo invariantes. También presentamos algunos contraejemplos para motivar y subrallar estos resultados. Pero, el resultado más importante que presentamos en este capítulo sin ninguna duda, es la siguiente proposición, que relaciona dos nociones de grupo invarianza.

Proposición 2. *Sea X un espacio normado, y sea G un grupo actuando sobre X. Entonces*

1. $f\colon X \to Y$ *es* G*-invariante si, y sólo si,* $Gr(f)$ *es* (G, Id)*-invariante.*

2. $f\colon X \to Y$ *es* G*-invariante si, y sólo si,* $Epi(f)$ *es* (G, Id)*-invariante.*

Resumen del Capítulo 3

Los contenidos de este capítulo han aparecido en el artículo publicado

[33] J. Falco, and D. Isert, Group invariant variational principles, *Revista de la real academia de ciencias exactas, físicas y naturales. Serie A. Matemáticas* **10**(3) (2024), pp. 443–474.

y en los dos trabajos enviados al proceso de revisión por pares

[34] J. Falco, and D. Isert, G-strong subdifferentiability and applications to norm attaining subspaces, *arXiv preprint arXiv:2403.18467* (2024).

[35] J. Falco, and D. Isert, Variational principles and applications to symmetric PDEs, *arXiv preprint arXiv:2403.18467* (2024).

Este capítulo es de vital importancia para el resto de la tesis, ya que contiene todas las definiciones necesarias y resultados previos necesarios en los capítulos posteriores. Vamos a proceder a continuación a realizar un estudio sección por sección del capítulo, y en cada sección anotaremos cuándo los contenidos de dicha sección van a ser relevantes.

Empezamos en la Sección 3.1 recordando algunos resultados de medida de Haar y integral de Bochner. Estos resultados los necesitaremos para introducir una de las nociones más importantes que va a estar presente en cada capítulo y sección de esta tesis. La definición de punto y funcional simetrizado. Esta definición es una generalización del operador de Reynolds definido en [16, Chapter 7.3]

Definición 3. Sea X un espacio de Banach y sea G un grupo topológico compacto actuando sobre X. Para $x \in X$ definimos el punto simetrizado de x con respecto de G, o el punto G-simetrizado, como

$$\overline{x} = \int_G g(x)\, d\mu(g),$$

donde la integral denota la integral de Bochner, y la medida es la medida de Haar asociada al grupo compacto G.

De forma similar, para un funcional $F\colon X \to \mathbb{R}$ definimos el funcional simétrico de F con respecto de G, o el funcional G-simétrico, como

$$\overline{F}(x) = \int_G F(g(x)) d\mu(g),$$

donde, de nuevo, la integral denota la integral de Bochner, y la medida es la medida de Haar asociada al grupo compacto G.

También presentamos las dos nociones de linearidad y convexidad con respecto del grupo, y damos dos ejemplos en los que observamos que estas dos nociones son un poco más restrictivas que las de linealidad y convexidad. Estas dos nociones aparecen de forma natural cuando estudiamos la generalización del principio variacional de Ekeland.

Definición 4. Sea X un espacio de Banach y G un grupo topológico compacto actuando sobre X. Sea $\varphi\colon X \to \mathbb{R}$ una función. Decimos que φ es convexa con respecto de G si

$$\varphi\left(\int_G g(x) d\mu(g)\right) \leqslant \int_G \varphi(g(x)) d\mu(g) \quad \forall\, x \in X.$$

Decimos que φ es lineal con respecto de G si

$$\varphi\left(\int_G g(x) d\mu(g)\right) = \int_G \varphi(g(x)) d\mu(g) \quad \forall\, x \in X.$$

A continuación, presentamos dos ejemplos para ilustrar las definiciones previas.

Ejemplo 5. Sean $X = \mathbb{R}^2$ con la topología euclidiana y $G = \{Id, \sigma\}$, donde

$$\begin{array}{rccc} \sigma: & \mathbb{R}^2 & \to & \mathbb{R}^2 \\ & (x,y) & \mapsto & (-x,y), \end{array}$$

y Id es la aplicación identidad. Ahora, definimos $f(x,y) = y^2$. Esta función es lineal con respecto del grupo G pero no lineal.

Ejemplo 6. Sea $X = \mathbb{R}$ con la topología euclidiana y sea $G = \{Id, -Id\}$. Definimos

$$h(x) = \begin{cases} -x-1 & \text{si } x \leqslant -1, \\ x+1 & \text{si } -1 < x < 0, \\ -1 & \text{si } x = 0, \\ -x+1 & \text{si } 0 < x < 1, \\ x-1 & \text{si } x \geqslant 1. \end{cases}$$

Esta es una función convexa con respecto del grupo pero no convexa.

La Sección 3.2 es un adelanto de un trabajo futuro en el que estamos trabajando. Tenemos dos objetivos principales en este estudio. Primero, generalizar una caracterización de un resultado bien conocido que garantiza que un espacio tiene la propiedad de Bishop-Phelps-Bollobás si, y sólo si, el espacio tiene la propiedad de Radon-Nikódym. Y también queremos dar un ejemplo negativo a la implicación inversa del resultado [16, Theorem 4.21]. Así pues, presentamos una adaptación al contexto G-invariante de muchos conceptos tales como puntos expuestos, fuertemente expuestos y extremales.

Definición 7. Sea X un espacio de Banach, G un grupo topológico compacto actuando sobre X, y $C \subseteq X$ un conjunto cerrado, convexo y

G-invariante. Decimos que un punto $x_0 \in C$ es G-expuesto si existe un funcional $x^* \in (X^* \setminus \{0\})_G$ tal que

1. $\langle x^*, x_0 \rangle = \sup_{x \in C} \langle x^*, x \rangle$.

2. $\langle x^*, x \rangle < \sup_{y \in C} \langle x^*, y \rangle$ para todo $x \in C \setminus \{x_0\}$.

Decimos que un punto $x_0 \in C$ es G-fuertemente expuesto si existe un funcional $x^* \in (X^* \setminus \{0\})_G$ tal que para toda sucesión $\{x_n\}_{n=1}^{+\infty} \subseteq C$ que satisface $\langle x^*, x_n \rangle \to \sup_{x \in C} \langle x^*, x \rangle$, entonces $\|\overline{x_n} - \overline{x_0}\| \to 0$.

Definición 8. Sea V un espacio vectorial, G un grupo topológico compacto actuando sobre V, y C un subconjunto convexo y G-invariant. Un punto $x_0 \in C$ es G-extremo si para todo $x_1, x_2 \in C$ y $0 < \lambda < 1$ tales que $\overline{x_0} = \lambda \overline{x_1} + (1 - \lambda) \overline{x_2}$, entonces $\overline{x_0} = \overline{x_1} = \overline{x_2}$. Denotamos el conjunto de puntos G-extremales en C como $\mathrm{Ext}_G(C)$.

También hablamos brevemente de las G-porciones, las cuales definimos de la siguiente forma.

Definición 9. Sea X un espacio de Banach, G un grupo topológico compacto actuando sobre X, y A un subconjunto G-invariante. Definimos la G-porción cómo

$$S_G(x^*, A, \alpha) = \left\{ x \in A_G \mid \langle x^*, x \rangle > \sup_{x \in A} \langle x^*, x \rangle - \alpha \right\},$$

dónde $x^* \in X_G^*$ y $\alpha > 0$.

Tras esto, observamos que la siguiente relación entre los puntos expuestos, fuertemente expuestos, y extremos, sigue siendo cierta en el caso grupo invariante.

Proposición 10. *Sea X un espacio de Banach, G un grupo topológico compacto actuando sobre X, y C un conjunto cerrado, convexo y G-invariante. Si $x_0 \in C$ es G-fuertemente expuesto, entonces x_0 es G-expuesto. Si x_0 es G-expuesto, entonces x_0 es G-extremo.*

Finalmente presentamos un contraejemplo a la versión G-invariante del teorema de Krein-Milman, y también presentamos la mejor alternativa que podemos obtener en este contexto.

Ejemplo 11. Tomamos en $\mathbb{R}^2$ el grupo $G = \{Id, \sigma\}$ donde $\sigma\colon \mathbb{R}^2 \to \mathbb{R}^2$ está definida como $\sigma(x, y) = (y, x)$. Consideramos la combinación convexa de los conjuntos $B_{\|\cdot\|_2}((1, -1), 1)$, $B_{\|\cdot\|_2}((-1, 1), 1)$, y $B_{\|\cdot\|_1}((0, 0), 1)$, y denotamos al conjunto resultante K. Entonces K es un conjunto compacto y G-invariante que no coincide con la clausura de la combinación convexa de sus puntos G-extremales.

Proposición 12. *Sea X un espacio de Banach, G un grupo topológico compacto actuando sobre X, y K un subconjunto compacto, convexo y G-invariante. Entonces*

$$K_G \subseteq \overline{conv}(Ext_G(K)) \subseteq K.$$

En [19] los autores estudian un teorema de separación de Hahn-Banach con funcionales grupo invariantes. En la sección 3.3 nos ocupamos de estudiar más resultados geométricos de separación de Hahn-Banach en nuestro contexto de aplicaciones grupo invariantes. Los resultados que presentamos en esta sección van a ser de gran utilidad en el Capítulo 4. Resumimos los resultados en el siguiente teorema.

Teorema 13. *Sea X un espacio de Banach, G un grupo topológico compacto actuando sobre X.*

1. *Sean A, B son dos conjuntos convexos y G-invariantes, donde A es abierto y $A \cap B = \varnothing$, entonces, existe un hiperplano G-invariante que separa A y B.*

2. *Si $A, B \subseteq X$ son dos conjuntos no vacíos, convexos y G-invariantes tales que A es cerrado, B es compacto y $A \cap B = \varnothing$. Entonces, existe un hiperplano G-invariant que separa estríctamente A y B.*

Los resultados estudiados en la Sección 3.4 van a ser de utilidad en el Capítulo 5. Aquí generalizamos las nociones de topología débil y topología débil-estrella, las cuales definimos de la siguiente forma.

Definición 14. Sea X un espacio normado y G un grupo topológico compacto actuando sobre X. Definimos la topología débil grupo invariant en X como la topología generado por los conjuntos

$$\{x \in X \ \mid \ \langle f_i, x - x_0 \rangle < \epsilon, \ \text{para } 1 \leqslant i \leqslant n\},$$

para cualquier elección de $x_0 \in X$, $f_1, \ldots, f_n \in X_G^*$ y $\epsilon > 0$. Denotamos esta topología por w_G o $\sigma_G(X, X^*)$.

Definimos la topología débil-estrella grupo invariant en X^* cómo la topología generado por los conjuntos

$$\{f \in X^* \ \mid \ \langle f - f_0, x_i \rangle < \epsilon, \ \text{para } 1 \leqslant i \leqslant n\},$$

para cualquier elección de $f_0 \in X^*$, $x_1, \ldots, x_n \in X_G$ y $\epsilon > 0$. Denotamos esta topología por w_G^* or $\sigma_G(X^*, X)$.

Una de las propiedades que hace tan útiles a las topologías débiles es que son Hausdorff. En esta dirección obtenemos justo el resultado opuesto para las topologías que acabamos de definir.

Proposición 15. *Sea X un espacio normado, G un grupo topológico compacto actuando sobre X. Si $X_G^* \subsetneq X^*$, entonces*

1. *La topología débil grupo invariante en X, w_G, es estríctamente más débil que la topología débil en X, w.*
2. *Las topologías w_G y w_G^* no son Hausdorff.*

Finalmente, estudiamos algunos resultados clásicos de reflexividad de un espacio de Banach, y de geometría de la bola. Con esta finalidad es necesario definir la noción de G-reflexividad.

Definición 16. Sea X un espacio de Banach y G un grupo topológico compacto actuando sobre X. Decimos que X es G-reflexivo si la inyección canónica $\pi\colon X \to X^{**}$ es G-sobreyectiva, y con esto queremos decir que, $\pi(X_G) = X^{**}_{G^{**}}$.

Resumimos a continuación los resultados que hemos obtenido.

Teorema 17. *Sea X un espacio de Banach y G un grupo topológico compacto actuando sobre X. Entonces*

1. *B_{X^*} es w^*_G-compacta.*
2. *$\overline{B_X}^{w^*_G} = B_{X^{**}_{G^{**}}}$.*
3. *X es G-reflexivo si, y sólo si, B_X es compacta en la topología $\sigma_G(X, X^*)$.*

En la sección 3.5 recordamos las definiciones de diferenciabilidad Fréchet y Gâteaux, y damos la definición de G-subdiferencial. Los resultados de esta sección van a ser necesarios en los dos capítulos siguientes, es decir en el Capítulo 4 y en el Capítulo 5.

Definición 18. Sea X un espacio de Banach, G un grupo topológico compacto actuando sobre X, y sea $f\colon X \to]-\infty, +\infty]$ una función propia. Para $x_0 \in \mathrm{Dom} f$ definimos la G-subdifferencial de f en el punto x_0 como

$$\partial_G f(x_0) = \{h \in X^*_G \ \mid \ \langle h, x - x_0\rangle \leqslant f(x) - f(x_0),\ \forall\, x \in X\}.$$

Si observamos más de cerca la definición de G-subdiferencial, podemos observar las siguientes tres formas equivalentes de definirla.

$$\begin{aligned}\partial_G f(x_0) &= \{x^* \in X^*_G \mid \langle x^*, x - x_0\rangle \leqslant f(x) - f(x_0)\ \ \forall\, x \in X\}\\ &= \{x^* \in X^*_G \mid \langle x^*, \overline{x} - x_0\rangle \leqslant f(\overline{x}) - f(x_0)\ \ \forall\, x \in X\}\\ &= \{x^* \in X^*_G \mid \langle x^*, x - x_0\rangle \leqslant f(x) - f(x_0)\ \ \forall\, x \in X_G\}\,.\end{aligned}$$

Resumen del Capítulo 4

La primera mitad del capítulo se puede encontrar en el artículo publicado

[33] J. Falco, and D. Isert, Group invariant variational principles, *Revista de la real academia de ciencias exactas, físicas y naturales. Serie A. Matemáticas* **10**(3) (2024), pp. 443–474.

Y la segunda mitad se puede encontrar en el trabajo en proceso de revisión por pares

[35] J. Falco, and D. Isert, Variational principles and applications to symmetric PDEs, *arXiv preprint arXiv:2403.18467* (2024).

Comenzamos este capítulo presentando el principio variacional de Ekeland en su versión G-invariante, G-EVP. El hecho más interesante aquí es que cuando intentamos generalizar este resultado para aplicaciones G-invariantes, necesitamos añadir la condición de convexidad con respecto del grupo en la aplicación de partida para que el resultado sea cierto.

Teorema 19. *Sea X un espacio de Banach y G un grupo topológico compacto actuando sobre X. Sea $\varphi\colon X \to]0, +\infty]$ una función propia, semicontínua inferiormente, acotada inferiormente, G-invariante y convexa con respecto del grupo G. Entonces, dado $\epsilon > 0$ y $\delta > 0$, existe un punto G-invariante $\tilde{x} \in X$ tal que*

$$\varphi(\tilde{x}) < \varphi(x) + \epsilon\|\overline{x} - \tilde{x}\| \quad \forall\, x \in X,\ x \neq \tilde{x}.$$

Además, si $x_0 \in X$ satisface que $\varphi(x_0) < \inf\{\varphi(x) \mid x \in X\} + \delta$, entonces podemos encontrar $\tilde{x}$ de forma que

$$\|\overline{x_0} - \tilde{x}\| < \frac{\delta}{\epsilon}.$$

También presentamos un ejemplo de función que no es convexa con respecto del grupo y que no satisface el teorema. A continuación, en la Sección 4.2 presentamos una serie de consecuendias del G-EVP. Una de las consecuencias más distinguidas que obtenemos es la que presentamos a continuación. Este resultado nos permite describir completamente el espacio dual formado por funcionales G-invariantes.

Corolario 20. *Sea X un espacio de Banach y G un grupo topológico compacto actuando sobre X. Sea φ una función contínua, diferenciable Gâteaux, meseta, G-invariante y lineal con respecto de G. Entonces*

$$X_G^* = \overline{Span}\{\partial\varphi(x) \quad | \quad x \in X_G\}.$$

Las últimas dos consecuencias que estudiamos del G-EVP son el teorema de Bishop-Phelps, y el teorema de Brønsted-Rockafellar, ambos en su respectiva versión G-invariante. A continuación mostramos los dos resultados.

Teorema 21 (Bishop-Phelps G-invariante)**.** *Sea X un espacio de Banach y G un grupo topológico compacto actuando sobre X. Sea $C \subseteq X$ es un subconjunto convexo, cerrado, acotado y G-invariante, entonces los funcionales que alcanzan la norma en C que son G-invariantes forman un conjunto denso en X_G^*.*

Teorema 22 (Brønsted-Rockafellar G-invariante)**.** *Sea X un espacio de Banach y G un grupo topológico compacto actuando sobre X. Supongamos que $f\colon X \to \mathbb{R} \cup \{+\infty\}$ es propia, convexa, semicontínua inferiormente y G-invariante. Entonces, para cualquier punto G-invariante x_0 y cualquier funcional x_0^* con $x_0 \in Dom(f)$ y $x_0^* \in \partial_\epsilon f(x_0)$, y para cualquier $\epsilon, \lambda > 0$, existen un punto G-invariante $z \in Dom(f)$ y un funcional $x^* \in X_G^*$ tales que*

$$x^* \in \partial_G f(x), \quad \|z - x_0\| \leqslant \frac{\epsilon}{\lambda}, \quad \|x^* - x_0^*\| \leqslant \lambda.$$

La segunda parte de este capítulo se centra en el estudio de diversas equivalencias del EVP. En la Sección 4.3 nos centramos en la equivalencia entre EVP, el teorema de la gota, y el teorema del pétalo todos ellos en su versión G-invariante. Presentamos a continuación los dos nuevos resultados que forman el centro de esta sección junto con el Teorema 19.

Teorema 23 (Teorema del pétalo G-invariante)**.** *Sea (M, d) un espacio métrico, $G \subseteq \mathcal{L}(M)$ un grupo topológico compacto de isometrías actuando sobre M, y $C \subset M$ un subconjunto completo G-invariante. Supongamos que d es (G, G)-invariante y convexa con respecto de G. Sean $x_0 \in C_G$, $b \in (M \backslash C)_G$, $r \leqslant d(b, C)$ y $s = d(b, x_0)$. Entonces, para todo $\gamma > 0$, existe un punto G-invariante $a \in C \cap P_\gamma(x_0, b)$ tal que $C \cap P_\gamma(a, b) = \{a\}$.*

Teorema 24 (Teorema de la gota G-invariante)**.** *Sea $(X, \|\cdot\|)$ un espacio normado, $G \subseteq \mathcal{L}(X)$ un grupo topológico compacto de isometrías actuando sobre X, y $C \subset X$ un subconjunto completo y G-invariante. Seupongamos que $x_0 \in C_G$, y $B = \overline{B(b, r)}$, donde $b \in X_G$ y $r < d(b, C)$. Entonces, existe un punto G-invariante $a \in C \cap D(x_0, B)$ tal que $C \cap D(a, B) = \{a\}$.*

Utilizando un enfoque alternativo en los dos resultados previos, podemos prescindir de la condición de que x_0 sea G-invariante, y utilizar el resultado clásico para cada elemento en la órbita de x_0, es decir, para $g(x_0)$ para cualquier $g \in G$, y así obtener resultados un poco más generales que los originales. Para ello, primero necesitamos definir una nueva función que usamos para cálcular cuál es la distáncia más pequeña entre un punto y todas sus traslaciones por elementos del grupo. La función es la siguiente.

Definición 25. Sean (M, d) un espacio métrico y G un grupo topológico compacto de isometrías actuando sobre M. Dado un punto $x \in M$ definimos

$$s_G(x) := \inf\{d(x, g(x)) \mid g \in G \text{ and } g(x) \neq x\}.$$

En los siguientes resultados, generalizaciones del teorema de la gota y del teorema del pétalo.

Proposición 26 (Teorema de la flor)**.** *Sea* (M, d) *un espacio métrico,* $G \subseteq \mathcal{L}(M)$ *un grupo topológico compacto de isometrías actuando sobre* M*, y* $C \subset M$ *un subconjunto de* M *completo y* G*-invariante. Sean* $x_0 \in C_G$ *y* $b \in M \backslash C$*. Entonces, para todo* $\gamma > 0$*, existe un punto* $a \in C \cap P_\gamma(x_0, b)$ *tal que*

$$C \cap P_\gamma(g(a), g(b)) = \{g(a)\} \text{ para toda } g \in G.$$

Además, para toda $g, g' \in G$ *verificando* $d(g(b), g'(b)) > 2d(b, C)$ *tenemos que*

$$P_\gamma(g(a), g(b)) \cap P_\gamma(g'(a), g'(b)) = \varnothing.$$

Proposición 27 (Teorema de la gota generalizado)**.** *Sea* $(E, \|\cdot\|)$ *un espacio normado,* $G \subseteq \mathcal{L}(E)$ *un grupo topológico compacto de isometrías actuando sobre* E*, y* $C \subset E$ *un subconjunto de* E *completo y* G*-invariante. Sean* $x_0 \in C_G$ *y* $b \in E \backslash C$*. Entonces, existe un punto* $a \in C \cap D(x_0, b)$ *tal que*

$$C \cap D(g(a), g(b)) = \{g(a)\} \text{ para toda } g \in G.$$

Además, dadas $g, g' \in G$ *verigicando* $d(g(b), g'(b)) > 2d(b, C)$ *tenemos que*

$$D(g(a), g(b)) \cap D(g'(a), g'(b)) = \varnothing$$

para cualesquiera $g, g' \in G$ *con* $g(b) \neq g'(b)$*.*

Observamos el papel que juega la función anterior para garantizar que todos los pétalos y todas las gotas que obtenemos sean disjuntas.

A continuación, en la Sección 4.4 presentamos la equivalencia entre EVP, el teorema de Caristi-Kirk, y el teorema de Takahashi, todos ellos en sus respectivas versiones G-invariantes. El resultado principal de esta sección es el siguiente.

Teorema 28. *Sean (M, d) un espacio métrico completo y $G \subseteq \mathcal{L}(M)$ un grupo topológico de isometrías compacto actuando sobre M. Entonces, los resultados siguientes son equivalentes:*

1. *Sea $U \subseteq M$ G-invariante que satisface*

$$\forall\, y \in S_0 \backslash U,\ \exists\, x \in M_G \text{ donde } x \neq \overline{y} \text{ y } f(\overline{y}, x) + d(\overline{y}, x) \leqslant 0.$$

   Entonces, existe un punto $\hat{x} \in (S_0 \cap U)_G$.

2. *Existe $\hat{x} \in S_0$ tal que $\hat{x}$ es G-invariante, y $f(\hat{x}, x) + d(\hat{x}, x) > 0$ para todo $x \in M$, $x \neq \hat{x}$.*

3. *Supongamos que $\forall\, y \in S_0$ con $\inf_{x \in M} f(\overline{y}, x) < 0$ existe*

$$x \in M_G \text{ con } x \neq \overline{y} \text{ y } f(\overline{y}, x) + d(\overline{y}, x) \leqslant 0.$$

   Entonces, existe $\hat{x} \in (S_0)_G$ tal que $f(\hat{x}, x) \geqslant 0$ para todo $x \in M_G$.

4. *Sea $T\colon M \to M$ una aplicación multivaluada tal que para todo $y \in S_0$ existe*

$$x \in (T(y))_G \text{ con } f(\overline{y}, x) + d(\overline{y}, x) \leqslant 0.$$

   Entonces, existe $\hat{x} \in (S_0)_G$ tal que $\hat{x} \in T(\hat{x})$.

Finalmente, en la Sección 4.5 estudiamos algunas aplicaciones del EVP. La primera es una aplicación geométrica relacionada con el cono contingente de un conjunto convexo. Esta aplicación es consecuencia del Teorema 27.

Teorema 29. *Sea X un espacio normado y $G \subseteq \mathcal{L}(X)$ un grupo topológico compacto de isometrías actuando sobre X. Supongamos que $C \subseteq X$ es un conjunto completo y G-invariante. Sea $x_0 \in C$, G-invariante, y consideremos $y \in X$ de forma que el segmento $[x_0, y]$ no* *está contenido en C. Entonces, para cada $\rho > 0$ y toda $g \in G$, existe $g(a) \in C$ tal que*

$$\|x_0 - g(a)\| \leqslant \|x_0 - g(y)\| + \rho, \;\; y \; g(y) \notin g(a) + K_C(g(a)).$$

Además, si $s_G > 2d(z, C)$, dónde z es tal que $z \in (E \backslash C)_G$ y $y = \lambda z$, entonces dadas $g, g' \in G$ cualesquiera, tenemos que

$$g(a) + K_C(g(a)) \cap g'(a) + K_C(g'(a)) = \varnothing.$$

En el texto también podemos encontrar una aplicación del Teorema 24 en la que obtenemos un resultado parecido al que terminamos de presentar. Seguidamente, pasamos a presentar unas aplicaciones del G-EVP en ecuaciones en derivadas parciales. El primer resultado que obtenemos nos asegura que el problema de Plateau perturbado tiene una única solución G-invariante.

Teorema 30. *Supongamos que Ω, y $v_0 \in W_0^{1,1}(\Omega)$ son G-invariantes. Entonces, existe en $W^{-1,\infty}(\Omega)$ un entrono del zero, y un subconjunto denso $\mathcal{T}$ en este entorno, tal que, para todo $T \in \mathcal{T}$, G-invariante, la ecuación de hipersuperfície minimal perturbada*

$$T = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \frac{\nabla v}{(1 + |\nabla v|^2)^{\frac{1}{2}}},$$

$$v - v_0 \in W_0^{1,1}(\Omega),$$

y el problema de Plateau perturbado

$$\inf \left(\int_\Omega 1 + |\nabla v|^2 dx \right)^{\frac{1}{2}} - \langle T, v \rangle,$$

$$v - v_0 \in W_0^{1,1}(\Omega),$$

ambos tienen una solución única G-invariante.

Finalizamos este capítulo presentando una aplicación del G-EVP a la teoría de control.

Teorema 31. *Supongamos que f satisface unas ciertas hipótesis, tomamos $h\colon \mathbb{R}^n \to \mathbb{R}$ una función diferenciable. Entonces, para todo $\epsilon > 0$, existe un control medible y G-invariante, cuya trayectoria es y, tal que*

$$\begin{cases} h(y(T)) \leqslant \inf h(x(T)) + \epsilon, \\ \langle f(t, y(t), v(t)), p(t)\rangle \leqslant \min_{u\in K}\langle f(t, y(t), u(t)), p(t)\rangle + \epsilon, \end{cases}$$

donde p es la solución del sistema diferencial

$$\begin{cases} \frac{dp_i}{dt}(t) = -\sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i}(t, y(t), v(t))p_j(t) \quad \forall 1 \leqslant i \leqslant n, \\ p(T) = h'(y(T)). \end{cases}$$

Resumen del Capítulo 5

Los contenidos de este capítulo aparecen en el trabajo enviado

[34] J. Falco, and D. Isert, G-strong subdifferentiability and applications to norm attaining subspaces, *arXiv preprint arXiv:2403.18467* (2024).

Parte del trabajo realizado en el Capítulo 3 y todo el trabajo realizado en el Capítulo 5 Sección 5.1 es para tener todos los prerequisitos necesarior para poder demostrar el Teorema 39. En la Sección 5.1 el primer resultado que estudiamos es el teorema de extensión de Hahn-Banach pero en el predual. El resultado es el siguiente.

Teorema 32. *Sea X un espacio de Banach, G un grupo topológico compacto actuando sobre X, y sea A un conjunto w^*_G-cerrado, convexo y G^*-invariante en X^*. Si $f \in X^*\backslash A$ es G-invariante, entonces existe un punto G-invariante, x, tal que $\sup_{h\in A}\langle h, x\rangle < \langle f, x\rangle$.*

A continuación, pasamos a estudiar el teorema de James G-invariante. Pero, antes de presentar dicho resultado, obtenemos el siguiente Teorema que resulta bastante natural. Este resultado nos asaegura que, para un espacio de Banach ser G-refleivo es una condición equivalente a que X_G sea reflexivo.

Teorema 33. *Sea X un espacio de Banach y G-un grupo topológico compacto actuando sobre X. Entonces X es G-reflexivo si, y sólo si, X_G es reflexivo.*

Con este resultado en mente, y [30, Theorem 6], podemos obtener finalmente el teorema de James G-invariante.

Teorema 34 (James G-invariante)**.** *Sea X un espacio de Banach y G un grupo topológico compacto actuando sobre X. Entonces X es G-reflexivo si, y sólo si, todo funcional G-invariante alcanza la norma.*

A continuación pasamos a estudiar el teorema de Moreau G-invariante. Este teorema simplemente es una restricción al espacio X_G del resultado clásico. En esta dirección, este es el mejor resultado que se puede obtener.

Proposición 35 (Moreau G-invariante)**.** *Sea X un espacio de Banach y G un grupo topológico compacto actuando sobre X. Sea $f\colon X \to \mathbb{R} \cup \{+\infty\}$ una función propia, convexa, G-invariante que es contínua en $x_0 \in Dom(f) \cap X_G$. Entonces,*

$$d^+ f(x_0)(x) = \sup \{\langle x^*, x\rangle \mid x^* \in \partial_G f(x_0)\} \quad \forall\, x \in X_G.$$

Además, el supremo se alcanza en un punto $x^* \in \partial_G f(x_0)$.

También presentamos un ejemplo en el que se muestra por qué el resultado anterior no se puede mejorar quitando la condición de G-invarianza de x. Dicho ejemplo es el siguiente.

Ejemplo 36. Sea $X = \mathbb{R}$ y $G = \{Id, -Id\} \subseteq \mathbb{R}$. Consideramos la función $f\colon \mathbb{R} \to \mathbb{R}$ definida por $f(x) = |x|$. Entonces, si $x_0 = 0$, que es el único punto G-invariante, y $x = 1$, que no es G-invariante, observamos que

$$d^+ f(0)(1) = 1, \text{ y } \partial_G |\cdot|(0) = \{0\}\,.$$

Por tanto, en general, la formula del máximo de Moreau G-invariante no es cierta si quitamos la condición $x \in X_G$.

Finalmente, presentamos la deifnición de conjunto G-polar y estudiamos algunas de sus propiedades, para obtener el Teorema 38.

Definición 37. Sea X un espacio de Banach, G un grupo topológico compacto actuando sobre X, y A un subconjunto de X. Definimos el conjunto polar G-invariante como

$$A^{\circ_G} = \{x^* \in X_G^* \mid |\langle x^*, x\rangle| \leqslant 1, \ \forall\, x \in A\}\,.$$

Teorema 38 (Krein-Smulian G-invariante)**.** *Sea X un espacio de Banach, G un grupo topológico compacto actuando sobre X, y $C \subseteq X^*$ un conjunto convexo y G^*-invariante. Si $\delta B_{X^*} \cap C$ es w_G^*-cerrado para todo $\delta > 0$, entonces C es w_G^*-cerrado en X^*.*

Por último, en la Sección 5.2 presentamos unas definiciones entre las cuales encontramos las de norma G-SSD, conjunto G-disipativo, G-aplicación dual, y estudiamos algunas relaciones entre ellas. Finalmente, obtenemos una condición necesaria para que un espacio sea G-reflexivo en términos de G-SSD. El resultado es el siguiente.

Teorema 39 (Resultado principal)**.** *Sea X un espacio de Banach y G un grupo topológico compacto actuando sobre X. Si X^* es G-SSD, entonces X es G-reflexivo.*

Cerramos el capítulo con dos aplicaciones de este resultado a la existencia de espacios vectoriales grandes dentro del conjunto de funcionales que alcanzan la norma, y una relación entre tener espacios vectoriales grandes de funcionales que alcanzan la norma y la propiedad del espacio de ser G-reflexivo.

Corolario 40. *Sea X un espacio de Banach y G un grupo topológico compacto actuando sobre X. Si X^* es G-SSD, entonces el conjunto de operadores que alcanzan la norma en X contiene, almenos, el espacio vectorial X^*_G.*

Proposición 41. *Sea X un espacio de Banach. Si el conjunto de funcionales que alcanzan la norma en X contiene un espacio de Banach de dimensión finita E, entonces existe un grupo topológico compacto G actuando sobre X tal que $E = X^*_G$. En particularm X es G-reflexivo.*

El documento concluye con un capítulo en el que presentamos algunas notas y preguntas abiertas. Al final del documento encontramos una lista extensa de referencias utilizadas en este trabajo.

Resum (Valencià)

En aquesta tesi estudiem les aplicacions grup invariants i obtenim generalitzacions de resultats clàssics de la teoria d'espais de Banach. Centrarem el nostre estudi en el cas de grups topològics compactes.

Els principals continguts d'aquest document estan organitzats en quatre capítols en els quals cobrim els continguts de l'article acceptat [32], l'article publicat [33], i els articles enviats a revisió per parelles [34, 35]. En aquesta secció resumirem en Valencià els continguts de cada capítol.

El primer capítol d'aquesta tesi és una traducció al Valencià del Capítol 2.

Resum del Capítol 2

Els continguts d'aquest capítol els podem trobar en l'article pendent de publicació en

[32] J. Falco, and D. Isert, Basic properties of the infinite dimensional group invariant points, sets and mappings, *Proceedings of XII Congreso del Máster en Investigación Matemática*, to appear.

El Capítol 2 té un doble propòsit, en primer lloc ens serveix com a capítol introductori. Expliquem quan comença a aparèixer la investigació d'aplicacions grup invariants, en la mesura del nostre coneixement, i fixem notació, adherint-nos a les notacions estàndard emprades habitualment en el camp. En segon lloc, ens serveix per a presentar algunes de les propietats més elementals que ens donaran una amplia visió de les aplicacions grup invariants. Açò també ens servirà per a adaptar el nostre punt de vista de l'escenari clàssic al nou escenari. Les tres nocions de grup invariants amb les que anem a treballar són les següents.

Definició 1. Siguen X, Y dos espais normats:

1. Un punt $x \in X$ és G-invariant, o invariant sota l'acció de G, si $g(x) = x$ per a tota $g \in G$.
2. Un conjunt $K \subset X$ és G-invariant si per a tota $g \in G$, $g(K) = K$.
3. Una aplicació $f\colon X \to Y$ és G-invariant si per a tot $x \in X$ i tota $g \in G$ se satisfà que
$$f(g(x)) = f(x).$$

Denotem el conjunt de tots els punts G-invariants per X_G, i el conjunt de tots els funcionals G-invariants per X^*_G.

És important subratllar ací que nosaltres anem a considerar que la norma dels espais de Banach amb els que treballem serà grup invariant.

Després d'aquestes definicions, presentem una selecció de resultats relatius a aplicacions i conjunts grup invariants. També presentem alguns contraexemples per a motivar i subratllar aquestos resultats. Però, el resultat més important que presentem en aquest capítol sense cap dubte és la següent proposició, que relaciona dos nocions de grup invariants.

Proposició 2. *Siga X un espai normat i siga G un grup actuant sobre X. Aleshores*

1. *$f\colon X \to Y$ és G-invariant si, i sols si, $Gr(f)$ és (G, Id)-invariant.*
2. *$f\colon X \to Y$ és G-invariant si, i sols si, $Epi(f)$ és (G, Id)-invariant.*

Resum del Capítol 3

Els continguts d'aquest Capítol apareixen com a part de l'article publicat

[33] J. Falco, and D. Isert, Group invariant variational principles, *Revista de la real academia de ciencias exactas, físicas y naturales. Serie A. Matemáticas* **10**(3) (2024), pp. 443–474.

i en els dos treballs enviats a revisió per parelles

[34] J. Falco, and D. Isert, G-strong subdifferentiability and applications to norm attaining subspaces, *arXiv preprint arXiv:2403.18467* (2024).

[35] J. Falco, and D. Isert, Variational principles and applications to symmetric PDEs, *arXiv preprint arXiv:2403.18467* (2024).

Aquest capítol és de vital importància per a la resta de la tesi, ja que conté totes les definicions necessàries i resultats previs necessaris els en capítols posteriors. Procedirem tot seguit a realitzar un estudi secció per secció del capítol, i en cada secció anotarem quan els continguts d'aquesta secció seran relevants.

Començem en la Secció 3.1 recordant alguns resultats de mesura de Haar i integral de Bochner. Aquests resultats els necessitrem per a introduir una de les nocions més importants que va a estar present en cada capítol i en cada secció de la tesi. La definició de punt i funcional simetritzat. Aquesta definició és una generalització de l'operador de Reynolds definit en [16, Chapter 7.3].

Definició 3. Siga X un espai de Banach i G un grup topològic compacte actuant sobre X. Per a $x \in X$ definim el punt simetritzat de x respecte de G, o el punt G-simetritzat, com

$$\overline{x} = \int_G g(x)\, d\mu(g),$$

on la integral és la integral de Bochner, i la mesura és la mesura de Haar associada al grup compacte G.

Similarment, per a un funcional $F\colon X \to \mathbb{R}$ definim el funcional simètric de F respecte de G, o el funcional G-simètric, com

$$\overline{F}(x) = \int_G F(g(x))d\mu(g),$$

on la integral és de nou la integral de Bochner i la mesura és la mesura de Haar associada al grup compacte G.

Presentem també les dos nocions de linealitat i convexitat respecte del grup, i donem dos exemples en els que observem que aquestes dos nocions són un poc més restrictives que les de linealitat i convexitat. Aquestes dos nocions ens apareixen de forma natural al estudiar la generalització del principi variacional d'Ekeland (EVP).

Definició 4. Siga X un espai de Banach i G un grup topològic compacte actuant sobre X. Siga $\varphi\colon X \to \mathbb{R}$ una funció. Diem que φ és convexa respecte del grup G si

$$\varphi\left(\int_G g(x)d\mu(g)\right) \leqslant \int_G \varphi(g(x))d\mu(g) \quad \forall\, x \in X.$$

I diem que φ és lineal respecte de G si

$$\varphi\left(\int_G g(x)d\mu(g)\right) = \int_G \varphi(g(x))d\mu(g) \quad \forall\, x \in X.$$

Tot seguit, presentem dos exemples per a ilustrar les definicions prèvies

Exemple 5. Siga $X = \mathbb{R}^2$ amb la topologia euclidiana i $G = \{Id, \sigma\}$, on

$$\begin{array}{rccc} \sigma\colon & \mathbb{R}^2 & \to & \mathbb{R}^2 \\ & (x,y) & \mapsto & (-x,y), \end{array}$$

i Id és l'aplicació identitat. Definim $f(x,y) = y^2$. Aquesta funció és lineal respecte de G però no lineal.

Exemple 6. Siga $X = \mathbb{R}$ amb la topologia euclidiana i siga $G = \{Id, -Id\}$. Definim

$$h(x) = \begin{cases} -x-1 & \text{si } x \leqslant -1, \\ x+1 & \text{si } -1 < x < 0, \\ -1 & \text{si } x = 0, \\ -x+1 & \text{si } 0 < x < 1, \\ x-1 & \text{si } x \geqslant 1. \end{cases}$$

Aquesta funció és convexa respecte del grup, però no és convexa.

La Secció 3.2 és part d'un treball que està actualment en progrès. Tenim dos objectius principals en aquest estudi, primer donar una caracterització d'un resultat ben conegut que garantiza que un espai té la propietat de Bishop-Phelps-Bollobás si, i solament si, l'espai té la propietat de Radon-Nikódym. I també volem obtindre un exemple negatiu a la implicació inversa del resultat [19, Theorem 4.21]. Així doncs, presentem adaptacions de molt conceptes tals com punts exposats, fortament exposats i extremals.

Definició 7. Siga X un espai de Banach, G un grup topològic compacte actuant sobre X, i $C \subseteq X$ un conjunt tancat, convex i G-invariant. Diem que un punt $x_0 \in C$ és G-exposat si existeix un funcional $x^* \in (X^* \setminus \{0\})_G$ tal que

1. $\langle x^*, x_0 \rangle = \sup_{x \in C} \langle x^*, x \rangle$.

2. $\langle x^*, x \rangle < \sup_{y \in C} \langle x^*, y \rangle$ per a tot $x \in C \setminus \{x_0\}$.

Diem que un punt $x_0 \in C$ és G-fortament exposat si existeix un funcional $x^* \in (X^* \setminus \{0\})_G$ de manera que per a tota successió $\{x_n\}_{n=1}^{+\infty} \subseteq C$ satisfent $\langle x^*, x_n \rangle \to \sup_{x \in C} \langle x^*, x \rangle$, aleshores $\|\overline{x_n} - \overline{x_0}\| \to 0$.

Definició 8. Siga V un espai vectorial, G un grup topològic compacte actuant sobre V, i C un subconjunt convex i G-invariant. Diem que un punt $x_0 \in C$ és G-extrem si donats $x_1, x_2 \in C$ i $0 < \lambda < 1$ tals que $\overline{x_0} = \lambda \overline{x_1} + (1 - \lambda) \overline{x_2}$, aleshores $\overline{x_0} = \overline{x_1} = \overline{x_2}$. Denotarem el conjunt de punts extrems en C com $\mathrm{Ext}_G(C)$.

També parlem breument de les G-porcions, les quals definim de la següent forma.

Definició 9. Siga X un espai de Banach, G un grup topològic compacte actuant sobre X, i A un subconjunt G-invariant. Definim la G-porció com

$$S_G(x^*, A, \alpha) = \left\{ x \in A_G \mid \langle x^*, x \rangle > \sup_{x \in A} \langle x^*, x \rangle - \alpha \right\}$$

on $x^* \in X_G^*$ i $\alpha > 0$.

Després d'açò, observem que la següent relació coneguda entre els punts exposats, fortament exposats i extrems continua sent certa en el context grup invariant.

Proposició 10. *Siga X un espai de Banach, G un grup topològic compacte actuant sobre X, i C un conjunt tancat, convex i G-invariant. Si $x_0 \in C$ és G-fortament exposat, aleshores x_0 és G-exposat. Si x_0 és G-exposat, aleshores x_0 és G-extrem.*

Finalment, presentem un contraexemple del teorema de Krein-Milman per a conjunts G-invariants, i obtenim la millor alternativa que podem obtindre en aquesta direcció.

Exemple 11. En $\mathbb{R}^2$ prenem el grup $G = \{Id, \sigma\}$ on $\sigma\colon \mathbb{R}^2 \to \mathbb{R}^2$ està definida com $\sigma(x,y) = (y,x)$. Considerem l'envoltura convexa dels conjunts $B_{\|\cdot\|_2}((1,-1),1)$, $B_{\|\cdot\|_2}((-1,1),1)$, i $B_{\|\cdot\|_1}((0,0),1)$, i anomenem al conjunt resultant K. Aleshores K és un conjunt compacte i G-invariant que no coincideix amb la clausura de l'envoltura convexa dels seus punts G-extrems.

Proposició 12. *Siga X un espai de Banach, G un grup topològic compacte actuant sobre X, i K un subconjunt convex, compacte i G-invariant. Aleshores:*

$$K_G \subseteq \overline{conv}(Ext_G(K)) \subseteq K.$$

En [19] els autors estudien un teorema de separació de Hahn-Banach. En la Secció 3.3 ens ocupem d'estudiar més resultats geomètrics de separació de Hahn-Banach en el nostre context d'aplicacions grup invariants. Els resultats que presentem en aquesta secció resultaran de gran utilitat en el Capítol 4. Resumim els resultats en el següent Teorema.

Teorema 13. *Siga X un espai de Banach i G un grup topològic compacte actuant sobre X.*

1. *Si A, B són dos conjunts convexos i G-invariants tals que A és obert i $A \cap B = \varnothing$, aleshores existeix un hiperplà G-invariant que separa A i B.*

2. *Siguen $A, B \subseteq X$ dos conjunts no buits, convexos i G-invariants tals que A és tancat, B és compacte i $A \cap B = \varnothing$, aleshores existeix un hiperplà G-invariant que separa estríctament A i B.*

Els resultats estudiats en la Secció 3.4 seran d'utilitat en el Capítol 5. Ací generalitzem les nocions de topologia dèbil i topologia dèbil-estrella, les quals definim de la següent forma.

Definició 14. Siga X un espai de Banach i G un grup topològic compacte actuant sobre X. Definim la topologia dèbil G-invariant en X com la topologia generada pels conjunts

$$\{x \in X \mid \langle f_i, x - x_0 \rangle < \epsilon, \text{ per a } 1 \leqslant i \leqslant n\},$$

per a tota tria de $x_0 \in X$, $f_1, \ldots, f_n \in X_G^*$ i $\epsilon > 0$. Denotem aquesta topologia per w_G ò $\sigma_G(X, X^*)$.

Definim la topologia dèbil-estrella G-invariant en X^* com la topologia generada pels conjunts

$$\{f \in X^* \mid \langle f - f_0, x_i \rangle < \epsilon, \text{ per a } 1 \leqslant i \leqslant n\},$$

per a tota tria de $f_0 \in X^*$, $x_1, \ldots, x_n \in X_G$ i $\epsilon > 0$. Denotem aquesta topologia per w_G^* ò $\sigma_G(X^*, X)$.

Una de les propietats que fan tan útils a les topologies dèbils és que són Hausdorff. En aquesta direcció obtenim el resultat completament opost per a les topologies que acabem de definir.

Proposició 15. *Siga X un espai de Banach i G un grup topològic compacte actuant sobre X. Si $X_G^* \subsetneq X^*$, aleshores*

1. *La topologia dèbil grup invariant en X és estríctament més dèbil que la topologia dèbil en X, w.*

2. *Les topologies w_G i w_G^* no són Hausdorff.*

Finalment, estudiem alguns resultats clàssics de reflexivitat d'un espai de Banach, i de geometria de la bola. Amb aquesta finalitat, necessitem definir la noció de G-reflexivitat.

Definició 16. Siga X un espai de Banach i G un grup topològic compacte actuant sobre X. Diem que X és G-reflexiu si la injecció canònica

$\pi\colon X \to X^{**}$ és G-sobrejectiva, i per açò ens referim a que, $\pi(X_G) = X^{**}_{G^{**}}$.

Resumim a continuació tots els resultats que hem adaptat.

Teorema 17. *Siga X un espai de Banach i G un grup topològic compacte actuant sobre X. Aleshores*

1. *B_{X^*} és w^*_G-compacta.*
2. *$\overline{B_X}^{w^*_G} = B_{X^{**}_{G^{**}}}$.*
3. *X és G-reflexiu si, i sols si, B_X és compacta en la topologia $\sigma_G(X, X^*)$.*

En la Secció 3.5 recordem les definicions de diferenciabilitat Fréchet i Gâteaux, i donem la definició de G-subdiferencial. Els resultats d'aquesta secció seran necessaris en els dos capítols següents, en el Capítol 4 i en el Capítol 5.

Definició 18. Siga X un espai de Banach, G un grup topològic compacte actuant sobre X, i siga $f\colon X \to]-\infty, +\infty]$ una funció pròpia. Per a $x_0 \in \mathrm{Dom} f$ definim la G-subdiferencial de f en x_0 com el següent conjunt

$$\partial_G f(x_0) = \{h \in X^*_G \ \mid \ \langle h, x - x_0\rangle \leqslant f(x) - f(x_0), \ \forall\, x \in X\}.$$

Si ens fixem amb més cura en la definició de G-subdiferencial, podem observar que tenim les següents tres formes equivalents de definir-la.

$$\begin{aligned}
\partial_G f(x_0) &= \{x^* \in X^*_G \mid \langle x^*, x - x_0\rangle \leqslant f(x) - f(x_0) \ \ \forall\, x \in X\} \\
&= \{x^* \in X^*_G \mid \langle x^*, \overline{x} - x_0\rangle \leqslant f(\overline{x}) - f(x_0) \ \ \forall\, x \in X\} \\
&= \{x^* \in X^*_G \mid \langle x^*, x - x_0\rangle \leqslant f(x) - f(x_0) \ \ \forall\, x \in X_G\}.
\end{aligned}$$

Resum del Capítol 4

Els resultats presentats en la primera meitat d'aquest capítol apareixen en l'article publicat

[33] J. Falco, and D. Isert, Group invariant variational principles, *Revista de la real academia de ciencias exactas, físicas y naturales. Serie A. Matemáticas* **10**(3) (2024), pp. 443–474.

I els resultats en la segona meitat del Capítol apareixen en el treball enviat

[35] J. Falco, and D. Isert, Variational principles and applications to symmetric PDEs, *arXiv preprint arXiv:2403.18467* (2024).

Començem aquest capítol presentant el principi variacional d'Ekeland en la seua versió G-invariant. El fet més interessant ací és que quan tractem de generalitzar aquest resultat per a aplicacions G-invariants, no podem obtindre un resultat positiu en general, necessitem afegir la condició de convexitat respecte del grup en l'aplicació de partida per a que el resultat es mantinga cert.

Teorema 19. *Siga X un espai de Banach i G un grup topològic compacte actuant sobre X. Siga $\varphi\colon X \to]0, +\infty]$ una funció pròpia, semicontinua inferiorment, acotada inferiorment, G-invariant, i convexa respecte del grup G. Aleshores, donats $\epsilon > 0$ i $\delta > 0$, existeix un punt G-invariant $\tilde{x} \in X$ tal que*

$$\varphi(\tilde{x}) < \varphi(x) + \epsilon\|\overline{x} - \tilde{x}\| \quad \forall\, x \in X,\ x \neq \tilde{x}.$$

A més a més, si $x_0 \in X$ satisfà que $\varphi(x_0) < \inf\{\varphi(x) \mid x \in X\} + \delta$, aleshores podem triar $\tilde{x}$ de manera que

$$\|\overline{x_0} - \tilde{x}\| < \frac{\delta}{\epsilon}.$$

També presentem un exemple de funció que no és convexa respecte del grup i que no satisfà el teorema. A continuació en la Secció 4.2 presentem una sèrie de conseqüències del G-EVP. Una de les conseqüències més distingudes que obtenim és la que presentem a continuació. Aquest resultat ens permet descriure completament l'espai dual format pels funcionals G-invariants.

Corol·lari 20. *Siga X un espai de Banach i G un grup topològic compacte actuant sobre X. Siga φ una funció continua, diferenciable Gâteaux, salt, G-invariant i lineal amb respecte de G. Aleshores*

$$X_G^* = \overline{Lin}\{\partial\varphi(x) \mid x \in X_G\}.$$

Les últimes dues conseqüències que estudiem del G-EVP són el teorema de Bishop-Phelps, i el teorema de Brønsted-Rockafellar, ambdós en la seua respectiva versió G-invariant. A continuació mostrem tots dos resultats.

Teorema 21 (Bishop-Phelps G-invariant)**.** *Siga X un espai de Banach real i G un grup topològic compacte actuant sobre X. Si $C \subseteq X$ és un subconjunt convex, tancat, acotat i G-invariant, aleshores els funcionals G-invariants que abasten la norma en C formen un conjunt dens en X_G^*.*

Teorema 22 (Brønsted-Rockafellar G-invariant)**.** *Siga X un espai de Banach i G un grup topològic compacte actuant sobre X. Siga $f\colon X \to \mathbb{R} \cup \{+\infty\}$ un funcional propi, convex, semicontinu inferiorment i G-invariant. Aleshores per a qualssevol punt G-invariant x_0, qualssevol funcional x_0^* que satisfan que $x_0 \in Dom(f)$ i $x_0^* \in \partial_\epsilon f(x_0)$, i per a tot $\epsilon, \lambda > 0$, existeixen un punt G-invariant $z \in Dom(f)$ i un funcional $x^* \in X_G^*$ tals que*

$$x^* \in \partial_G f(x), \quad \|z - x_0\| \leqslant \frac{\epsilon}{\lambda}, \quad \|x^* - x_0^*\| \leqslant \lambda.$$

La segona part d'aquest capítol es centra en l'estudi de diverses equivalències de l'EVP. En la Secció 4.3 ens centrem en l'equivalència entre l'EVP, el teorema de la gota i el teorema del pètal, tots ells en la seua respectiva versió G-invariant. Presentem a continuació els dos nous resultats que formen l'epicentre d'aquesta secció juntament amb el Teorema 19.

Teorema 23 (Teorema del Pètal G-invariant)**.** *Siga (M, d) un espai mètric, $G \subseteq \mathcal{L}(M)$ un grup topològic d'isometries compacte actuant sobre M, i $C \subset M$ un subconjunt complet i G-invariant. Suposem que d és (G, G)-invariant i convexa respecte de G. Siguen $x_0 \in C_G$, $b \in (M \backslash C)_G$, $r \leqslant d(b, C)$ i $s = d(b, x_0)$. Aleshores, per a tot $\gamma > 0$, existeix un punt G-invariant $a \in C \cap P_\gamma(x_0, b)$ tal que $C \cap P_\gamma(a, b) = \{a\}$.*

Teorema 24 (Teorema de la gota G-invariant)**.** *Siga $(X, \|\cdot\|)$ un espai normat, $G \subseteq \mathcal{L}(X)$ un grup topològic d'isometries compacte actuant sobre X, i $C \subset X$ un subconjunt complet i G-invariant. Suposem que $x_0 \in C_G$, i $B = \overline{B(b, r)}$, on $b \in X_G$ i $r < d(b, C)$. Aleshores existeix un punt G-invariant $a \in C \cap D(x_0, B)$ tal que $C \cap D(a, B) = \{a\}$.*

Utilitzant un enfoc alternatiu en els dos resultats previs, podem prescindir de la condició sobre x_0 de ser G-invariant i utilitzar el resultat clàssic per a cada element $g(x_0)$ i cada $g \in G$. D'aquesta forma obtenim resultats més generals que els originals. Però, primer necessitem definir una nova funció que emprem per a calcular la distància més curta entre un punt i els translladats per tots els elements del grup d'ell mateix. La funció és la següent.

Definició 25. Siga (M, d) un espai mètric i G un grup topològic d'isometries compacte actuant sobre M. Per a un punt $x \in E$ definim

$$s_G(x) := \inf\{d(x, g(x)) \mid g \in G \text{ i } g(x) \neq x\}.$$

En els següents resultats, generalitzacions del teorema de la gota i del teorema del pètal.

Proposició 26 (Teorema de la flor)**.** *Siga* (M, d) *un espai mètric,* $G \subseteq \mathcal{L}(M)$ *un grup topològic d'isometries compacte actuant sobre* M*, i* $C \subset M$ *un subconjunt de* M *complet i* G*-invariant. Siga* $x_0 \in C_G$*,* $b \in M \backslash C$*. Aleshores, per a tot* $\gamma > 0$*, existeix un punt* $a \in C \cap P_\gamma(x_0, b)$ *tal que*

$$C \cap P_\gamma(g(a), g(b)) = \{g(a)\} \text{ per a tota } g \in G.$$

A més a més, per a totes dues $g, g' \in G$ *amb* $d(g(b), g'(b)) > 2d(b, C)$ *es satisfà que*

$$P_\gamma(g(a), g(b)) \cap P_\gamma(g'(a), g'(b)) = \varnothing.$$

Proposició 27 (Teorema de la gota generalitzat)**.** *Siga* $(E, \|\cdot\|)$ *un espai normat,* $G \subseteq \mathcal{L}(E)$ *un grup topològic d'isometries compacte actuant sobre* E*, i* $C \subset E$ *un subconjunt de* E *complet i* G*-invariant. Siguen* $x_0 \in C_G$*,* $b \in E \backslash C$*. Aleshores, existeix un punt* $a \in C \cap D(x_0, b)$ *tal que*

$$C \cap D(g(a), g(b)) = \{g(a)\} \text{ per a tota } g \in G.$$

A més a més, per a totes dues $g, g' \in G$ *satisfent* $d(g(b), g'(b)) > 2d(b, C)$*, tenim que*

$$D(g(a), g(b)) \cap D(g'(a), g'(b)) = \varnothing$$

per a $g, g' \in G$ *amb* $g(b) \neq g'(b)$*.*

Observem el paper que juga la funció anterior per a garantitzar que tots els pètals i les gotes que obtenim siguen disjunts.

Tot seguit, en la Secció 4.4 presentem l'equivalència entre l'EVP, el teorema de Caristi-Kirk i el teorema de Takahashi, tots ells en les seues respectives versions G-invariants. El resultat principal d'aquesta secció és el següent.

Teorema 28. *Siga (M, d) un espai mètric complet i $G \subseteq \mathcal{L}(M)$ un grup topològic d'isometries compacte actuant sobre M. Aleshores, els següents resultats són equivalents:*

1. *Siga $U \subseteq M$ un conjunt G-invariant satisfent que*

$$\forall\, y \in S_0 \backslash U,\ \exists\, x \in M_G \text{ amb } x \neq \overline{y} \text{ i } f(\overline{y}, x) + d(\overline{y}, x) \leqslant 0.$$

   Aleshores, existeix un punt $\hat{x} \in (S_0 \cap U)_G$.

2. *Existeix un punt $\hat{x} \in S_0$ tal que $\hat{x}$ és G-invariant, i $f(\hat{x}, x) + d(\hat{x}, x) > 0$ per a tot $x \in M$, $x \neq \hat{x}$.*

3. *Suposem que $\forall\, y \in S_0$ satisfent $\inf_{x \in M} f(\overline{y}, x) < 0$ existeix un punt*

$$x \in M_G \text{ tal que } x \neq \overline{y} \text{ i } f(\overline{y}, x) + d(\overline{y}, x) \leqslant 0.$$

   Aleshores, existeix $\hat{x} \in (S_0)_G$ tal que $f(\hat{x}, x) \geqslant 0$ per a tot $x \in M_G$.

4. *Siga $T \colon M \to M$ una aplicació multivaluada tal que per a tot $y \in S_0$ existeix*

$$x \in (T(y))_G \text{ tal que } f(\overline{y}, x) + d(\overline{y}, x) \leqslant 0.$$

   Aleshores, existeix un punt $\hat{x} \in (S_0)_G$ tal que $\hat{x} \in T(\hat{x})$.

Finalment, en la Secció 4.5 estudiem algunes aplicacions de l'EVP. La primera és una aplicació geomètrica relacionada amb el con contingent d'un conjunt convex. Aquesta aplicació és conseqüència del Teorema 27.

Teorema 29. *Siga X un espai normat i $G \subseteq \mathcal{L}(X)$ un espai topològic d'isometries compacte actuant sobre X. Suposem que $C \subseteq X$ és un conjunt complet i G-invariant. Siga $x_0 \in C$, G-invariant, i triem $y \in X$ tal que el segment $[x_0, y]$ no està en C. Aleshores, per a cada $\rho > 0$ i per a tota $g \in G$, existeix $g(a) \in C$ tal que*

$$\|x_0 - g(a)\| \leqslant \|x_0 - g(y)\| + \rho, \text{ i } g(y) \notin g(a) + K_C(g(a)).$$

A més a més, si $s_G > 2d(z, C)$, sent z tal que $z \in (E \backslash C)_G$ i $y = \lambda z$, aleshores per a totes dues $g, g' \in G$, tenim que

$$g(a) + K_C(g(a)) \cap g'(a) + K_C(g'(a)) = \varnothing.$$

En el text també podem trobar una aplicació del Teorema 24 en el que obtenim un resultat paregut al que acabem de presentar. Seguidament, passem a estudiar unes aplicacions del G-EVP en el context d'equacions amb derivades parcials. El primer resultat que obtenim ens assegura que el problema de Plateau pertorbat té una única solució G-invariant.

Teorema 30. *Siguen Ω, i $v_0 \in W_0^{1,1}(\Omega)$ G-invariants. Aleshores, existeix en $W^{-1,\infty}(\Omega)$ un entorn de l'origen, i un subconjunt $\mathcal{T}$ dens en aquest entorn, tal que, per a cada operador G-invariant $T \in \mathcal{T}$, l'equació d'hipersuperfície minimal pertorbada*

$$T = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \frac{\nabla v}{(1 + |\nabla v|^2)^{\frac{1}{2}}},$$

$$v - v_0 \in W_0^{1,1}(\Omega),$$

i el problema de Plateau pertorbat

$$\inf \left(\int_\Omega 1 + |\nabla v|^2 dx \right)^{\frac{1}{2}} - \langle T, v \rangle,$$

$$v - v_0 \in W_0^{1,1}(\Omega),$$

ambdós tenen una única solució G-invariant.

Concloem aquest capítol presentant una aplicació del G-EVP a la teoria de control.

Teorema 31. *Suposem que f satisfà unes certes hipòtesi, i siga $h \colon \mathbb{R}^n \to \mathbb{R}$ una funció diferenciable. Aleshores, per a tot $\epsilon > 0$, existeix una* *mesura de control G-invariant, v, la trajectoria de la qual és y tal que*

$$\begin{cases} h(y(T)) \leqslant \inf h(x(T)) + \epsilon, \\ \langle f(t, y(t), v(t)), p(t)\rangle \leqslant \min_{u\in K}\langle f(t, y(t), u(t)), p(t)\rangle + \epsilon, \end{cases}$$

on p és la solució del sistema diferencial

$$\begin{cases} \frac{dp_i}{dt}(t) = -\sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i}(t, y(t), v(t))p_j(t) \quad \forall 1 \leqslant i \leqslant n, \\ p(T) = h'(y(T)). \end{cases}$$

Resum del Capítol 5

Els continguts d'aquest Capítol apareixen en l'article en procés de revisió per parelles

[34] J. Falco, and D. Isert, G-strong subdifferentiability and applications to norm attaining subspaces, *arXiv preprint arXiv:2403.18467* (2024).

Part del treball realitzat en el capítol 3 i tot el treball realitzat en el Capítol 5 Secció 5.1 és per a tindre tots els prerequisits necessaris per a poder demostrar el Teorema 39. En la Secció 5.1 el primer resultat que estudiem és el teorema d'extensió de Hahn-Banach però en el predual. El resultat és el següent.

Teorema 32. *Siga X un espai de Banach, G un grup topològic compacte actuant sobre X, i A un conjunt w_G^*-tancat, convex i G^*-invariant en X^*. Si $f \in X^* \backslash A$ és G-invariant, aleshores existeix un punt G-invariant, x, tal que* $\sup_{h\in A}\langle h, x\rangle < \langle f, x\rangle$.

A continuació, pasem a estudiar el teorema de James G-invariant. Tanmateix, abans de presentar dit resultat, obtenim el següent teorema que resulta ser prou natural. Aquest resultat ens assegura que, per a un espai de Banach, ser G-reflexiu és una condició equivalent a que X_G siga reflexiu.

Teorema 33. *Siga X un espai de Banach i G un grup topològic compacte actuant sobre X. Aleshores, X és G-reflexiu si, i sols si, X_G és reflexiu.*

Amb aquest resultat en ment i [30, Theorem 6], podem obtindre com a conseqüència el teorema de James G-invariant.

Teorema 34 (James G-invariant)**.** *Siga X un espai de Banach i G un grup topològic compacte actuant sobre X. Aleshores X és G-reflexiu si, i sols si, tot funcional G-invariant abasta la norma.*

A continuació, passem a estudiar el teorema de Moreau G-invariant. Aquest resultat és simplement una restricció del resultat clàssic a l'espai de punts G-invariants. En aquesta direcció, aquest és el millor que podem obtindre.

Proposició 35 (Moreau G-invariant)**.** *Siga X un espai de Banach i G un grup topològic compacte actuant sobre X. Siga $f\colon X \to \mathbb{R} \cup \{+\infty\}$ una funció pròpia, convexa, G-invariant i continua en $x_0 \in Dom(f) \cap X_G$. Aleshores,*

$$d^+ f(x_0)(x) = \sup\{\langle x^*, x\rangle \mid x^* \in \partial_G f(x_0)\} \quad \forall\, x \in X_G.$$

A més a més, aquest suprem s'assoleix en algun punt $x^* \in \partial_G f(x_0)$.

També donem un exemple en el que il·lustrem per qué el resultat anterior no es pot millorar prescindit de la condició de G-invariança sobre el punt x.

Exemple 36. Siga $X = \mathbb{R}$ i $G = \{Id, -Id\} \subseteq \mathbb{R}$. Considerem la funció $f\colon \mathbb{R} \to \mathbb{R}$ definida per $f(x) = |x|$. Aleshores, si $x_0 = 0$, que és l'únic punt G-invariant, i $x = 1$, que no és G-invariant, observem que

$$d^+ f(0)(1) = 1, \text{ i } \partial_G |\cdot|(0) = \{0\}.$$

Aleshores, en general, la fòrmula del màxim de Moreau G-invariant no es manté certa si llevem la condició de $x \in X_G$.

Finalment, presentem la definició de conjunt G-polar i estudiem algunes de les seues propietats, i passem a demostrar el Teorema 38, Krein-Smulian G-invariant.

Definició 37. Siga X un espai de Banach, G un grup topològic compacte actuant sobre X, i A un subconjunt de X. Definim el conjunt polar G-invariant com

$$A^{\circ_G} = \{x^* \in X_G^* \mid |\langle x^*, x\rangle \leqslant 1, \ \forall\, x \in A|\}\,.$$

Teorema 38 (Krein-Smulian G-invariant)**.** *Siga X un espai de Banach, G un grup topològic compacte actuant sobre X, i $C \subseteq X^*$ un conjunt convex i G^*-invariant. Si $\delta B_{X^*} \cap C$ és w_G^*-tancat per a tot $\delta > 0$, aleshores C és w_G^*-tancat en X^*.*

Per últim en la Secció 5.2 presentem unes definicions entre les quals trobem la de norma G-SSD, conjunt G-dissipatiu, G-aplicació dualitat, i estudiem algunes de les relacions entre aquestes. Finalment, obtenim una condició necessaria per a que un espai siga G-reflexiu en termes de la G-SSD. El resultat és el següent.

Teorema 39 (Resultat principal)**.** *Siga X un espai de Banach i G un grup topològic compacte actuant sobre X. Si X^* és G-SSD, aleshores X és G-reflexiu.*

Tanquem aquest capítol amb dues aplicacions d'aquest resultat a la existència d'espais vectorials grans dins del conjunt de funcionals que alcançen la norma, i una relació entre tindre espais vectorials grans de funcionals que alcançen la norma i la propietat sobre l'espai de ser G-reflexiu.

Corol·lari 40. *Siga X un espai de Banach i G un grup topològic compacte actuant sobre X. Si X^* és G-SSD, aleshores el conjunt d'operadors que alcançen la norma en X conté, almenys, l'espai vectorial X^*_G.*

Proposició 41. *Siga X un espai de Banach. Si el conjunt de funcionals que alcançen la norma en X conté un espai de Banach finit dimensional E, aleshores existeix un grup topològic compacte G actuant sobre X tal que $E = X^*_G$. En particular, X és G-reflexiu.*

Aquest document conclou amb un capítol de conclusió en el que deixem algunes notes i preguntes obertes, Al final del document podem trobar una extensa llista de referències emprades en aquest treball.

Capítol 1

Introducció al concepte de grup invariància

Buscant representacions alternatives d'algunes àlgebres de funcions holomorfes, en 2018 Aron, Falco, Garcia i Maestre en [3] van emprar arguments de grups d'invariançes amb aplicacions pròpies holomorfes. Aquell any, mentre treballaven en l'anomenat article, Aron, Falco i Maestre s'adonaren que no hi havien teoremes de separació de tipus Hahn-Banach per a polinomis grup invariants. Així, mentre teballavaen en el primer article, obtingueren resultats en aquesta direcció de separació de polinomis i conjunts, i els varen publicar en [5]. No obstant això, ells treballaven en $\mathbb{C}^n$, aleshores utilitzaven grups finits els elements dels quals foraven una base de Gröbner. Per a més informació en bases de Gröbner veure [16, Chapter 2]. Donant una mirada més profunda als arguments que van emprar, van observar que podien emprar tècniques similars si consideraven grups topològics compactes. Fins on sabem, aquest tipus d'arguments no s'haviet fet servir mai en la teoria de la geometria d'espais de Banach fins [30], on s'obté una generalització d'un dels resultats més importants en teoria d'alcançar la norma, el teorema de Bishop-Phelps-Bollobas, al context grup invariant. Les tècniques que va emprar en aquest article van ser ben rebudes en la comunitat investigadora i, des de la seua publicació, s'han publicat alguns articles més amb tècniques grup invariants, apart dels que s'han publicat per a aquesta tesi, veure [19] i [20].És important subratllar que hi ha treball realitzat d'aplicacions grup invariants en espais de Banach, però en el context de teoria de la mesura, veure [37, Chapter 22]. Però, aquesta és l'única referència, fins on arriba el nostre coneixement, que empra nocions grup invariants en espais de Banach, aleshores tot el que podem dir és que ens trobem front a una nova, inexplorada, i preciosa línea d'investigació.

En aquest capítol introduïrem les definicións inicials de grup invariant, i estudiarem algunes de les propietats més bàsiques que se'n deriven d'aquestes. També fixarem la notació que emprarem al llarg de la tesi. Malgrat que els resultats presentats en aquest capítols són molt elementals, els incloem pel be de la completitud. Tot el treball presentat en aquest capítol pot trobar-se en l'article pendent de publicació en

[32] J. Falco, and D. Isert, Basic properties of the infinite dimensional group invariant points, sets and mappings, *Proceedings of XII Congreso del Máster en Investigación Matemática*, to appear.

Començem fixant notació sent consistents en la notació estàndard emprada usualment en el camp.

Siga X un espai normat, usualment un espai de Banach. L'espai de totes les aplicacions lineals i continues de X en si mateix el denotem per $\mathcal{L}(X)$. Els conjunts X_G, B_X, S_X representen, respectivament els punts grup invariants, la bola unitat i l'esfera unitat.

Al cor d'aquest treball tenim el concepte de grup topològic que a continuació introduïm com el pilar central de la tesi.

Definició 1. Un grup topològic G és un espai topològic que és també un grup, tal que l'operació de grup xy i l'aplicació inversa x^{-1} són aplicacions continues.

En molts exemples, emparem grups topològics que venen naturalment de l'àlgebra, com el següent exemple senzill. Considerem $X = \mathbb{R}^2$ amb la topologia euclidiana, un possible grup topològic és $G = \{Id, \sigma\}$ on $\sigma = (1, 2)$ és la permutació de les primeres dos coordenades d'un punt en X. Observar la relació natural d'aquest grup topològic amb el grup clàssic en àlgebra Σ_2, donat per la identitat i la permutació de dos elements.

Per a continuar, presentem la definició d'acció de grup sobre un conjunt.

Definició 2. Donat un grup G i un conjunt K, denotem l'acció de grup sobre el conjunt K per

$$\langle G, K \rangle = \{gx \colon g \in G, x \in K\} .$$

D'ara en avant els grups topològics, G, consistiran en isometries, particularment, $G \subseteq \mathcal{L}(X)$. A més a més, la lletra G queda reservada per a denotar un grup topològic d'aplicacions lineals, continues i invertibles amb la topologia dotada per la topologia relativa de $\mathcal{L}(X)$. De la mateixa manera, la lletra g s'emprarà per a referir-nos a elements del grup G.

Amb tot açò, ja estem en condicions per a donar les tres definicions fonamentals d'aquesta tesi, les nocions de grup invariant.

Definició 3. Siguen X, Y dos espais normats:

1. Un punt $x \in X$ és G-invariant, o invariant sota l'acció de G, si $g(x) = x$ per a tot $g \in G$.

2. Un conjunt $K \subseteq X$ és G-invariant si per a tot $g \in G$, $g(K) = K$.

3. Una aplicació $f\colon X \to Y$ és G-invariant si per a tot $x \in X$ i tota $g \in G$ tenim que
$$f(g(x)) = f(x).$$

Denotem el conjunt de punts G-invariant com segueix

$$X_G = \{x \in X \mid x \text{ és } G\text{-invariant}\}.$$

Denotem el conjunt de funcionals lineals i continus que són G-invariants com
$$X_G^* = \{f \in X^* \mid f \text{ és } G\text{-invariant }\}.$$

Com que estem considerant que els elements de G són isometries, queda clar que la norma de l'espai X és G-invariant donat que

$$\|g(x)\| = \|x\| \quad \forall\, g \in G.$$

Passem ara a definir quan la distància és grup invariant. Observem, en primer lloc, que la funció distància $d\colon X \times X \to \mathbb{R}^+$ té domini $X \times X$, aleshores no té sentit parlar de G-invariança. Tot i a això, la definició pot ser adaptada naturalment de la següent manera.

Definició 4. Diem que l'aplicació distància $d\colon X \times X \to \mathbb{R}^+$ és (G, G)-invariant si, per a tota $g \in G$

$$d(g(x), g(y)) = d(x, y) \quad \forall\, x, y \in X.$$

Observem d'ací que $\|g(x)\| = d(g(x), g(0))$ i $\|x\| = d(x, 0)$. Aleshores la definició de (G, G)-invariant per a la funció distància és consistent amb la definició de norma G-invariant.

D'ara en avant sempre assumirem que els elements del grup amb els que estem treballant són sempre isometries, és a dir, la norma de l'espai sempre assumirem que és G-invariant.

Exemple 5. Considerem $\mathbb{R}^2$ amb la topologia euclidiana, i el grup $G = \{Id, \sigma\}$ on σ actua en l'espai de la següent forma:

$$\sigma(x, y) = (y, x).$$

Si observem un moment la Figura 1.1 podem veure que els dos quadrats són subconjunts G-invariants de $\mathbb{R}^2$, i la linia $y = x$ són els punts G-invariants de $\mathbb{R}^2$.

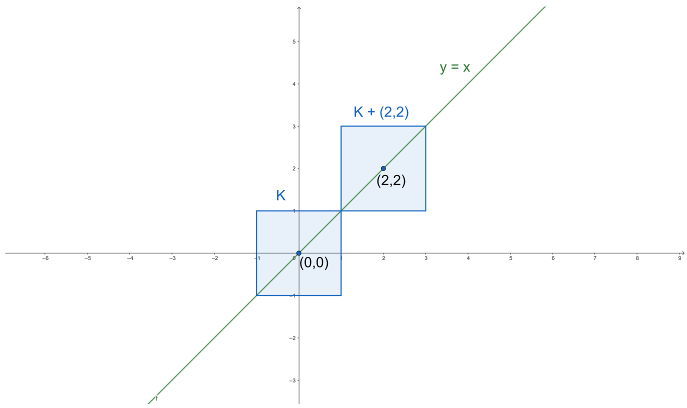


Figura 1.1. Translacio d'un quadrat G-invariant per un punt G-invariant

Com podem observar en la Figura 1.1, el quadrat G-invariant centrat en $(2, 2)$ i longitud de costat 2, és la translació del quadrat G-invariant centrat en $(0, 0)$ i longitud de costat 2 pel punt G-invariant $(2, 2)$. Aquest és un cas particular de la propietat següent.

Propietat 6. *Siga X un espai normat i G un grup actuant en X. Si $K \subseteq X$ i $x \in X$ són ambdós G-invariants, aleshores $K + x$ és també G-invariant.*

Demostració. Siguen $G \subseteq \mathcal{L}(X)$, i K, x G-invariants. Per a $y \in K$ es satisfà

$$g(y + x) = g(y) + g(x) = g(y) + x \in g(K) + x = K + x \quad \forall\, g \in G.$$

Aleshores per a tota $g \in G$, $g(K + x) \subseteq K + x$. Per tal d'aconseguir l'altra inclusió, observem que tot element del grup té un únic element invers, llavors

$$K + x \subseteq G^{-1}(K + x).$$

Per tant

$$K + x \subseteq G(K + x).$$

Aleshores, $K + x$ és G-invariant. □

Com a conseqüència directa de la propietat que acabem de provar, obtenim el següent resultat.

Corol·lari 7. *Siga X un espai normat, i G un grup actuant sobre X. Si K és un subespai vectorial de X que és G-invariant, i x és també G-invariant, aleshores $K + x$ és un subespai afí G-invariant de X.*

En els següents dos exemples volem il·lustrar que les condicions de G-invariança sobre K i x són ambdues necessàries.

Exemple 8. Siga $G = \{Id, \sigma\}$ un grup actuant sobre $\mathbb{R}^2$ tal i com l'hem definit a l'Exemple 5.

(i) Considerem K el quadrat centrat en $(0, 0)$ i de longitud de costat 2, que és G-invariant, i tria el punt $x = (3, 0)$, que no és G-invariant. Aleshores $K + x$ no és G-invariant, ja que $\sigma(3, 0) = (0, 3) \notin K + (3, 0)$.

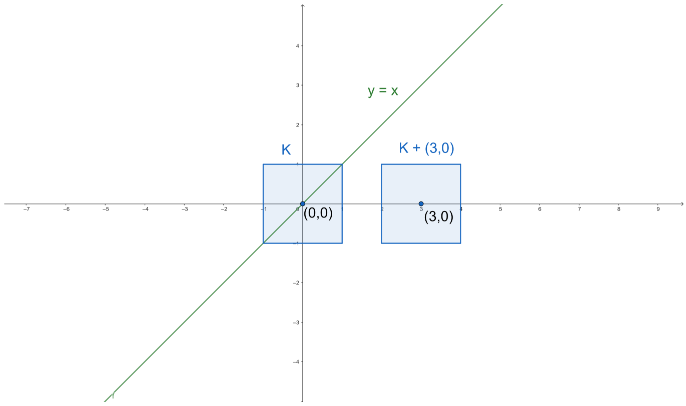


Figura 1.2. Translació d'un subconjunt G-invariant per un punt no G-invariant.

(ii) Siga K el quadrat centrat en $(1, 0)$ i de longitud de costat 2, el qual no és G-invariant, i siga $x = (1, 1)$ que sí és un punt G-invariant. Aleshores $K+x$ no és G-invariant ja que $\sigma(3, 1) = (1, 3) \notin K+(1, 1)$.

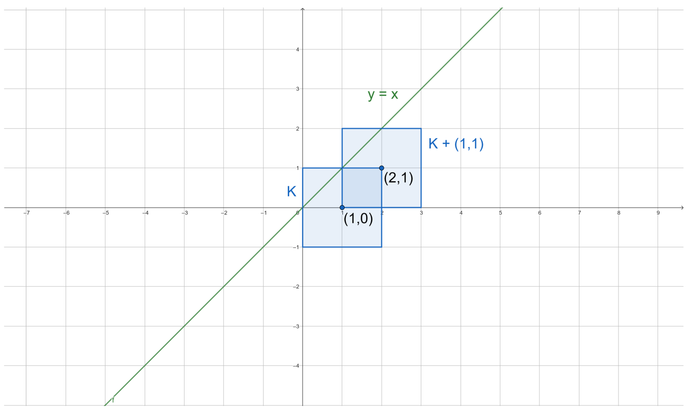


Figura 1.3. Translació d'un subconjunt no G-invariant per un punt G-invariant.

(iii) De la mateixa manera, podem trobar un subconjunt, K, i un punt, x, els quals no són G-invariants, però $K + x$ sí és G-invariant. Siga K el quadrat centrat en $(1, 0)$ i de longitud de costat 2, que no és G-invariant, i siga $x = (0, 1)$ que tampoc és G-invariant. Tot i això, $K + x$ sí és G-invariant.

Tanmateix, la implicació contrària de la propietat 6 és certa si el punt x és G-invariant.

Propietat 9. *Siga X un espai de Banach, i G un grup actuant sobre X. Si $K + x$ és G-invariant i x és G-invariant, aleshores K és G-invariant.*

Demostració. Siga $g \in G$, aleshores per a tot $y \in K$ tenim que

$$y + x \in K + x.$$

Per la G-invariança de $K + x$ tenim que

$$g(y + x) \in K + x.$$

Però sabem per linealitat i G-invariança de x que $g(y + x) = g(y) + g(x)$. Aleshores

$$g(y) + x \in K + x,$$

d'on concloem que $g(y) \in K$ per a tot $y \in K$. En particular hem opbtès que $g(K) \subseteq K$. Prenent ara elements inversos obtenim l'altra inclusió, llavors K és G-invariant. □

Anem ara a veure que en la proposició anterior, el fet que els conjunts $K + x$ i K siguen G-invariant no són suficients per concloure que el punt x és G-invariant.

Exemple 10. Siga $G = \{Id, \sigma\}$ en $\mathbb{R}^2$, on $\sigma(x, y) = (y, x)$. Definim el conjunt $K = \{(x, y) \mid x \in \mathbb{Z} \text{ ò } y \in \mathbb{Z}\}$ que és clarament G-invariant, i triem el punt no G-invariant $x = (1, 0)$. Observem que el conjunt $K + x$ és, de nou, G-invariant.

També tenim aquesta propietat relativa a funcions G-invariants.

Propietat 11. *Siga X un espai de Banach i G un grup actuant sobre G. Si $\varphi\colon X \to \mathbb{R}$ és G-invariant, aleshores φ^n també ho és per a tot $n \in \mathbb{Z}$.*

Demostració. Si $n \in \mathbb{N}$, per la G-invariança de φ:

$$\varphi^n(g(x)) = \varphi(g(x)) \cdot \dots \cdot \varphi(g(x)) = \varphi(x) \cdot \dots \cdot \varphi(x) = \varphi^n(x).$$

Si $n = 0$, és clar ja que les constants són sempre G-invariants. Si $n \leqslant -1$, de nou per la G-invariança de φ tenim que:

$$\varphi^n(g(x)) = \frac{1}{\varphi^{-n}(g(x))} = \frac{1}{\varphi(g(x)) \cdot \dots \cdot \varphi(g(x))} =$$

$$= \frac{1}{\varphi(x) \cdot \dots \cdot \varphi(x)} = \frac{1}{\varphi^{-n}(x)} = \varphi^n(x).$$

□

En la següent proposició volem relacionar la G-invariança d'una funció amb la G-invariança de la seua gràfica. Recordem primer que si X i Y són dos espais normats i f és una aplicació de X en Y, aleshores la gràfica de f es defineix com

$$Gr(f) = \{(x, f(x))\colon x \in X\} \subseteq X \times Y.$$

I si $Y = \mathbb{R}$ aleshores, l'epígraf de f es deineix com

$$Epi(f) = \{(x, \lambda) \in X \times \mathbb{R}\colon f(x) \leqslant \lambda\}.$$

Notem que el grup $G \subseteq \mathcal{L}(X)$ pot ser inclós naturalment en $\mathcal{L}(X \times Y)$ simplement associant a l'aplicació $g \in G$, l'aplicació

$$\begin{array}{rccc} (g, Id)\colon & X \times Y & \to & X \times Y \\ & (x, y) & \mapsto & (g(x), y). \end{array}$$

Açò defineix naturalment una acció de G en $X \times Y$, i en particular per a $(x, f(x)) \in Gr(f)$ i $g \in G$ tenim que

$$g(x, f(x)) = (g(x), f(x)) \quad \forall\, g \in G.$$

Obserem en la Figura 1.4 que, si prenem $X = \mathbb{R}$, considerem el grup $G = \{Id, -Id\}$ actuant en $\mathbb{R}$ i prenem $f(x) = x^2$, que és també una funció G-invariant, aleshores, la gràpica i l'epígraf de f són també (G, Id)-invariants.

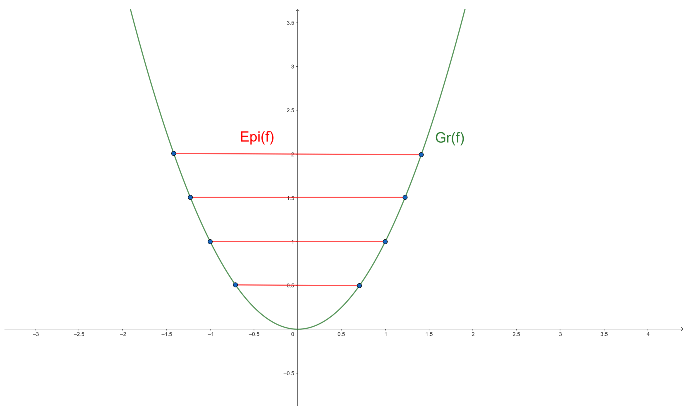


Figura 1.4. La gràfica i l'epígraf de la funció $f(x) = x^2$.

De fet, aquesta propietat és una equivalència, tal i com mostrem en la Proposició següent.

Proposició 12. *Siga X un espai normat, i siga G un grup actuan sobre X. Aleshores $f\colon X \to Y$ és G-invariant si, i solament si, $Gr(f)$ és (G, Id)-invariant.*

Demostració. Suposem primer que f és G-invariant. Volem veure que

$$(G, Id)Gr(f) = Gr(f).$$

Començem primer comprovant l'inclusió següent: $Gr(f) \subseteq (G, Id)Gr(f)$.

Triem $(x, f(x)) \in Gr(f)$ i $g \in G$. Aleshores $z = g^{-1}(x) \in X$. Aplicant la G-invariança de f tenim que

$$(x, f(x)) = (g(z), f(g(z))) = (g(z), f(z)) =$$

$$= (G, Id)(z, f(z)) \in (G, Id)Gr(f).$$

Aleshores $Gr(f) \subseteq (G, Id)Gr(f)$. L'altra inclussió és clara emprant elements inversos, aleshores $Gr(f)$ és (G, Id)-invariant.

Suposem ara que $Gr(f)$ és (G, Id)-invariant. Volem veure que la funció f és G-invariant. Triem $(x, f(x)) \in Gr(f)$ i $g \in G$. Com que $(g, Id)(x, f(x)) \in Gr(f)$, aleshores existeix un punt $z \in X$ tal que

$$(z, f(z)) = (g, Id)(x, f(x)) = (g(x), f(x)).$$

D'on deduïm que $z = g(x)$ i $f(z) = f(x)$. Ara és clar que

$$f(x) = f(z) = f(g(x)).$$

Llavors $f(x) = f(g(x))$ per a tota $g \in G$ i tot $x \in X$. Aleshores f és G-invariant. □

Com a conseqüència obtenim la següent caracterització.

Corol·lari 13. *Siga X un espai normat, i siga G un grup actuant sobre X. Aleshores $f\colon X \to Y$ és G-invariant si, i solament si, $Epi(f)$ és (G, Id)-invariant.*

Nota 14. Observem ací l'analogia amb els conjunts convexos, on tenim que una funció és convexa si, i solament si, el seu epígraf és convex. La Proposició 12 i el Corol·lari 13 són dos resultats interessants ja que relaciones dues nocions G-invariants, la de G-invariant per a funcions i la de G-invariants per a conjunts, i afirmen que ambdues són equivalents.

Chapter 2

Introduction to the concept of group invariance

When searching for alternative representations of some algebras of holomorphic functions, in 2018 Aron, Falco, Garcia and Maestre in [3] used group invariant arguments for proper holomorphic mappings. During the preparation of this paper, Aron, Falco and Maestre, realized that, surprisingly, there were no separation theorems for group invariant polynomials. So while working on the first paper, they obtained results on this direction of separation theorems, and they published them in [5]. However, they were working in $\mathbb{C}^n$, so the group that they used was a finite group whose elements were actually a Gröbner basis. For more information on this we recommend [16, Chapter 2]. A close look at the arguments they used shows that similar techniques can be used if we consider compact topological groups. As far as we know, this type of arguments were never before used in the theory of geometry of Banach spaces until [30], where a generalization of one of the most important results in the theory of norm attaining, the Bishop-Phelps-Bollobas theorem, to the group invariant case is obtained. The techniques used in this paper were well-recieved, and since its publication, some more papers that used group invariant techniques have appeared, apart from the ones presented in this work, see for instance [19] and [20]. It is important to remark that some work was done before on group invariant mappings in Banach spaces, but in the context of measure theory, see [37, Chapter 22]. But this is the only reference, to the extent of our knowledge, that used group invariant notions in the theory of Banach spaces, so all we can say is that we are in front of a new, unexplored, and beautiful way of research.

This chapter lays the foundation for the study of group-invariant concepts. We introduce their basic definitions, explore key introductory properties, and establish the notation that will be used throughout the thesis. Although these results are elementary, they are included for the sake of completeness. All of the material presented here can be found in the work to be published at

[32] J. Falco, and D. Isert, Basic properties of the infinite dimensional group invariant points, sets and mappings, *Proceedings of XII Congreso del Máster en Investigación Matemática*, to appear.

We start by establishing the notation, adhering to the standard conventions commonly used in the field. Let X be a normed space, typically a Banach space. The space of all linear and continuous mappings from X to itself is denoted by $\mathcal{L}(X)$. The sets B_X and S_X represent, respectively, the unit ball, and the unit sphere of X.

At the heart of this work lies the concept of a topological group, which we now introduce as the cornerstone of the thesis.

Definition 1. A topological group G is a topological space that is also a group, such that the group operation xy and the inverse mapping x^{-1} are continuous mappings.

In many examples, we will use topological groups that rise naturally from algebra, like the following basic example. Considering $X = \mathbb{R}^2$ with the euclidean topology. A possible topological group is $G = \{Id, \sigma\}$ where $\sigma = (1, 2)$ is the permutation of the two coordinates of a point in X. Note the natural relation of this topological group with the classical group in algebra Σ_2, given by the identity and the permutations of two elements.

To continue, we recall the definition of action of a group on a set.

Definition 2. Given a group G and a set K, we denote the action of the group on the set K by

$$\langle G, K\rangle = \{gx \colon g \in G, x \in K\}.$$

Hereinafter, the topological groups G considered will consist on isometries, particularly $G \subseteq \mathcal{L}(X)$. The term G is reserved to denote a topological group of invertible bounded linear mappings, equipped with the relative topology induced by $\mathcal{L}(X)$. Similarly, the word g will refer to the elements of the group G.

With this definition in place, we are now prepared to introduce the three fundamental definitions of group invariance.

Definition 3. Let X, Y be two normed spaces:

1. A point $x \in X$ is G-invariant, or invariant under the action of G if $g(x) = x$ for all $g \in G$.

2. A set $K \subset X$ is G-invariant if for every $g \in G$, $g(K) = K$.

3. A mapping $f \colon X \to Y$ is G-invariant if for every $x \in X$ and every $g \in G$ we have that
$$f(g(x)) = f(x).$$

We denote the set of points that are G-invariant as follows

$$X_G = \{x \in X \mid x \text{ is } G\text{-invariant}\}.$$

We will denote the set of continuous linear G-invariant functionals by

$$X_G^* = \{f \in X^* \mid f \text{ is } G\text{-invariant}\}.$$

Since we are considering the elements of G to be isometries, it is clear that the norm of the space X is G-invariant given that

$$\|g(x)\| = \|x\| \quad \forall g \in G.$$

Let us define now when a distance is group invariant. Observe first that the distance function $d\colon X \times X \to \mathbb{R}^+$ has domain $X \times X$, so it makes no sense talking of G-invariance. However, the definition can be naturally adapted as follows.

Definition 4. We say that a distance $d\colon X \times X \to \mathbb{R}^+$ is (G,G)-invariant if, for all $g \in G$

$$d(g(x), g(y)) = d(x, y) \quad \forall x, y \in X.$$

Notice from here that $\|g(x)\| = d(g(x), g(0))$ and $\|x\| = d(x, 0)$. So the definition of (G,G)-invariance for a distance is consistent with the norm being group invariant.

From now on we will always assume that the elements of the group that we are working with are isometries, i.e, the norm is always G-invariant for us.

Example 5. Pick, on $\mathbb{R}^2$ with the euclidean topology, the group $G = \{Id, \sigma\}$ where σ acts on the space as

$$\sigma(x, y) = (y, x).$$

If we take a look at Figure 2.1 we can see that the two represented squares are G-invariant subsets of $\mathbb{R}^2$, and the line $y = x$ are all the G-invariant points of $\mathbb{R}^2$.

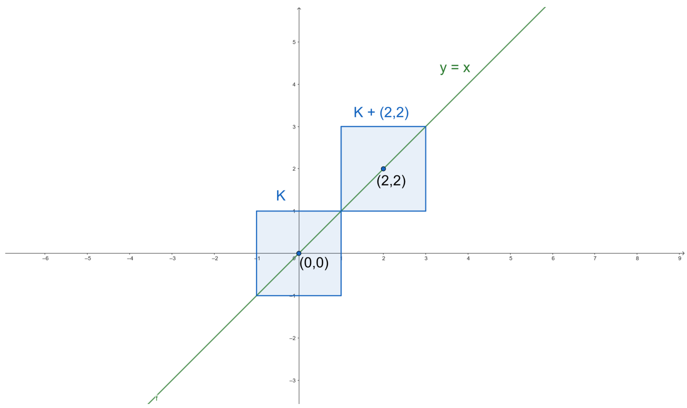


Figure 2.1. Translation of a G-invariant square by a G-invariant point

Observe that in Figure 2.1, the G-invariant square centered at $(2, 2)$ and side length 2 is the translation of the G-invariant square centered at $(0, 0)$ and side length 2 by the G-invariant point $(2, 2)$. This is a particular case of the following property.

Property 6. *Let X be a normed space and G a group acting on X. If both $K \subseteq X$ and $x \in X$ are G-invariants, then $K + x$ is also G-invariant.*

Proof. Assume that $G \subseteq \mathcal{L}(X)$, and that K and x are G-invariant. Take $y \in K$. For every $g \in G$ we have that

$$g(y + x) = g(y) + g(x) = g(y) + x \in g(K) + x = K + x.$$

Therefore, for every $g \in G$, $g(K + x) \subseteq K + x$. In order to get the other inclusion, observe that every element of a group has a unique inverse element, so

$$K + x \subseteq G^{-1}(K + x),$$

thus

$$K + x \subseteq G(K + x).$$

Therefore, $K + x$ is G-invariant. □

As a direct consequence of the property above we get the following.

Corollary 7. *Let X be a normed space, and G be a group acting on X. If K is a vector subspace of X which is G-invariant, and x is also G-invariant, then $K + x$ is an afine G-invariant subspace of X.*

On the following two examples we want to illustrate that the conditions on K and x of being G-invariants are both required.

Example 8. Let $G = \{Id, \sigma\}$ be a group acting on $\mathbb{R}^2$ as defined in Example 5.

(i) Pick K to be the square centered at $(0, 0)$ and side length 1, which is G-invariant, and choose $x = (3, 0)$, which is not G-invariant. Then $K + x$ is not G-invariant, since $\sigma(3, 0) = (0, 3) \notin K + (3, 0)$.

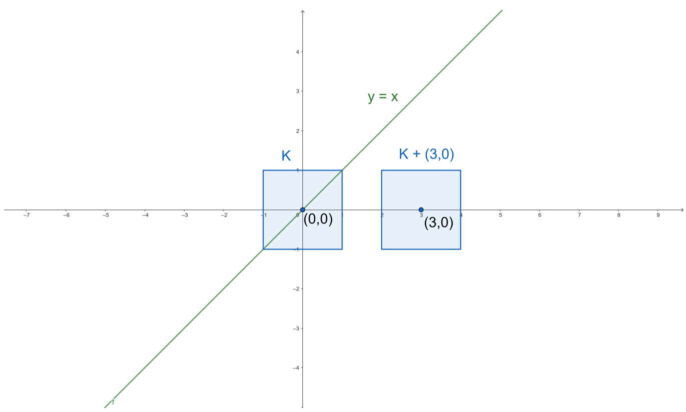


Figure 2.2. Translation of a G-invariant subset by a non G-invariant point.

(ii) Let K be the square centered at $(1, 0)$ and side length 1, which is not G-invariant, and let $x = (1, 1)$ which is G-invariant. Then $K + x$ again is not G-invariant, since $\sigma(3, 1) = (1, 3) \notin K + (1, 1)$.

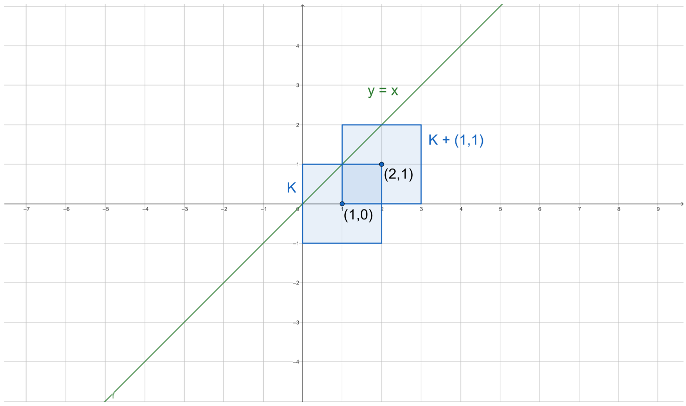


Figure 2.3. Translation of a non G-invariant subset by a G-invariant point.

(iii) On the same way, we can find a subset and a point which are not G-invariant but $K + x$ is G-invariant. Let K be the square centered at $(1, 0)$ and side length 2, which is not G-invariant, and let $x = (0, 1)$ which is also not G-invariant. However, $K + x$ is indeed G-invariant.

Nevertheless the reverse implication of the last property 6 can be obtained if the point x is G-invariant.

Property 9. *Let X be a Banach space, and G be a group acting on X. If $K + x$ is G-invariant and x is G-invariant, then K is G-invariant.*

Proof. Let $g \in G$. For all $y \in K$ we have that

$$y + x \in K + x.$$

By the G-invariance of $K + x$

$$g(y + x) \in K + x.$$

But we know by linearity and G-invariance of x that, $g(y + x) = g(y) + g(x) = g(y) + x$. Therefore

$$g(y) + x \in K + x,$$

from where we conclude that $g(y) \in K$ for all $y \in K$. In particular $g(K) \subseteq K$. By taking now the inverse of g we obtain the other inclusion, hence K is G-invariant. □

Let us continue by remarking that even if the sets $K + x$ and K are G-invariant, we cannot conclude that the point x is G-invariant.

Example 10. Let $G = \{Id, \sigma\}$ on $\mathbb{R}^2$ with the Euclidean topology, where $\sigma(x, y) = (y, x)$. Define the set $K = \{(x, y) \mid x \in \mathbb{Z} \text{ or } y \in \mathbb{Z}\}$ which is clearly G-invariant, and pick the non G-invariant point $x = (1, 0)$. Notice that the set $K + x$ is again G-invariant

Also, we have this simple property concerning G-invariant functions.

Property 11. *Let X be a Banach space and G a group acting on X. If $\varphi\colon X \to \mathbb{R}$ is G-invariant, then, so is the n-th power of φ, $(\varphi(x))^n$, for all $n \in \mathbb{Z}$.*

Proof. If $n \in \mathbb{N}$, by the G-invariance of φ:

$$\varphi^n(g(x)) = \varphi(g(x)) \cdot \dots \cdot \varphi(g(x)) = \varphi(x) \cdot \dots \cdot \varphi(x) = \varphi^n(x).$$

If $n = 0$, it is clear because the constants are always G-invariant, and if $n \leqslant -1$, again by the G-invariance of φ we get:

$$\varphi^n(g(x)) = \frac{1}{\varphi^{-n}(g(x))} = \frac{1}{\varphi(g(x)) \cdot \dots \cdot \varphi(g(x))} =$$

$$= \frac{1}{\varphi(x) \cdot \dots \cdot \varphi(x)} = \frac{1}{\varphi^{-n}(x)} = \varphi^n(x).$$

□

In the next proposition we want to relate the G-invariance of a function with the G-invariance of its graph. This allows us to connect notions 2 and 3 of Definition 3. For this, we recall that if X, and Y are two normed spaces and f is a mapping from X to Y, then the graph of f is defined as

$$Gr(f) = \{(x, f(x)) \colon x \in X\} \subseteq X \times Y,$$

and if $Y = \mathbb{R}$ then the epigraph of f is defined as

$$Epi(f) = \{(x, \lambda) \in X \times \mathbb{R} \colon f(x) \leqslant \lambda\}.$$

Note that the group $G \subseteq \mathcal{L}(X)$ can be naturally embedded in $\mathcal{L}(X \times Y)$ by using the mapping

$$\begin{array}{rccc} (g, Id) \colon & X \times Y & \to & X \times Y \\ & (x, y) & \mapsto & (g(x), y). \end{array}$$

This naturally defines an action of G on $X \times Y$, and in particular for $(x, f(x)) \in \mathrm{Gr}(f)$ and $g \in G$ we have that

$$g(x, f(x)) = (g(x), f(x)) \quad \forall\, g \in G.$$

We observe in Figure 2.4 that, if we take the group $G = \{Id, -Id\}$ acting on $\mathbb{R}$ and take $f(x) = x^2$, which is a G-invariant function, then, the graph and epigraph of f are also (G, Id)-invariant

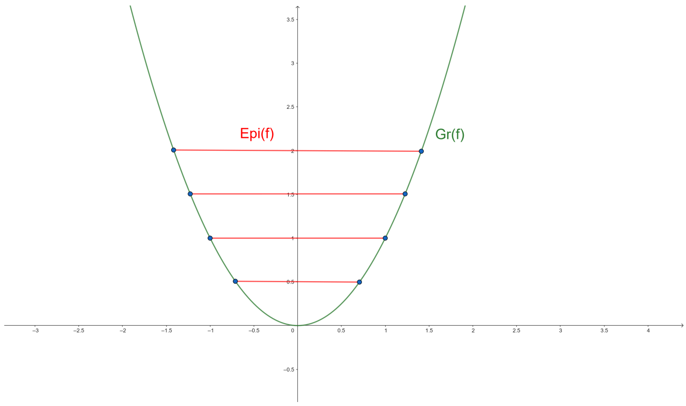


Figure 2.4. The graph and epigraph of the function $f(x) = x^2$.

In fact, this relation always holds, as we show in the following proposition.

Proposition 12. *Let X be a normed space, and let G be a group acting on X. Then $f\colon X \to Y$ is G-invariant if, and only if, $Gr(f)$ is (G, Id)-invariant.*

Proof. Let us show first the if condition. Suppose that f is G-invariant. We want to see that $(G, Id)Gr(f) = Gr(f)$. Let us start checking the inclusion $Gr(f) \subseteq (G, Id)Gr(f)$.

Pick $(x, f(x)) \in Gr(f)$ and $g \in G$. Then $z = g^{-1}(x) \in X$. By applying the G-invariance of f we get that

$$(x, f(x)) = (g(z), f(g(z))) = (g(z), f(z)) \in (G, Id)Gr(f).$$

Therefore $Gr(f) \subseteq (G, Id)Gr(f)$. The other inclusion is straightforward by using the inverse element, thus $Gr(f)$ is (G, Id)-invariant.

To show the only if condition, suppose that $Gr(f)$ is (G, Id)-invariant. We want to check that f is G-invariant. Fix $(x, f(x)) \in Gr(f)$ and $g \in G$. Since $(g, Id)(x, f(x)) \in Gr(f)$, then there exists $z \in X$ with

$$(z, f(z)) = (g, Id)(x, f(x)) = (g(x), f(x)).$$

We deduce that $z = g(x)$ and $f(z) = f(x)$, now it is clear that

$$f(x) = f(z) = f(g(x)).$$

Therefore $f(x) = f(g(x))$ for every $g \in G$ and every $x \in X$. So f is G-invariant. $\square$

And as a consequence we get the next characterization.

Corollary 13. *Let X be a normed space, and let G be a group acting on X. Then $f\colon X \to Y$ is G-invariant if, and only if, $Epi(f)$ is (G, Id)-invariant.*

Remark 14. It is worth noting the analogy with convex sets, where we have that a function is convex if, and only if, its epigraph is convex. This result is relevant because it relates two G-invariant notions, the G-invariance for functions and the G-invariance for sets, and we are saying that both of them are equivalent.

Chapter 3

Generalization of classical concepts

In Chapter 2 we have setted our framework, in this Chapter we are going to clarify all the previous definitions and results that we are going to use in the rest of the work. Most of the definitions presented here are generalizations to the group-invariant setting of previously known concepts, and many of the results are generalizations to our context of known results to our context. We will also emphasize the differences between the classical and the group-invariant definitions.

The structure of the chapter is as follows, in Section 3.1 we will study the most important definition of this work, the symmetrized operator. Section 3.2 is dedicated to the study of G-extreme, G-exposed and G-strongly exposed points. In section 3.3 we will study G-invariant Hahn-Banach separation theorems. In section 3.4 we will define the G-weak and G-weak star topologies, study some of their important properties, generalize some useful classical results, and deepen our study in the spaces X, X_G, X_G^*, and $X_{G^{**}}^{**}$. Finally in Section 3.5 we will recall the definitions of Gâteaux and Fréchet differentiability, and define the notion of G-subdifferential.

This chapter cannot be found just in one paper. This is a compilation of results that have been published in many articles and that all together glue all the work done. The results in sections 3.1, 3.3, 3.4, and 3.5 of this chapter can be found in the published work

> [33] J. Falco, and D. Isert, Group invariant variational principles, *Revista de la real academia de ciencias exactas, físicas y naturales. Serie A. Matemáticas* **10**(3) (2024), pp. 443–474.

and in the submitted works

> [34] J. Falco, and D. Isert, G-strong subdifferentiability and applications to norm attaining subspaces, *arXiv preprint arXiv:2403.18467* (2024).

> [35] J. Falco, and D. Isert, Variational principles and applications to symmetric PDEs, *arXiv preprint arXiv:2403.18467* (2024).

And Section 3.2 has not been published yet since it is a part of an ongoing research.

3.1 Symmetrization

The symmetrized point and operator is one of the key definitions of this work. The symmetrized operator has the particularity that it is G-invariant, so in most of the results in which we need to obtain a G-invariant point, we will require to do a symmetrization argument in order to restrict ourselves to the space X_G. This symmetrized operator is a generalization of the Reynolds operator defined in [16, Chapter 7.3], and it was first defined, to our knowledge, in [4, Theorem 2.3] in the context of holomorphic functions. Then, it was revisited, again in the context of complex analysis, in [3, Section 4: Proposition 4]. Our definition of symmetrized operator follows the same ideas used in those articles, but we define it in a bit more general setting for any Banach space. We will also need in our definition to use the Haar measure and the Bochner integral, so we are going to give here the basic results that we will need to take into account to present this definition.

Definition 3.1.1. Let G be a locally compact group, and let μ be a nonzero regular Borel measure on G. Then, μ is a left Haar measure if it is invariant under left translations, in the sense that $\mu(xA) = \mu(x)$ holds for each $x \in G$ and each $A \in \mathcal{B}(G)$.

First, we will start with results concerning the Haar measure. For the proof, and more details, of the following classical results we recommend [15, Chapter 9].

Theorem 3.1.2 (Existence of a Haar measure)**.** *Let G be a locally compact group. Then, there is a left Haar measure on G.*

Theorem 3.1.3 (Uniqueness of the Haar measure)**.** *Let G be a locally compact group, and μ and ν two left Haar measures on G. The, there exists a positive constant $c \in \mathbb{R}$ such that $\nu = c\mu$.*

Theorem 3.1.4 (Finiteness on compact groups)**.** *Let G be a locally compact group, and let μ be a left Haar measure on G. Then, μ is finite if, and only if, G is compact.*

Now, we will give a few notes on the Bochner integral. For the proof, and more details, on the following results see [15, Appendix E] and the references therein.

Let $(M, \mathcal{A})$ be a measurable space, let E be a real or complex Banach space, and let $\mathcal{B}(E)$ be the σ-algebra of borel subsets of E.

Recall that, a function $f\colon M \to X$ is Borel measurable if it is measurable with respect to $\mathcal{A}$ and $\mathcal{B}(E)$, and it is strongly measurable if it is Borel measurable and $f(X)$ is separable.

Definition 3.1.5. Let $(M, \mathcal{A}, \mu)$ be a measurable space and X a real or complex Banach space. A function $f\colon M \to X$ is integrable if it is strongly measurable and the function $x \mapsto |f(x)|$ is integrable.

We now proceed to define the Bochner integral.

Definition 3.1.6. Let $(M, \mathcal{A}, \mu)$ be a measurable space and X a real or complex Banach space. Suppose that $f\colon M \to X$ is simple and integrable. Let $a_1, \ldots, a_n$ be the nonzero values of f, and suppose that these values are attained on the sets $A_1, \ldots, A_n$. Then, each A_i has finite measure under μ, and we define the integral of f to be:

$$\int f\, d\mu = \sum_{i=1}^{n} a_i \mu(A_i).$$

Let now f be an arbitrary integrable function, and choose a sequence $\{f_n\}$ of simple integrable functions such that

1. $f(x) = \lim_n f_n(x)$ holds at each $x \in M$.
2. $x \mapsto \sup_n |f_n(x)|$ is integrable.

Then, the dominated convergence theorem implies that

$$\lim_n \int |f_n - f|\, d\mu = 0,$$

and hence that $\lim_{m,n} \int |f_m - f_n|\, d\mu = 0$. Hence, $\{f_n\}$ is a Cauchy sequence, so it is convergent. We define the Bochner integral of f, and we denote it by $\int f\, d\mu$, to be the limit of the sequence $\left\{\int f_n\, d\mu\right\}$.

Let us present a few properties of this integral.

Proposition 3.1.7. *Let $(M, \mathcal{A}, \mu)$ be a measurable space and X a real or complex Banach space. Suppose that $f, g\colon M \to X$ are integrable and that a and b are real or complex values. Then,*

1. *$af + bg$ is integrable and*

$$\int (af + bg)\, d\mu = a \int f\, d\mu + b \int g\, d\mu.$$

2. *$\left|\int f\, d\mu\right| \leqslant \int |f|\, d\mu$.*

The following property of the Bochner integral will be used several times later.

Proposition 3.1.8. *Let $(M, \mathcal{A}, \mu)$ be a measurable space, let X be a real or complex Banach space, and let $f\colon M \to X$ be integrable. Then,*

$$\int \varphi \circ f\, d\mu = \varphi\left(\int f\, d\mu\right)$$

holds for each $\varphi \in X^*$.

Now we are ready to define the symmetrization[1] operator. Roughly speaking this operator takes one point and, by averaging its orbit under the action of G, produces a G-invariant point.

Definition 3.1.9. Let X be a Banach space and let G be a compact topological group acting on X. For $x \in X$ we define the symmetrization point of x with respect to G, or the G-symmetrization point, to be

$$\overline{x} = \int_G g(x)\, d\mu(g),$$

where the integral is the Bochner integral, and the measure is the Haar measure associated to the compact group G. Similarly for a functional

[1] The word symmetrization refers to the well-known process of applying Definition 3.1.9 to the group of permutations $G = \Sigma_n$ acting on $\mathbb{R}^n$.

$F\colon X \to \mathbb{R}$ we define the symmetric functional of F with respect to G, or the G-symmetric functional, to be

$$\overline{F}(x) = \int_G F(g(x))d\mu(g),$$

where again the integral is the Bochner integral and the measure is the Haar measure associated to the compact group G.

Remark 3.1.10. Observe that if a functional $h\colon X \to \mathbb{R}$ is linear and continuous, then we have that

$$h(\overline{x}) = \int_G h(g(x))d\mu(g).$$

Indeed, if we apply Proposition 3.1.8, then we obtain the desired equality.

It is easy to see now that the G-symmetrized point is G-invariant. For every $g \in G$, by linearity,

$$g(\overline{x}) = g\left(\int_G h(x)d\mu(h)\right) = \int_G gh(x)d\mu(h).$$

But now, since G is a group, it is clear that $G = \{gh \mid g \in G\}$, which means that

$$g(\overline{x}) = \int_G gh(x)d\mu(g) = \int_G h(x)d\mu(h) = \overline{x}.$$

We can use the same argument to check that the G-symmetric functional is G-invariant.

While extending Ekeland's variational principle to G-invariant functions, we will require some additional condition for the result to be true, we will see that in Chapter 4. This condition that we will need to add is the notion of convexity with respect to the group. We will now define, and study, some useful properties about convexity and linearity with respect to the group. Both of this definitions will be crucial in Chapter 4.

Definition 3.1.11. Let X be a Banach space and G a compact topological group acting on X. Let $\varphi\colon X \to \mathbb{R}$ be a function. We say that φ is convex with respect to G given that

$$\varphi\left(\int_G g(x)d\mu(g)\right) \leqslant \int_G \varphi(g(x))d\mu(g) \quad \forall\, x \in X.$$

And we say that φ is linear with respect to G if

$$\varphi\left(\int_G g(x)d\mu(g)\right) = \int_G \varphi(g(x))d\mu(g) \quad \forall\, x \in X.$$

It is easy to see that if φ is G-invariant and convex (resp. linear) with respect to G then we get that

$$\varphi(\overline{x}) \leqslant \varphi(x) \quad (\text{resp. } \varphi(\overline{x}) = \varphi(x))\,.$$

For a function, being convex or linear with respect to the group is a weaker condition than being convex or linear, as the following result and example show.

Property 3.1.12. *Let X be a Banach space and G a compact topological group acting on X, if $\varphi\colon X \to \mathbb{R}$ is*

1. *Linear and continuous, then it is linear with respect to G.*

2. *Convex, then it is convex with respect to G.*

Proof. Suppose that φ is linear, then we already know that we can interchange the Bochner integral with the function φ by Remark 3.1.10, so it is clear that φ is linear with respect to G.

Suppose now that φ is convex. By Proposition 3.1.8, we obtain that φ is convex with respect to G. □

We are going to give now two examples to show that the reverse implications are, in general, not true.

Example 3.1.13. Let $X = \mathbb{R}^2$ with the Euclidean topology and $G = \{Id, \sigma\}$, where

$$\begin{array}{cccc} \sigma: & \mathbb{R}^2 & \to & \mathbb{R}^2 \\ & (x,y) & \mapsto & (-x,y). \end{array}$$

Note that

$$\overline{(x,y)} = \frac{(x,y)+(-x,y)}{2} = (0,y), \quad \forall\, x,y \in \mathbb{R}.$$

Now, define $f(x,y) = y^2$. Observe that, on the one hand,

$$f\left(\overline{(x,y)}\right) = f(0,y) = y^2.$$

And, on the other hand,

$$\int_G f(g(x))d\mu(g) = \frac{f(x,y)+f(-x,y)}{2} = y^2.$$

Hence, it is clear that f is linear with respect to G, but it is also clear that f is not linear.

Example 3.1.14. Let $G = \{Id, -Id\}$, and define

$$f(x) = \begin{cases} -x-1 & \text{if } x \leqslant -1, \\ x+1 & \text{if } -1 < x < 0, \\ -1 & \text{if } x = 0, \\ -x+1 & \text{if } 0 < x < 1, \\ x-1 & \text{if } x \geqslant 1. \end{cases}$$

This function is clearly convex with respect to the group G, since $f(\overline{x}) = f(0) = -1 \leqslant \int_G f(g(x))d\mu(g)$. But also it is clear that f is not convex since its epigraph is not convex, see Figure 3.1.

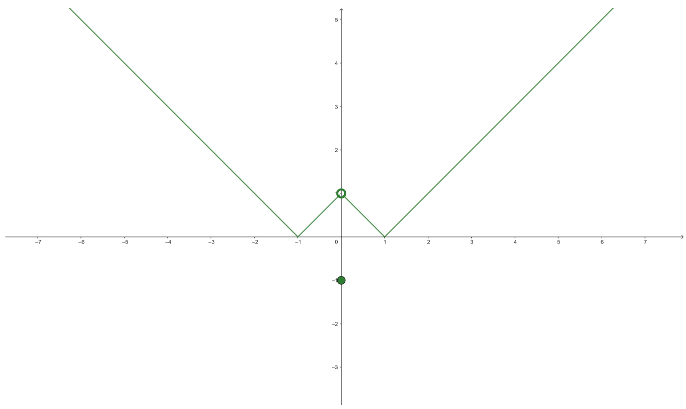


Figure 3.1. A nonconvex function that is convex with respect to the group G.

It is not difficult to modify the previous construction to obtain the analogous result, but with a continuous function.

Property 3.1.15. *Let X be a Banach space, G a compact topological group acting on X, and $\varphi\colon X \to \mathbb{R}$ be G-invariant. If φ is linear with respect to G then $\frac{1}{\varphi^2}$ is convex with respect to G.*

Proof. Define $\psi\colon X \to \mathbb{R}$ as follows

$$\psi(x) = \begin{cases} \frac{1}{\varphi^2(x)} & \text{if } \varphi(x) \neq 0, \\ +\infty & \text{if } \varphi(x) = 0. \end{cases}$$

Fix $x \in X$. Observe that

$$\psi\left(\int_G g(x)d\mu(g)\right) \leqslant \int_G \psi(g(x))d\mu(g),$$

is equivalent, by definition, to

$$\frac{1}{\varphi^2\left(\int_G g(x)d\mu(g)\right)} \leqslant \int_G \frac{1}{\varphi^2(g(x))}d\mu(g).$$

By G-invariance of φ, Property 11 and definition of the symmetrized point, this is equivalent to

$$\frac{1}{\varphi^2\left(\overline{x}\right)} \leqslant \int_G \frac{1}{\varphi^2(x)}d\mu(g) = \frac{1}{\varphi^2(x)}.$$

This is also equivalent to,

$$\varphi^2(x) \leqslant \varphi^2(\overline{x}).$$

By linearity of φ this inequality is satisfied. □

Corollary 3.1.16. *Let φ be G-invariant, φ satisfies that $\varphi(x) \leqslant \varphi(\overline{x})$ if, and only if, φ^{-2k} is convex with respect to G for all $k \in \mathbb{Z}^+$.*

Corollary 3.1.17. *Let φ be G-invariant. If φ is linear with respect to G then φ^{-2k} is convex with respect to G for all $k \in \mathbb{Z}^+$.*

From now on, all the groups that we are going to work with will be compact groups, unless otherwise said. The main reason for asking the group to be compact is to guarantee that there exists $\overline{x}$. In the following example we want to illustrate that the condition on G being compact cannot be completely removed, otherwise the space of G-invariant points may reduce only to the zero, i.e, $X_G = \{0\}$.

Example 3.1.18. Consider

$$X_G = \{x \in X \mid x \text{ is } G\text{-invariant } \},$$

$$X_G^* = \{f \in X^* \mid f \text{ is } G\text{-invariant}\}.$$

Set $X = l_1$, A a countable set and $\{N_i\}_{i\in A}$ a pairwise disjoint partition of the natural numbers with $|N_i| = +\infty$. Consider the group

$$G = \langle\{(\alpha, \beta) \mid \alpha, \beta \in N_i, \text{ for some } i \in A\}\rangle,$$

that is not compact. Then, clearly $X_G = \{0\}$ since $|N_i| = +\infty$ for all $i \in A$. However

$$X_G^* = \{f \in l_\infty \mid f_j = \alpha_i \text{ for some } \alpha_i \in \mathbb{R},\ i \in A, \text{ and } \forall\, j \in N_i\}\,.$$

Thus $\dim(X_G^*) = |A|$.

3.2 Extremal, exposed and strongly exposed

This section is devoted to generalize to the context of G-invariance the notions of extremal, exposed, and strongly exposed point. We will study the relations between these three concepts and at the end of the section we will study the G-invariant Krein-Milman theorem which, surprisingly, does not hold in this context. We will give an example of when this result does not hold, and we will give the best result that we can hope to achieve in this context.

Definition 3.2.1. Let X be a Banach space, G a compact topological group acting on X, and $C \subseteq X$ a closed, convex , and G-invariant set. We say that a point $x_0 \in C$ is G-exposed if there exists a functional $x^* \in (X^*\setminus\{0\})_G$ such that

1. $\langle x^*, x_0\rangle = \sup_{x\in C}\langle x^*, x\rangle$.

2. $\langle x^*, x\rangle < \sup_{y\in C}\langle x^*, y\rangle$ for all $x \in C\setminus\{x_0\}$.

We say that a point $x_0 \in C$ is G-strongly exposed if there exists a functional $x^* \in (X^* \setminus \{0\})_G$ such that for every sequence $\{x_n\}_{n=1}^{+\infty} \subseteq C$ with $\langle x^*, x_n \rangle \to \sup_{x \in C} \langle x^*, x \rangle$, then $\|\overline{x_n} - \overline{x_0}\| \to 0$.

Remark 3.2.2. Let X be a Banach space and G a compact topological group acting on X. Recall that, if a function $f\colon X \to \mathbb{R}$ is convex with respect to the group, then the following equality holds:

$$\sup \{f(x) \mid x \in X\} = \sup \{f(x) \mid x \in X_G\}.$$

The proof of this can be found in [33, Theorem 12].

By just putting the definitions together and taking into account Remark 3.2.2, we can easily obtain the following result.

Proposition 3.2.3. *Let X be a Banach space, G a compact topological group acting on X, and $C \subseteq X$ a closed, convex, and G-invariant subset. Then, a point $x_0 \in C$ is G-exposed in C if, and only if, $\overline{x_0}$ is exposed in C_G in X_G.*

Also, a point $x_0 \in C$ is G-strongly exposed in C if, and only if, $\overline{x_0}$ is strongly exposed in C_G in X_G.

Let X be a Banach space, and $A \subseteq X$ a nonempty subset. Recall that a slice of A is defined as

$$S(x^*, A, \alpha) = \left\{ x \in A \mid \langle x^*, x \rangle > \sup_{x \in A} \langle x^*, x \rangle - \alpha \right\},$$

where $x^* \in X^*$ and $\alpha > 0$.

Property 3.2.4. *Assume that X is a Banach space and G a compact topological group acting on X. If A and x^* are G-invariant, then so is $S(x^*, A, \alpha)$.*

Proof. Let $x \in S(x^*, A, \alpha)$, then

$$\langle x^*, g(x) \rangle = \langle x^*, x \rangle > \sup_{x \in A} \langle x^*, x \rangle - \alpha = \sup_{x \in A} \langle x^*, g(x) \rangle - \alpha.$$

Where we have used first that x^* is G-invariant, secondly that x is an element of the slice, and for the last equality that both A and x^* are G-invariant. □

The following property gives us a necessary condition for the slices to be convex sets. This is going to be very useful in what follows since we will need to do symmetrization arguments, and we know that when a point x lies in a convex set, then symmetrization of x remains in the set, see [19, Proposition 2.2].

Property 3.2.5. *Let X be a Banach space, and $C \subseteq X$ be a convex set. Then $S(x^*, C, \alpha)$ is convex.*

The previous result motivates the following definition of G-slice of a G-invariant set.

Definition 3.2.6. Let X be a Banach space, G a compact topological group acting on X, and A a G-invariant subset. We define the G-slice of A as

$$S_G(x^*, A, \alpha) = \left\{ x \in A_G \mid \langle x^*, x \rangle > \sup_{x \in A} \langle x^*, x \rangle - \alpha \right\},$$

where $x^* \in X^*_G$ and $\alpha > 0$.

Remark 3.2.7. Note that if the set A is convex, then $S_G(x^*, A, \alpha)$ is non-empty.

Example 3.2.8. Consider $X = \mathbb{R}^2$, and consider $G = \{Id, \sigma\} \subseteq \mathbb{R}^2$, where $\sigma(x, y) = (-x, y)$. Observe that the condition of $x \in A_G$ in the definition of G-slice is required to determine that when a point is G-strongly exposed, then the diameter of the G-slices goes to zero too.

Take in $\mathbb{R}^2$ the set $C = B_{\|\cdot\|_\infty}((0,0), 1)$. Then the G-strongly exposed points of the set are all those points that lie in the intersection between the horizontal lines $y = 1$ and $y = -1$, and C. Recall that

$$\text{diam}S(x^*, C, \alpha) = \sup\{\|u - v\| \mid u, v \in S(x^*, C, \alpha)\}.$$

So, if we define the G-slice to be:

$$S_G(x^*, A, \alpha) = \left\{x \in A \mid \langle x^*, x\rangle > \sup_{x\in A}\langle x^*, x\rangle - \alpha\right\},$$

from the fact that $\|\overline{x_n} - \overline{x}\| \to 0$ we cannot conclude that $\text{diam}S(x^*, C, \alpha) \to 0$. Because it can happen that $x_0 = (1, 1)$, which is a G-strongly exposed point, but all of the x_n lie in the vertical line $x = -1$ when $-1 \leqslant y \leqslant 1$, in which case, $S_G(x^*, A, \alpha)$ would tend to 2 when $\alpha \to 0$.

Proposition 3.2.9. *Let X be a Banach space, G a compact topological group acting on X, and $C \subseteq X$ a closed, convex, and G-invariant subset. The following are equivalent:*

1. *x_0 is G-strongly exposed.*

2. *There exists a functional $x^* \in (X^* \setminus \{0\})_G$ such that $x_0 \in S_G(x^*, C, \alpha)$ for all $\alpha > 0$, and diam$S_G(x^*, C, \alpha) \to 0$ when $\alpha \to 0$.*

Proof. By Proposition 3.2.3 we have that x_0 is G-strongly exposed if, and only if, $\overline{x_0}$ is strongly exposed in C_G in X_G. Thus, the diameter of $S(x^*, C, \alpha) \to 0$ when $\alpha \to 0$ in X_G. Then, the diameter of the G-slice goes to 0.

Conversely, let $\{x_n\}_{n=1}^{+\infty} \subseteq C$ such that $\langle x^*, x_n\rangle \to \sup_{x\in C}\langle x^*, x\rangle$, $\epsilon > 0$ and $m \in \mathbb{N}$. We know that there exists $n_0 \in \mathbb{N}$ such that

$$\left|\langle x^*, x_n\rangle - \sup_{x\in C}\langle x^*, x\rangle\right| < \frac{1}{m} \quad \forall n \geqslant n_0.$$

Then, by definition, $x_n \in S_G(x^*, C, \frac{1}{m})$ for all $n \geqslant n_0$. In particular, $\overline{x_n} \in S_G(x^*, C, \frac{1}{m})$ for all $n \geqslant n_0$. Since $\text{diam}S_G(x^*, C, \alpha) \to 0$ when

$\alpha \to 0$, there exists $m_0 \geqslant n_0$ such that

$$\|\overline{x_n} - \overline{x_0}\| \leqslant \sup \left\{ \|u - v\| \mid u, v \in S_G(x^*, C, \frac{1}{m}) \right\} \leqslant \epsilon \quad \forall m \geqslant m_0.$$

□

Proposition 3.2.10. *Let X be a Banach space, G a compact topological group acting on X, and $C \subseteq X$ a closed, convex and G-invariant subset. If $x_0 \in C$ is G-strongly exposed, then x_0 is G-exposed.*

Proof. We know that $x_0 \in C$ is G-strongly exposed if, and only if, $\overline{x_0} \in C_G$ is strongly exposed in X_G. Then, $\overline{x_0} \in C_G$ is exposed in X_G. Hence, $x_0 \in C$ is G-exposed. □

Now we present the definition of G-extreme point.

Definition 3.2.11. Let V be a vectorial space, G a compact topological group acting on V, and C a convex and G-invariant subset of V. A point $x_0 \in C$ is G-extreme if for $x_1, x_2 \in C$ and $0 < \lambda < 1$ such that $\overline{x_0} = \lambda \overline{x_1} + (1 - \lambda)\overline{x_2}$, then $\overline{x_0} = \overline{x_1} = \overline{x_2}$. We will denote the set of G-extremal points in C as $\mathrm{Ext}_G(C)$.

Observe that, by just putting the definitions in order, we have the following equivalence.

Proposition 3.2.12. *Let V be a vectorial space, G a compact topological group acting on V, and C a convex and G-invariant subset of V. A point $x_0 \in C$ is G-extreme if, and only if, $\overline{x_0} \in C_G$ is extreme in X_G*

With a similar argument to the one used in Proposition 3.2.10, we obtain the following equivalence.

Proposition 3.2.13. *Let X be a Banach space, G a compact topological group acting on X, and $C \subseteq X$ a closed convex and G-invariant set. If x_0 is G-exposed, then x_0 is G-extreme.*

Definition 3.2.14. Let X be a Banach space, G a compact topological group acting on X, and $C \subseteq X$ a convex subset. A G-face, F, in C is a closed, convex, and G-invariant subset of C, whose elements can only be written as convex combinations of F.

The following two results can be proved by repeating the same argument we made in Proposition 3.2.10, and applying the classical version of this results. For the original proof of this results we refer the reader to [29, Lemma 3.61, Theorem 3.65] and [47, Lemma 2.10.5, Theorem 2.10.6]

Proposition 3.2.15. *Let V be a vectorial space, G a compact topological group acting on V, C a convex and G-invariant subset, and $f : V \to \mathbb{R}$ a G-invariant functional that attains its supreme. If the set*

$$F = \left\{ x \in C \mid f(x) = \sup_{y \in C} f(y) \right\}$$

is non-empty, then it is a G-face of C.

Proposition 3.2.16. *Let V be a real vectorial space, G a compact topological group acting on V, and C a convex and G-invariant subset. Then:*

1. *$x_0 \in C$ is a G-extreme point if, and only if, $\{\overline{x_0}\}$ is a G-face of C.*
2. *If F is a G-face of C, and F' is a G-face of F, then F' is a G-face of C.*

Proposition 3.2.17. *Let X be a Banach space and G a compact topological group acting on X. Every compact, convex and G-invariant subset of X has, at least, one G-extreme point.*

The Krein-Milman theorem is an important result in geometry of Banach spaces. This result states that, in a Banach space, every compact convex subset is the closure of the convex hull of its extreme points. We are going to give now an example where we can clearly see that this result does not hold, in general, on the G-invariant context.

Example 3.2.18. Take in $\mathbb{R}^2$ the group $G = \{Id, \sigma\}$ where $\sigma\colon \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $\sigma(x, y) = (y, x)$. Consider the convex hull of the sets $B_{\|\cdot\|_2}((1, -1), 1)$ and $B_{\|\cdot\|_2}((-1, 1), 1)$, and call the resulting set K.

Observe that the G-extreme points in this set are all the points that lie in the lines $y = -x + \sqrt{2}$ and $y = -x - \sqrt{2}$, i.e, all the points that lie in the green segments in Figure 3.2. Therefore the closed convex hull of the G-extreme points is the orange set in Figure 3.2. Observe that $K \neq \overline{\text{conv}}(\text{Ext}_G(K))$, indeed take $(-1 - \frac{\sqrt{2}}{2}, 1 + \frac{\sqrt{2}}{2}) \in K$ but $(-1 - \frac{\sqrt{2}}{2}, 1 + \frac{\sqrt{2}}{2}) \notin \overline{\text{conv}}(\text{Ext}_G(K))$. Also, it is clear that, $K_G \neq \overline{\text{conv}}(\text{Ext}_G(K))$, take $(-1, 1) \notin K_G$, but $(-1, 1) \in \overline{\text{conv}}(\text{Ext}_G(K))$.

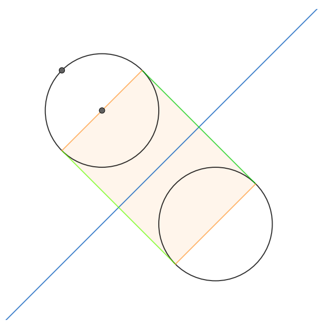


Figure 3.2. Counterexample of Krein-Milman's theorem in the G-invariant context.

So, in view of Example 3.2.18, and related to the Krein-Milman theorem, this is the best we can hope to achieve.

Proposition 3.2.19. *Let X be a Banach space, G a compact topological group acting on X, and K a compact, convex and G-invariant subset.*

Then:

$$K_G \subseteq \overline{conv}(Ext_G(K)) \subseteq K.$$

Proof. On the one hand observe that, by definition, $\mathrm{Ext}_G(K)$ is a subset of K. Hence, $\mathrm{conv}(\mathrm{Ext}_G(K)) \subseteq K$ by convexity of K. Moreover, since K is closed, we have that $\overline{\mathrm{conv}}(\mathrm{Ext}_G(K)) \subseteq K$.

On the other hand, observe that $K_G = K \cap X_G$, and K_G is convex by convexity of K. Also, since K_G is a closed subset of the compact set K, we have that K_G is compact. Now, by the Krein-Milman theorem in X_G we have that

$$K_G = \overline{\mathrm{conv}}(\mathrm{Ext}_G(K_G)).$$

By definition, it is clear that $\mathrm{Ext}(K_G) \subseteq \mathrm{Ext}_G(K)$ □

Remark 3.2.20. Observe that, in the previous proposition, it may happen that $K_G = \overline{\mathrm{conv}}(Ext_G(K))$. Indeed, suppose that we are working in $\mathbb{R}^2$ with the euclidean topology, and that $G = \{Id, (1,2)\}$, where $(1,2)$ indicates the permutation of the two coordinates. Take $K = B_{\|\cdot\|_\infty}((0,0),1)$. The only two G-extreme points are $x_0 = (1,1)$, and $x_1 = (-1,-1)$, and $K_G = \{(x,x) \mid -1 \leqslant x \leqslant 1\}$. It is clear that $K_G = \overline{\mathrm{conv}}(x_0, x_1)$.

3.3 Geometric forms of the Hahn-Banach theorem

We will continue now showing the G-invariant Hahn-Banach geometric theorems. This results will be required for the proofs of Bishop-Phelps and Brønsted-Rockafellar theorems in Chapter 4. But, before moving to the first Hahn-Banach geometric theorem let us recall the definition of the Minkowski functional of a convex set.

Definition 3.3.1. Let X be a Banach space and $C \subseteq X$ be an open convex set with $0 \in C$. For every $x \in X$ the functional

$$p(x) = \inf \left\{ \alpha > 0 \mid \alpha^{-1} x \in C \right\}$$

is called the Minkowski functional of C.

We will make use of the following properties of the Minkowski functional, the proof of this result can be found in [11, Lemma 1.2].

Property 3.3.2. *The Minkowski functional p of an open convex set C satisfies the following properties*

1. $p(\lambda x) = \lambda p(x)$ *for every* $x \in X$, $\lambda > 0$.
2. $p(x + y) \leqslant p(x) + p(y)$ *for every* $x, y \in X$.
3. *There is a constant* $M > 0$ *such that* $0 \leqslant p(x) \leqslant M\|x\|$ *for every* $x \in X$.
4. $C = \{x \in X \mid p(x) < 1\}$.

In [19, Theorem 3.1] it is proved that the Minkowski functional is G-invariant. Let us continue with the following lemma.

Lemma 3.3.3. *Let X be a Banach space, G a compact topological group acting on X, $C \subseteq X$ a G-invariant open convex set. If there exists a G-invariant point $x_0 \in X \backslash C$, then we can find $f \in X_G^*$ such that $f(x) < f(x_0)$ for all $x \in C$.*

Proof. Given $x \in C$ we know by [19, Proposition 2.2] that $\overline{x} \in C$. Consider $C - \overline{x}$, which is a G-invariant set, and the point $x_0 - \overline{x}$. We can assume that $0 \in C$, otherwise we can do a translation by a G-invariant point of C and obtain that the translated set is G-invariant and does contain 0, and tje new x_0 is also G-invariant. Consider the subspace $H = \mathbb{R}x_0$ and the functional

$$\begin{aligned} h\colon \quad H &\to \mathbb{R} \\ tx_0 &\mapsto t. \end{aligned}$$

Observe that H is a G-invariant subspace, since x_0 is G-invariant, and so is the functional h since

$$h(g(tx_0)) = h(tg(x_0)) = h(tx_0) \quad \forall\, g \in G,\, \forall\, t \in \mathbb{R}.$$

Moreover, by doing a study of cases when $t > 0$ and $t \leqslant 0$, is clear that $h(x) \leqslant p(x)$ for all $x \in H$, where p is the Minkowski functional associated to C. Then, we can apply the G-invariant Hahn-Banach theorem, [30, Proposition 1], to obtain a functional $f \in X_G^*$ such that

1. $f_{|_H} = h$,
2. $f(x) \leqslant p(x)$ for every $x \in X$.

In particular $f(x_0) = 1$ and $f(x) \leqslant p(x) < 1$ for all $x \in C$. □

Theorem 3.3.4. *Let X be a Banach space and G a compact topological group acting on X. If A, B are two convex and G-invariant sets, with A open and $A \cap B = \varnothing$, then there exists a G-invariant hyperplane that separates A and B.*

Proof. Define $C = A - B$ and notice that C is convex and G-invariant, since A and B are G-invariant. Also, C is open since we can write

$$C = \bigcup_{y \in B} A - y,$$

and A is open. Moreover, 0 is clearly G-invariant and does not belong to C since $A \cap B = \varnothing$. Therefore, we can apply Lemma 3.3.3 to obtain a functional $f \in X_G^*$ with

$$f(z) < 0 \quad \forall\, z \in C.$$

Since $C = A - B$ and f is linear, we deduce that

$$f(x) < f(y) \quad \forall\, x \in A,\ \forall\, y \in B.$$

Therefore, there exists an $\alpha \in \mathbb{R}$ such that

$$\sup_{x \in A} f(x) \leqslant \alpha \leqslant \inf_{y \in B} f(y).$$

□

The classical version of the previous theorem is usually called the first Hahn-Banach geometric form. Now we want to give a proof of the so-called second Hahn-Banach geometric form for G-invariant functionals. To do so we will require to use the following lemma whose proof can be found in [19, Theorem 3.1].

Lemma 3.3.5. *Let X be a Banach space and G a compact topological group acting on X. If C is a closed convex G-invariant subset of X and x_0 is a G-invariant point not in C, then there exists $f \in X_G^*$ such that*

$$f(x_0) > \sup_{x \in C} f(x).$$

The main difference with Lemma 3.3.3 is the closeness of the set C, that makes the separation between the point and the set strict.

Theorem 3.3.6. *Let X be a Banach space and G a compact topological group acting on X. If $A, B \subseteq X$ are nonempty, convex and G-invariant sets such that A is closed, B is compact and $A \cap B = \varnothing$, then there exists a G-invariant hyperplane that strictly separates A and B.*

Proof. Pick a G-invariant point $x_0 \in B$ such that

$$d(x_0, A) = d(B, A).$$

Note that this point exists because B is compact. Indeed, it is clear that there exists $x \in B$ with $d(x, A) = d(A, B)$ by the G-invariance of B. Observe that $\overline{x} \in B$ and it is straightforward to show that $d(x, A) = d(\overline{x}, A)$.

Then, since $A \cap B = \varnothing$, by Lemma 3.3.5 there exists $f \in X_G^*$ such that

$$f(x_0) > f(x) \quad \forall\, x \in A.$$

Now, the only problem that one can have is that the set B is not contained on one of the two half spaces defined by the hyperplane $f^{-1}(f(x_0))$. But that would be a contradiction with the fact that $d(x_0, A) = d(B, A)$, since, by the convexity of B, we could find another point x_1 such that $d(x_1, A) < d(x_0, A)$. □

To finish this section we will give a direct consequence of the previous result.

Corollary 3.3.7. *Let $X \neq \{0\}$ be a Banach space and G a compact topological group acting on X. Let $F \subseteq X$ be a G-invariant linear subspace such that $\overline{F} \neq X$, then there exists a nonzero functional $f \in X_G^*$ such that*

$$\langle f, x\rangle = 0 \quad \forall\, x \in F.$$

Proof. Observe that $\overline{F}$ is again G-invariant because F is G-invariant and all the points in $\overline{F}$ are obtained as limit points of elements of F. Indeed, if $x_n \to x \in \overline{F}$ with $x_n \in F$ for all $n \in \mathbb{N}$, then $g(x_n) \in F$ for all natural number n and $g(x_n) \to g(x)$ for every $g \in G$. Now let $x_0 \in X \backslash \overline{F}$ be G-invariant, by the previous result we know that there exists $f \in X_G^*$, $f \not\equiv 0$, that strictly separates $\overline{F}$ and $\{x_0\}$, i.e., there exists a constant

$\alpha \in \mathbb{R}$ such that:

$$\langle f, x \rangle < \alpha < \langle f, x_0 \rangle \quad \forall\, x \in F,$$

for some $\alpha \in \mathbb{R}$.

If $F = \{0\}$, since $X \neq \{0\}$, any non-zero G-invariant functional of X^* satisfies the conclusion. Otherwise, for any $x \in F, x \neq 0$ and using the last inequality we deduce that

$$\langle f, x \rangle = 0 \quad \forall\, x \in F,$$

since $\langle f, \lambda x \rangle = \lambda \langle f, x \rangle < \alpha$ for all $\lambda \in \mathbb{R}$. □

3.4 weak and weak-star topologies

To continue, our goal is to define two new topologies that appear naturally when working with group invariant mappings. This will allow us to obtain the group invariant version of several classical results in geometry of Banach spaces.

Let us recall first the definition of adjoint operator.

Definition 3.4.1. Let X, Y be two Banach spaces, and $T\colon X \to Y$ be a bounded operator. The adjoint operator of T, denoted by $T^*\colon Y^* \to X^*$, is the operator defined by

$$\langle x, T^* y^* \rangle = \langle Tx, y^* \rangle \quad \forall\, x \in X,\ y^* \in Y^*.$$

The proof of this result follows from the norm-to-norm continuity of the adjoint operator and the compacity of G.

Proposition 3.4.2. *If $G \subseteq L(X)$ is a compact topological group, then so is $G^* \subseteq L(X^*)$.*

Remark 3.4.3. Dantas, Falco and Jung proved in [19, Proposition 2.4], that an operator $T \in L(X,Y)$ is G-invariant if, and only if, T^{**} is G^{**}-invariant.

To clarify the notation that will appear in this section, we recall that if $G \subseteq \mathcal{L}(X)$:

$$G^{*} = \{g^{*} : g \in G\},$$

$$G^{**} = \{g^{**} : g \in G\},$$

$$X^{**}_{G^{*}} = \{f \in X^{**} \ \mid \ f \text{ is } G^{*}\text{-invariant}\},$$

$$X^{**}_{G^{**}} = \{x^{**} \in X^{**} \ \mid \ g^{**}(x^{**}) = x^{**}\},$$

where in the first case we are considering functionals that are G^{*} invariant and in the second case we are considering points that are G^{**} invariant.

Let us start by showing that the previous two spaces coincide.

Proposition 3.4.4. *Let X be a Banach space and G a compact topological group acting on X. Then, $X^{**}_{G^{**}} = X^{**}_{G^{*}}$.*

Proof. Pick $x^{**} \in X^{**}_{G^{**}}$ and $y^{*} \in X^{*}$, then:

$$\langle x^{**}, g^{*}(y^{*})\rangle = \langle g^{**}(x^{**}), y^{*}\rangle = \langle x^{**}, y^{*}\rangle.$$

So, $x^{**} \in X^{**}_{G^{*}}$

Let now $f \in X^{**}_{G^{*}}$ and $x^{*} \in X^{*}$, observe that

$$\langle g^{**}(f), x^{*}\rangle = \langle f, g^{*}(x^{*})\rangle = \langle f, x^{*}\rangle.$$

So, $f \in X^{**}_{G^{**}}$. □

The following result establishes a fundamental relation between G^{*}-invariant functionals and G-invariant points. Specifically, it demonstrates that any G^{*}-invariant functional can be derived from a G-invariant point, and conversely, every G-invariant point induces a corresponding G^*-invariant functional.

Proposition 3.4.5. *Let X be a Banach space and G a compact topological group acting on X. Then, $x \in X_G \Leftrightarrow x^{**} \in X^{**}_{G^*}$*

Proof. Take $x \in X_G$, observe that, for $y \in X^*$,

$$\langle x^{**}, g^*(y)\rangle = \langle g^*(y), x\rangle = \langle y, g(x)\rangle = \langle y, x\rangle = \langle x^{**}, y\rangle,$$

where we have used that x is G-invariant.

Take now $x^{**} \in X^{**}_{G^*}$, it is clear that

$$\langle y, g(x)\rangle = \langle x^{**}, g^*(y)\rangle = \langle x^{**}, y\rangle = \langle y, x\rangle \quad \forall\, y \in X^*.$$

Since it is true for all $y \in X^*$, we deduce that $g(x) = x$. □

We move now to present the definition of the G-weak and G-weak-star topologies.

Definition 3.4.6. Let X be a normed space and G a compact topological group acting on X. We define the weak group invariant topology on X as the topology generated by the sets

$$\{x \in X \ \mid\ \langle f_i, x - x_0\rangle < \epsilon, \ \text{ for } 1 \leqslant i \leqslant n\},$$

for all choices of $x_0 \in X$, $f_1, \ldots, f_n \in X^*_G$ and $\epsilon > 0$. We denote this topology by w_G or $\sigma_G(X, X^*)$.

Definition 3.4.7. Let X be a normed space and G a compact topological group acting on X. We define the weak-star group invariant topology on X^* as the topology generated by the sets

$$\{f \in X^* \ \mid\ \langle f - f_0, x_i\rangle < \epsilon, \ \text{ for } 1 \leqslant i \leqslant n\},$$

for all choices of $f_0 \in X^*$, $x_1, \ldots, x_n \in X_G$ and $\epsilon > 0$. We denote this topology by w_G^* or $\sigma_G(X^*, X)$.

Notice the following relation between the w topology and the w_G topology.

Proposition 3.4.8. *Let X be a normed space and G a compact topological group acting on X. If $X_G^* \subsetneq X^*$, then the weak group invariant topology on X, w_G, is strictly weaker than the weak topology of X, w.*

Proof. If $X_G^* \subsetneq X^*$, then, there exists $f \in X^* \backslash X_G^*$, $x_0 \in X$ and $g \in G$ such that $\langle f, x_0 \rangle \neq \langle f, g(x_0) \rangle$. Let $\epsilon < |\langle f, g(x_0)\rangle - \langle f, x_0 \rangle|$ and consider

$$U = \{x \in X \ \mid \ \langle f, x \rangle - \langle f, x_0 \rangle < \epsilon\},$$

we claim that $U \in w \backslash w_G$. By definition $U \in w$. We proceed by contradiction to show that $U \notin w_G$. If $U \in w_G$, then $U = \cup_\alpha U_\alpha$ where

$$U_\alpha = \{x \in X \ \mid \ \langle f_{i,\alpha}, x - y_\alpha \rangle < \epsilon_\alpha, \text{ for } 1 \leqslant i \leqslant n\},$$

for some $y_\alpha \in X$ and $f_{i,\alpha} \in X_G^*$. But observe now that, $x_0 \in U$, and since $U \in w_G$, $g(x_0) \in U$. However $|\langle f, g(x_0) \rangle - \langle f, x_0 \rangle| > \epsilon$, a contradiction. □

Also, we would like to highlight this result which is quite different to what happens with the topologies in the non group invariant version.

Property 3.4.9. *Let X be a Banach space and G a compact topological group acting on X. If $X_G^* \subsetneq X^*$ then the topologies w_G and w_G^* are not Hausdorff.*

Proof. We are only going to show the result for the w_G topology. The w_G^* case follows analogously. Pick $x \in X$ which is not G-invariant. Then,

$x \neq g(x)$ for some $g \in G$. By definition, any open set containing x contains an element

$$U = \{y \in X \ \mid \ \langle f_i, y - x\rangle < \epsilon, \ \text{ for } 1 \leqslant i \leqslant n\}$$

for some $f_1, \ldots, f_n \in X_G^*$ and $\epsilon > 0$. Now, any open set containing $g(x)$, again, by definition, contains a set

$$V = \{y \in X \ \mid \ \langle h_i, y - g(x)\rangle < \epsilon', \ \text{ for } 1 \leqslant i \leqslant m\}$$

for some $h_1, \ldots, h_m \in X_G^*$ and $\epsilon' > 0$. Observe that $x \in U \cap V \neq \emptyset$, by the G-invariance of $h_1, \ldots, h_m$. Thus, we cannot find two disjoint neighborhoods of x and $g(x)$. □

Example 3.4.10. Define in $\mathbb{R}^2$ with the euclidean topology the group $G = \{Id, \varphi\}$, where $\varphi \colon \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $\varphi(x, y) = (x, -y)$. Observe that $X_G = \mathbb{R} \times \{0\}$. Then, the function $f(x, y) = y$ is not G-invariant, but f belongs to every w_G^*-neighbourhood of 0 since

$$(f - 0)(x, 0) = 0 \quad \forall x \in \mathbb{R}.$$

Proposition 3.4.11. *Let X be a normed space and G a compact topological group acting on X.*

1. *Let $\{f_n\}_n \subseteq X^*$, $f \in X^*$. Then $f_n \xrightarrow{w_G^*} f$ if, and only if, $\langle f_n, x\rangle \to \langle f, x\rangle$ when $n \to +\infty$ for every $x \in X_G$.*
2. *Let $\{x_n\}_n \subseteq X$, $x \in X$. Then $x_n \xrightarrow{w_G} g(x)$ for every $g \in G$ if, and only if, $\lim_{n\to+\infty}\langle f, x_n\rangle = \langle f, x\rangle$ for every $f \in X_G^*$.*

Our next result is the group invariant version of the classical Banach-Alaoglu-Bourbaki. The proof of this result follows from the Banach-Alaoglu-Bourbaki theorem and Proposition 3.4.8.

Theorem 3.4.12 (Group invariant Banach-Alaoglu-Bourbaki's theorem)**.** *Let X be a Banach space and G a compact topological group acting on X. Then B_{X^*} is w^*_G-compact.*

To continue, let us introduce the notion of G-reflexive space.

Definition 3.4.13. Let X be a Banach space and G a compact topological group acting on X. We say that X is G-reflexive if the canonical injection $\pi\colon X \to X^{**}$ is G-surjective, and by this we mean that, $\pi(X_G) = X^{**}_{G^{**}}$.

Before we continue, let us recall the following lemma that can be found in [11, Lemma 3.3]. This result will be needed in the proof of Goldstine's theorem.

Lemma 3.4.14 (Helly)**.** *Let X be a Banach space. Let $f_1, \dots, f_k \in X^*$, and let $\gamma_1, \dots, \gamma_k \in \mathbb{R}$. The following properties are equivalent:*

1. *$\forall \epsilon > 0$, $\exists x_\epsilon \in B_X$ such that*

$$|\langle f_i, x_\epsilon \rangle - \gamma_i| < \epsilon \quad \forall\, 1 \leqslant i \leqslant k.$$

2. *$\left|\sum_{i=1}^k \beta_i \gamma_i\right| \leqslant \left\|\sum_{i=1}^k \beta_i f_i\right\|$ for all $\beta_1, \dots, \beta_k \in \mathbb{R}$.*

Theorem 3.4.15 (Group invariant Goldstine's theorem)**.** *Let X be a Banach space and G a compact topological group acting on X. Then $\overline{B_X}^{w^*_G} = B_{X^{**}_{G^{**}}}$.*

Proof. Let $\xi \in B_{X^{**}_{G^{**}}}$ and V be a neighborhood of ξ for the w^*_G topology. We want to see that $V \cap \pi(B_X) \neq \varnothing$. Notice that, for some given $f_1, \dots, f_k \in X^*_G$, and $\epsilon > 0$:

$$V' = \{\eta \in X^{**} \;\mid\; |\langle \eta - \xi, f_i \rangle| < \epsilon,\;\; \forall\, 1 \leqslant i \leqslant k,\;\; f_i \in X^*_G\} \subseteq V.$$

Therefore, it is enough to find an $x \in B_X$ such that $\pi(x) \in V'$, i.e.,

$$|\langle f_i, x\rangle - \langle \xi, f_i\rangle| < \epsilon \quad \text{for all } 1 \leqslant i \leqslant k.$$

Define $\gamma_i = \langle \xi, f_i\rangle$. By Lemma 3.4.14 it suffices to check that

$$\left|\sum_{i=1}^{k} \beta_i \gamma_i\right| \leqslant \left\|\sum_{i=1}^{k} \beta_i f_i\right\|.$$

But this is clear since:

$$\left|\sum_{i=1}^{k} \beta_i \gamma_i\right| = \left|\sum_{i=1}^{k} \beta_i \langle \xi, f_i\rangle\right| = \left|\left\langle \xi, \sum_{i=1}^{k} \beta_i f_i \right\rangle\right| \leqslant \left\|\sum_{i=1}^{k} \beta_i f_i\right\|,$$

where we have used that $\|\xi\| \leqslant 1$. □

Theorem 3.4.16 (Group invariant Kakutani's theorem)**.** *Let X be a Banach space and G a compact topological group acting on X. Then X is G-reflexive if, and only if, B_X is compact in the $\sigma_G(X, X^*)$ topology.*

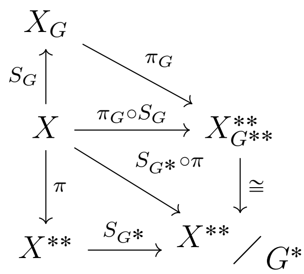
Proof. To prove the if statement, assume X is G-reflexive. Then, since $w_G \subseteq w$ and we know that B_X is compact in the $\sigma(X^{**}, X^*)$ topology, we have that B_X is compact in the $\sigma_G(X^{**}, X^*)$ topology. So, it only remains to show that π^{-1} is continuous from $(X^{**}, \sigma_G(X^{**}, X^*))$ to $(X, \sigma_G(X, X^*))$. For this, fix $f \in X^*$. If we show that $\langle f, \pi^{-1}\xi\rangle$ is continuous on $(X^{**}, \sigma_G(X^{**}, X^*))$, then it is clear that π^{-1} is a continuous mapping. Notice that:

$$\langle f, \pi^{-1}\xi\rangle = \langle \xi, f\rangle.$$

And the mapping $\xi \mapsto \langle \xi, f\rangle$ is continuous on X^{**} for the w_G^* topology. Hence, B_X is w_G-compact.

For the reverse implication, we know by Theorem 3.4.15 that $\pi(B_{X_G}) = B_{X^{**}_{G^{**}}}$. As a consequence, $\pi(X_G) = X^{**}_{G^{**}}$. □

To conclude this section, we aim to emphasize the relations among X, X_G, X^{**}, $X^{**}_{G^{**}}$, and ${X^{**}}\big/{G^*}$. To illustrate these connections, we will demonstrate that the following commutative diagram holds:



where π denotes the canonical injection from X into X^{**}, π_G denotes the canonical injection of Definition 3.4.13, S_G denotes the symmetrization of the points in X, and S_{G^*} denotes the symmetrization of the points in X^{**}.

3.5 Differentiability

In Chapters 4 and 5 we will need differentiability notions. In Chapter 4 for exploring some consequences of the Ekeland's variational principle, and in Chapter 5 for studying strongly subdifferentiable mappings. We will present here the basic definitions and properties required in those chapters.

Let's start by recalling the definitions of Gâteaux and Fréchet differentiability. Let X be a Banach space, $U \subseteq X$ a nonempty and open set, and $F\colon U \to \mathbb{R}$. The Gâteaux variation of F at $u_0 \in U$ in the direction of h is the limit

$$\lim_{t\to 0} \frac{F(u_0 + th) - F(u_0)}{t}.$$

If it exists and is finite we denote this limit by $\delta F(u_0)(h)$. If the Gâteaux variation of F is linear and continuous, we say that F is Gâteaux differentiable.

We say that F is Fréchet differentiable at u_0 if there exists a linear and continuous mapping $L\colon X \to \mathbb{R}$ such that

$$\lim_{h\to 0} \frac{F(u_0+h) - F(u_0) - L(h)}{\|h\|}.$$

It is well known that if L satisfies the definition of Fréchet differentiable, then L is unique, and we will denote it by $F'(u_0)$. Our main result here establishes that the G-invariance of the function is preserved by the Gâteaux and Fréchet differentiability.

Property 3.5.1. *Let X be a Banach space, and G be a group acting on X. If $F\colon X \to \mathbb{R}$ is Gâteaux (Fréchet) differentiable at u_0 in the direction of h, u_0 is G-invariant and F is G-invariant, then $\delta F(u_0)(h)$ ($F'(u_0)$) is G-invariant.*

Proof. Fix $g \in G$. Applying first that u_0 is G-invariant and secondly that F is G-invariant we obtain the result:

$$\begin{aligned}
\delta F(u_0)(g(h)) &= \lim_{t\to 0} \frac{F(u_0 + tg(h)) - F(u_0)}{t} \\
&= \lim_{t\to 0} \frac{F(g(u_0 + th)) - F(u_0)}{t} \\
&= \lim_{t\to 0} \frac{F(u_0 + th) - F(u_0)}{t} = \delta F(u_0)(h).
\end{aligned}$$

The proof of $F'(u_0)$ being G-invariant is analogous. □

In the following example we are going to see that the reverse implication does not hold in general.

Example 3.5.2. In $\mathbb{R}$ let $G = \{Id, -Id\}$. Pick

$$\begin{array}{rccc} F\colon & \mathbb{R} & \rightarrow & \mathbb{R} \\ & x & \rightarrow & F(x) = \frac{x^3}{3}, \end{array}$$

this function is clearly not G-invariant, but its Gâteaux and Fréchet derivatives, $F'(x) = x^2$, are G-invariant.

Our next goal is to study the concept of G-invariance in the context of subdifferentiability. Let us start by recalling the definition of the right directional derivative of an operator.

For two Banach spaces X, Y, and a subset $U \subseteq X$, the right directional derivative of $f\colon U \rightarrow Y$ at $x_0 \in \mathrm{int}(U)$ is defined as

$$d^+ f(x_0)(x) = \lim_{t \rightarrow 0^+} \frac{f(x_0 + tx) - f(x_0)}{t}.$$

To continue, let us provide some sufficient conditions to ensure that the right directional derivative is group invariant.

Property 3.5.3. *Let X, Y be two Banach spaces, $U \subseteq X$ a subset, G a compact topological group acting on X, and $f\colon U \rightarrow Y$ a mapping. Let $x_0 \in int(U) \cap X_G$, and let f be G-invariant, then so is $d^+ f(x_0)(x)$.*

Proof. Let $g \in G$:

$$\begin{aligned} d^+ f(x_0)(g(x)) &= \lim_{t \rightarrow 0^+} \frac{f(x_0 + tg(x)) - f(x_0)}{t} \\ &= \lim_{t \rightarrow 0^+} \frac{f(g(x_0 + tx)) - f(x_0)}{t} \\ &= \lim_{t \rightarrow 0^+} \frac{f(x_0 + tx) - f(x_0)}{t} \\ &= d^+ f(x_0)(x), \end{aligned}$$

where we have used first the G-invariance of x_0 and then the G-invariance of f. □

Recall that for a function $f\colon X \to \mathbb{R} \cup \{+\infty\}$ proper, convex and lower semicontinuous, the ϵ-subdifferential and the subdifferential of f at x_0 are defined as

$$\partial_\epsilon f(x_0) = \{h \in X^* \mid \langle h, x - x_0\rangle \leqslant f(x) - f(x_0) + \epsilon \;\; \forall\, x \in X\}\,.$$

$$\partial f(x_0) = \{h \in X^* \mid \langle h, x - x_0\rangle \leqslant f(x) - f(x_0), \, \forall\, x \in X\}.$$

Also we say that $h \in X^*$ is a subgradient of f at $x_0 \in \mathrm{Dom}(f)$ if

$$\langle h, x - x_0\rangle \leqslant f(x) - f(x_0) \quad \forall\, x \in X.$$

Observe that the subdifferential and the ϵ-subdifferential both are subsets of X^*. So we will study when they are G^*-invariant.

Property 3.5.4. *Let X be a Banach space, G a compact topological group acting on X, $f\colon X \to \mathbb{R} \cup \{+\infty\}$ a proper function, and $x_0 \in Dom(f)$ a G-invariant point. Then, the ϵ-subdifferential and the subdifferential of f at x_0 are G^*-invariant.*

Proof. We are going to show the G-invariance of the $\partial f(x_0)$. Let $g^* \in G^*$, we want to see that if we choose $g^* \in G^*$, then $g^*(\partial f(x_0)) = \partial f(x_0)$. Observe that, by the definition of the adjoint of an operator, the linearity of g and the G-invariance of x_0 we have that

$$\langle g^* u, x - x_0\rangle = \langle u, g(x - x_0)\rangle = \langle u, g(x) - g(x_0)\rangle = \langle u, g(x) - x_0\rangle.$$

And now, since $u \in \partial f(x_0)$, and f is G-invariant it is clear that

$$\langle g^* u, x - x_0\rangle = \langle u, g(x) - x_0\rangle \leqslant f(g(x)) - f(x_0) = f(x) - f(x_0).$$

This means that $g^* u \in \partial f(x_0)$, so we deduce that $g^*(\partial f(x_0)) \subseteq \partial f(x_0)$. Using the inverse mapping of g, we deduce the other inclusion. Then, we have that $g^*(\partial f(x_0)) = \partial f(x_0)$ □

So now, we define the G-subdifferential

Definition 3.5.5. Let X be a Banach space, G a group acting on X, and let $f\colon X \to]-\infty, +\infty]$ be a proper function. For $x_0 \in \mathrm{Dom} f$ we define the G-subdifferential of f at x_0 as

$$\partial_G f(x_0) = \{h \in X_G^* \;\mid\; \langle h, x - x_0\rangle \leqslant f(x) - f(x_0),\, \forall\, x \in X\}.$$

In the case that G is compact, it is obvious that if f is convex with respect to the group G and continuous

$$\partial_G f(x_0) \subseteq \partial_G f(\overline{x_0}).$$

Notice also that, if f is linear with respect to the group G and continuous

$$\partial_G f(x_0) = \partial_G f(\overline{x_0}),$$

and if in addition x_0 is G invariant, then

$$\begin{aligned}\partial_G f(x_0) &= \{x^* \in X_G^* \mid \langle x^*, x - x_0\rangle \leqslant f(x) - f(x_0) \;\; \forall\, x \in X\} \\ &= \{x^* \in X_G^* \mid \langle x^*, \overline{x} - x_0\rangle \leqslant f(\overline{x}) - f(x_0) \;\; \forall\, x \in X\} \\ &= \{x^* \in X_G^* \mid \langle x^*, x - x_0\rangle \leqslant f(x) - f(x_0) \;\; \forall\, x \in X_G\}\,.\end{aligned}$$

Chapter 4

Group invariant variational principles and applications

In this chapter we study one of the most relevant results in optimization theory, Ekeland's variational principle (EVP). This result was proved by Ekeland in 1974, see [25]. His motivation was to guarantee the existence of approximate solutions to control theory problems. In particular condition 3 in Corollary 4.2.1 will help us in the absence of an exact minimum to find a sequence which almost minimize the function, and almost satisfy the first-order necessary conditions. Of course, Ekeland's variational principle has been very useful in other areas like optimization for its versatility for finding solutions, or approximate solutions, to PDE's problems. However, the potential of Ekeland's result was yet to be discovered, because not only it was useful in control and optimization theory, it was also very useful in geometry of Banach spaces. Since the publication of the Bishop-Phelps theorem, [6], many results in geometry of Banach spaces used the same geometric construction of generating a convex cone in the space, associate this cone to a partial ordering and use then a transfinite induction argument. Ekeland's variational principle was no exception to this. But, surprisingly Ekeland's result was a double-edged sword. On the one hand it was useful for giving alternative proofs to theorems such as Brønsted-Rockafellar, Bishop-Phelps theorem, Banach fixed point theorem, etc. And, on the other hand, it was the glue for finding equivalences between many results in geometry of Banach spaces.

It is worth pointing out that during the years, Ekeland's original proof, see [25], has been modified finding more concise proofs which do not require transfinite induction, see for instance [26], or [29] for a more geometric one. Also, since the result was so important, researchers tried to generalize it to a more general scenario. For example in 1976 Brézis and Browder extended Ekeland's result to a general principle on ordered sets, see [12], and they used this generalization in semigroup theory for finding upper bounds on semigroups. There were also other strengthenings of the original result as we can see in [40].

The goal of this chapter is to study both of these faces of the Ekeland's result but in the group invariant case. For this, in section 4.1 we are going to present Ekeland's result, in section 4.2 we will present some of its consequences, in sections 4.3 and 4.4 we will present some of its equivalences, and finally, in section 4.5 we will present some applications to geometry of Banach spaces and PDEs, all of this in the G-invariant setting.

Sections 4.1 and 4.2 have appeared in the published work

> [33] J. Falco, and D. Isert, Group invariant variational principles, *Revista de la real academia de ciencias exactas, físicas y naturales. Serie A. Matemáticas* **10**(3) (2024), pp. 443–474.

Sections 4.3, 4.4, and 4.5 have appeared in the work

> [35] J. Falco, and D. Isert, Variational principles and applications to symmetric PDEs, *arXiv preprint arXiv:2403.18467* (2024).

4.1 Ekeland variational principle for G-invariant functions

In this section we are going to focus on the generalization of the Ekeland variational principle to the context of G-invariant functionals, and on some consequences of this principle. On the proof of the Theorem 4.1.4 it is required the following group invariant Cantor set theorem. The proof of this result can be obtained directly from the Cantor set theorem, and a small observation. We include here the details, for the sake of completeness.

Proposition 4.1.1. *Let (M, d) be a complete metric space, and let G be a topological group acting on X. If $\{C_n\}_{n=1}^{+\infty}$ is a countable family of compact sets such that $C_n \neq \emptyset$, $C_{n+1} \subseteq C_n$ and C_n is G-invariant for every $n \in \mathbb{N}$, then*

$$\bigcap_{n=1}^{+\infty} C_n \quad \text{is nonempty and } G\text{-invariant.}$$

Proof. By the Cantor set theorem we know that $\cap_{n=1}^{+\infty} C_n \neq \emptyset$. It only remains to show that $\cap_{n=1}^{+\infty} C_n$ is G-invariant. For this, we want to show that

$$g\left(\bigcap_{n=1}^{+\infty} C_n\right) = \bigcap_{n=1}^{+\infty} C_n \quad \forall\, g \in G.$$

Since every C_n is G-invariant, we get that $g(C_n) \subseteq C_n$ for all $n \in \mathbb{N}$. Therefore, $g\left(\bigcap_{n=1}^{+\infty} C_n\right) \subseteq C_n$ for all $n \in \mathbb{N}$. Thus,

$$g\left(\bigcap_{n=1}^{+\infty} C_n\right) \subseteq \bigcap_{n=1}^{+\infty} C_n$$

Using g^{-1} we obtain the reverse inclusion, hence $\cap_{n=1}^{+\infty} C_n$ is G-invariant. □

As a direct consequence, we obtain that if the diameter of the sets converges to zero, then the intersection is a G-invariant point.

Corollary 4.1.2. *Let (M, d) be a complete metric space, and let G be a topological group acting on X. If $\{C_n\}_{n=1}^{+\infty}$ is a countable family of compact sets such that $C_n \neq \varnothing$, $C_{n+1} \subseteq C_n$, $\delta(C_n) \to 0$ and C_n is G-invariant for every $n \in \mathbb{N}$, then*

$$\bigcap_{n=1}^{+\infty} C_n = \{x\} \text{ is } G\text{-invariant},$$

where $\delta(C_n)$ denotes the diameter of the set C_n.

Before we present the Ekeland variational principle, let us recall some definitions that we will use.

Definition 4.1.3. Let X be a topological space. A function $f\colon X \to \mathbb{R}$ is lower semicontinuous if for every $\lambda \in \mathbb{R}$ the set

$$\{x \in X \mid f(x) \leqslant \lambda\}$$

is closed.

A function $f\colon X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is proper if it has nonempty domain, never takes on the value $-\infty$, and it is not identically equals to $+\infty$.

To continue we present the main result of this section, that is, the group invariant Ekeland's variational principle. The proof that we are giving here is not based on the proof given by Ekeland in [25] but on the proof given in [29, Theorem 7.39].

Theorem 4.1.4 (G-Ekeland's variational principle)**.** *Let X be a Banach space and G a compact topological group acting on X. Let $\varphi\colon X \to]0, +\infty]$ be proper, lower semicontinuous, bounded below, G-invariant,* *and convex with respect to the group G. Then, given $\epsilon > 0$ and $\delta > 0$, there exists a G-invariant point $\tilde{x} \in X$ such that*

$$\varphi(\tilde{x}) < \varphi(x) + \epsilon \|\overline{x} - \tilde{x}\| \quad \forall\, x \in X,\ x \neq \tilde{x}.$$

Moreover, if $x_0 \in X$ satisfies that $\varphi(x_0) < \inf\{\varphi(x) \mid x \in X\} + \delta$, then we can choose $\tilde{x}$ to be such that

$$\|\overline{x_0} - \tilde{x}\| < \frac{\delta}{\epsilon}.$$

Proof. Fix $\delta > 0$. We want to define a sequence $\{x_n\}_{n=1}^{+\infty} \subseteq X$ of G-invariant points that is convergent to our desired $\tilde{x}$.

Define $b := \inf\{\varphi(x) \mid x \in X\}$, and let us verify that

$$b = \inf\{\varphi(x) \mid x \in X\} = \inf\{\varphi(x) \mid x \in X \text{ is } G\text{-invariant}\}.$$

Since the set on the right-hand side of the equality is smaller than the set on the left-hand side, it is obvious that

$$\inf\{\varphi(x) \mid x \in X\} \leqslant \inf\{\varphi(x) \mid x \in X \text{ is } G\text{-invariant}\}.$$

For the other inequality choose $x \in X$, and consider $\overline{x}$, the G-symmetric point of x. Let's prove that $\varphi(\overline{x}) \leqslant \varphi(x)$. Using the definition of $\overline{x}$, the convexity of φ with respect to the group and that φ is G-invariant we have that

$$\varphi(\overline{x}) = \varphi\left(\int_G g(x)d\mu(g)\right) \leqslant \int_G \varphi(g(x))d\mu(g) = \int_G \varphi(x)d\mu(g) = \varphi(x).$$

Therefore, we obtain the desired equality:

$$\inf\{\varphi(x) \mid x \in X\} = \inf\{\varphi(x) \mid x \in X \text{ is } G\text{-invariant}\}.$$

Define $b_0 := b$. Since it is an infimum, we can choose a G-invariant point $x_0 \in X$ such that

$$\varphi(x_0) < b_0 + \frac{\delta}{2}.$$

If x_k is defined for $k = 0, \ldots, n$ and is G-invariant, we define

$$b_n = \inf\left\{t \in \mathbb{R} \mid \exists\, x \in X \ G\text{-invariant}\colon (x, t) \in \operatorname{epi} \varphi \cap ((x_n, \varphi(x_n)) + K_\epsilon)\right\},$$

and choose $x_{n+1} \in X$, G-invariant, satisfying the following properties

$$(x_{n+1}, \varphi(x_{n+1})) \in (x_n, \varphi(x_n)) + K_\epsilon, \tag{4.1.1}$$

$$\varphi(x_{n+1}) < b_n + \frac{\delta}{2^{n+2}}, \tag{4.1.2}$$

where we define

$$K_\epsilon = \{(x, t) \in X \times \mathbb{R} \mid t \leqslant -\epsilon \|x\|\}.$$

Observe that, since $\|\cdot\|$ is G-invariant, then so is K_ϵ. Moreover by (4.1.1) we get:

$$(x_{n+1} - x_n, \varphi(x_{n+1}) - \varphi(x_n)) \in K_\epsilon.$$

By definition of K_ϵ we know that

$$\varphi(x_{n+1}) - \varphi(x_n) \leqslant -\epsilon \|x_{n+1} - x_n\|,$$

so

$$\varphi(x_n) - \varphi(x_{n+1}) \geqslant \epsilon \|x_{n+1} - x_n\| \geqslant 0,$$

and then

$$\varphi(x_n) \geqslant \varphi(x_{n+1}) \quad \forall n \in \mathbb{N}.$$

Which means that $\{\varphi(x_n)\}_{n=0}^{+\infty}$ is non increasing.

Note that $\{(x_n, \varphi(x_n)) + K_\epsilon\}_{n=0}^{+\infty}$ is a sequence of nested sets and b_n is nondecreasing.

Now we will show that the sequence $\{x_n\}_{n=0}^{+\infty}$ is convergent to some point. By (4.1.1) and (4.1.2), we have that

$$b_{n-1} \leqslant b_n \leqslant \varphi(x_{n+1}) \leqslant \varphi(x_n) < b_{n-1} + \frac{\delta}{2^{n+1}} \quad \forall\, n \in \mathbb{N}.$$

Therefore

$$\varphi(x_n) - \varphi(x_{n-1}) < \frac{\delta}{2^{n+1}} \quad \forall\, n \in \mathbb{N}.$$

And from this we can obtain that

$$\epsilon \|x_n - x_{n+1}\| \leqslant \varphi(x_n) - \varphi(x_{n-1}) < \frac{\delta}{2^{n+1}} \quad \forall\, n \in \mathbb{N},$$

hence

$$\|x_n - x_{n+1}\| \leqslant \frac{\delta}{\epsilon 2^{n+1}} \quad \forall\, n \in \mathbb{N}.$$

Thus, we obtain that $\{x_n\}_{n=0}^{+\infty}$ is a Cauchy sequence, but since X is a Banach space, the sequence is convergent to some $\tilde{x} \in X$.

Finally, we are going to check that we can apply Corollary 4.1.2. We already know that $(x_n, \varphi(x_n)) + K_\epsilon$ are nested, and since φ is lower semicontinuous, it is clear that epi $\varphi \cap ((x_n, \varphi(x_n)) + K_\epsilon)$ is closed for every $n \in \mathbb{N}$. Let us estimate their diameter in $(X \times \mathbb{R}, \rho)$ where $\rho(x, y) = \|x\| + |t|$ is the metric defined in the product space. Fix $n \in \mathbb{N} \setminus \{0\}$, given $(x, t) \in (x_n, \varphi(x_n)) + K_\epsilon$, by the definition of K_ϵ we have that

$$\epsilon \|x - x_n\| \leqslant \varphi(x_n) - t,$$

and by the construction of the sequence

$$\varphi(x_n) - t \leqslant \varphi(x_n) - b_n < \frac{\delta}{2^{n+1}}.$$

Then
$$\|x - x_n\| \leqslant \frac{\delta}{\epsilon 2^{n+1}}.$$
This proves that $\operatorname{diam}(\text{epi } \varphi \cap ((x_n, \varphi(x_n)) + K_\epsilon)) \to 0$. Applying now Corollary 4.1.2 we get that

$$\text{epi } \varphi \bigcap \left(\bigcap_{n=1}^{+\infty} (x_n, \varphi(x_n)) + K_\epsilon \right) = \{(\tilde{x}, \varphi(\tilde{x}))\} \text{ and is } (G, Id)\text{-invariant.}$$

In particular we have that $\tilde{x}$ is G-invariant.

By construction we have that $\text{epi } \varphi \cap ((\tilde{x}, \varphi(\tilde{x})) + K_\epsilon) = (\tilde{x}, \varphi(\tilde{x}))$, and in particular

$$\varphi(\tilde{x}) < \varphi(x) + \epsilon\|x - \tilde{x}\| \quad \forall\, x \in X,\ x \neq \tilde{x}.$$

Moreover, we have that

$$\|x_0 - x_n\| \leqslant \|x_0 - x_1\| + \cdots + \|x_{n-1} - x_n\| \leqslant \frac{\delta}{\epsilon}\left(\frac{1}{2} + \cdots + \frac{1}{2^n}\right),$$

therefore $\|x_0 - \tilde{x}\| \leqslant \frac{\delta}{\epsilon}$. □

We are going now to give an example that shows that the convexity of the functional with respect to the group cannot be completely removed.

Example 4.1.5. Consider the group $G = \{Id, -Id\}$ acting on $\mathbb{R}$, then the function
$$f(x) = \begin{cases} 2x + 1 & \text{if } -\frac{1}{2} \leqslant x \leqslant 0, \\ -2x + 1 & \text{if } 0 \leqslant x \leqslant \frac{1}{2}, \\ 0 & \text{otherwise}, \end{cases}$$

is not convex with respect to the group G. Observe that the only G-invariant point in $\mathbb{R}$ is 0. Then, for given $\epsilon < 1$ and for $x = \frac{1}{2}$, we have

that
$$1 = f(0) \geqslant f\left(\frac{1}{2}\right) + \epsilon\frac{1}{2}.$$

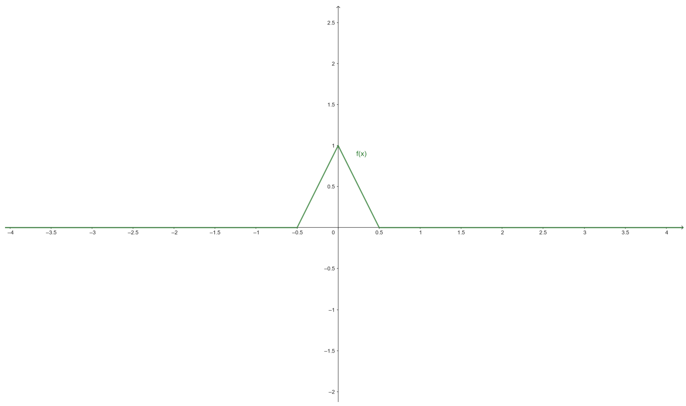


Figure 4.1. Non convex function with respect to the group G.

4.2 Consequences of Ekeland variational principle

The first direct consequence of the Ekeland variational principle is the Palais-Smale minimizing sequences. Let us continue by generalizing this result.

Corollary 4.2.1. *Let X be a Banach space and G be a compact topological group. Let $\varphi\colon X \to \mathbb{R}$ be Gâteaux differentiable, bounded below, G-invariant and convex with respect to the group. Then, there exists a sequence $\{x_n\}_{n=1}^{+\infty} \subseteq X$ such that*

(i) x_n *is G-invariant for all* $n \in \mathbb{N}$,

(ii) $\varphi(x_n) \to \inf\{\varphi(x) \mid x \in X\}$,

(iii) $\|\delta\varphi(x_n)\| \to 0$.

Proof. Choose $\epsilon = \delta = \frac{1}{n}$. By Theorem 4.1.4 we can find a sequence $\{x_n\}_{n=1}^{+\infty} \subseteq X$ of G-invariant points such that

$$\varphi(x_n) < \inf_{x\in X} \varphi(x) + \frac{1}{n},$$

and such that

$$\varphi(x_n) \leqslant \varphi(x) + \frac{1}{n}\|x - x_n\| \quad \forall\, x \in X.$$

Therefore

$$\varphi(x_n) - \varphi(x) \leqslant \frac{1}{n}\|x - x_n\| \quad \forall\, x \in X.$$

Now let's check that $\|\delta\varphi(x_n)\| \to 0$. We know that

$$\|\delta\varphi(x_n)\| = \sup_{y\in X, \|y\|\leqslant 1} |\delta\varphi(x_n)(y)|.$$

Fix $y \in B_X$. Defining $x = x_n + ty$, for $t > 0$, we have that

$$\varphi(x_n) - \varphi(x_n + ty) < \frac{1}{n}\|ty\|,$$

therefore

$$\frac{\varphi(x_n) - \varphi(x_n + ty)}{t} \leqslant \frac{1}{n}\|y\| \quad \forall\, y \in B_X.$$

Taking now limits when $t \to 0$ we get that

$$-\delta\varphi(x_n)(y) = \lim_{t\to 0} \frac{\varphi(x_n) - \varphi(x_n + ty)}{t} \leqslant \frac{\|y\|}{n} \quad \forall\, y \in B_X.$$

Then

$$\delta\varphi(x_n)(y) \geqslant -\frac{\|y\|}{n} \quad \forall\, y \in B_X.$$

Since we have the inequality for every y, considering $-y$ and using the linearity of the Gâteaux differential, we have the following inequality

$$\delta\varphi(x_n)(y) \leqslant \frac{\|y\|}{n} \quad \forall\, y \in B_X.$$

So

$$|\delta\varphi(x_n)(y)| \leqslant \frac{\|y\|}{n} \leqslant \frac{1}{n} \quad \forall\, y \in B_X.$$

Taking now limits when $n \to +\infty$ we obtain the desired result. □

Corollary 4.2.2. *Let X be a Banach space and G a compact topological group. Let $F\colon X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function, Gâteaux differentiable, bounded below, G-invariant, convex with respect to G and so that there exists constants $k, c > 0$ with*

$$F(x) \geqslant k\|x\| + c \quad \forall\, x \in X.$$

Then, the range of $\delta F(x)$ is dense on kB_G, where B_G is the closed unit ball in X_G^*.

Proof. Choose $u \in kB_G$. We want to see if there exists a sequence of G-invariant points $\{u_n\}_{n=1}^{+\infty} \subseteq X$ such that

$$\|\delta F(u_n) - u\| \leqslant \epsilon \quad \forall\, \epsilon > 0.$$

Define

$$\begin{aligned} H\colon\ X &\to \mathbb{R} \\ x &\mapsto F(x) - \langle u, x\rangle. \end{aligned}$$

It is easy to see that H is Gâteaux differentiable, G-invariant and convex with respect to G. Note that

$$\begin{aligned}\inf_{x\in X} H(x) &= \inf_{x\in X} F(x) - \langle u, x\rangle \\ &\geqslant \inf_{x\in X} k\|x\| + c - \langle u, x\rangle \\ &\geqslant \inf_{x\in X} k\|x\| + c - \|u\|\,\|x\| \\ &= \inf_{x\in X} (k - \|u\|)\,\|x\| + c \\ &\geqslant c > -\infty,\end{aligned}$$

where we have used the hypothesis on the first inequality, and that $u \in kB_G$ on the last inequality. Then, H is bounded below, and it satisfies the hypothesis of Corollary 4.2.1. Therefore there exists a sequence of G-invariant points $\{u_n\}_{n=1}^{+\infty}$ such that $\|\delta H(u_n)\| \leqslant \epsilon$, and in particular

$$\|\delta F(u_n) - u\| \leqslant \epsilon.$$

□

Corollary 4.2.3. *Let X be a Banach space and G a compact topological group. Let $F\colon X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function, Gâteaux differentiable, bounded below, G-invariant, convex with respect to G and suppose that there exists $\phi\colon \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty\}$ such that $\frac{\phi(t)}{t} \to +\infty$ when $t \to +\infty$ and*

$$F(x) \geqslant \phi(\|x\|) \quad \forall x \in X.$$

Then, the range of $\delta F(x)$ is dense on X_G^*.

Proof. It is clear that for all $k > 0$, there exists $c \in \mathbb{R}$ such that F satisfies that

$$F(x) \geqslant k\|x\| + c.$$

Then, $\delta F(x)$ is dense on every closed ball of X_G^* □

The following corollary is another consequence of Theorem 4.1.4. But, first let us introduce one definition.

Definition 4.2.4. Let X be a Banach space, we say that a function $\phi\colon X \to \mathbb{R}$ is a bump function on X if it has bounded nonempty support.

Corollary 4.2.5. *Let X be a Banach space and G a compact topological group acting on X. Let φ be a continuous function, Gâteaux differentiable, bump, G-invariant and linear with respect to G. Then*

$$X_G^* = \overline{Span}\left\{\partial\varphi(x) \;\mid\; x \in X_G\right\}.$$

Proof. Define $\psi\colon X \to \mathbb{R} \cup \{+\infty\}$ as follows

$$\psi(x) = \begin{cases} \frac{1}{\varphi(x)^2} & \text{if } \varphi(x) \neq 0, \\ +\infty & \text{if } \varphi(x) = 0. \end{cases}$$

Let $f \in X_G^*$, then it is clear that $\psi(x) - f(x)$ is lower semicontinuous and bounded below. Notice also that ψ is Gâteaux differentiable and by Property 3.1.15 is convex with respect to the group. Therefore, by Theorem 4.1.4, we have that there exists a G-invariant point $x_0 \in X$ such that for every $h \in X$ and $t > 0$ we have that

$$\psi(x_0 + th) - f(x_0 + th) \geqslant \psi(x_0) - f(x_0) - \epsilon t\|h\|.$$

Hence
$$\frac{\psi(x_0 + th) - \psi(x_0)}{t} \geqslant \frac{f(x_0 + th) - f(x_0)}{t} - \epsilon\|h\|,$$

and by linearity of f we obtain

$$\frac{\psi(x_0 + th) - \psi(x_0)}{t} \geqslant f(h) - \epsilon\|h\|.$$

Taking now limits when $t \to 0^+$ we finally have that

$$\partial\psi(x_0)(h) = \lim_{t\to 0^+} \frac{\psi(x_0 + th) - \psi(x_0)}{t} \geqslant f(h) - \epsilon\|h\| \quad \forall\, h \in X.$$

Considering $-h$ and using the linearity of f and $\partial\psi$, it follows that

$$\partial\psi(x_0)(h) \leqslant f(h) + \epsilon\|h\| \quad \forall\, h \in X.$$

Therefore

$$|\partial\psi(x_0)(h) - f(h)| \leqslant \epsilon\|h\| \quad \forall\, h \in X.$$

Finally, we can conclude that

$$\|\partial\psi(x_0) - f\| = \left\| -\frac{\partial\varphi(x_0)}{\varphi(x_0)^3} - f \right\| \leqslant \epsilon \quad \forall\, \epsilon > 0.$$

□

On 1981 Sullivan showed that the validity of Ekeland's variational principle on a metric space, say (M, d), is equivalent to its completeness, see [56]. As a consequence, one can show the same for Banach spaces by replacing the distance d by the norm of the space. Our next result is based on this characterization, and tells us exactly when the G-invariant normed spaces are indeed Banach spaces.

Theorem 4.2.6. *Let $(X, \|\cdot\|)$ be a normed space, and G a compact topological group acting on X. X_G is a Banach space if, and only if, for every $f\colon X \to \mathbb{R}$ that is bounded below, Lipschitz continuous, G-invariant, convex with respect to G and for all $\epsilon > 0$, there exists $x_0 \in X_G$ such that*

$$f(x_0) < \inf_X f + \epsilon,$$

and

$$f(x_0) \leqslant f(x) + \epsilon\|x - x_0\| \quad \forall\, x_0 \neq x \in X.$$

Proof. The direct implication is clear by applying Theorem 4.1.4 with $\delta = \epsilon^2$.

For the sufficient condition, fix a Cauchy sequence $\{x_n\}_{n=1}^{+\infty} \subseteq X_G$. Then for all $\epsilon < 1$ there exists $N_0 \in \mathbb{N}$ such that

$$\|x_n - x_m\| < \epsilon \quad \forall n > m \geqslant N_0.$$

Define now the functional

$$\begin{array}{rccc} f\colon & X & \to & \mathbb{R} \\ & x & \mapsto & \lim\limits_{n\to+\infty} \|x_n - \overline{x}\|. \end{array}$$

observe that it is non negative and satisfies that $\inf\{f(x) \mid x \in X\} = 0$. Let's see that it is well defined. We know that

$$|\, \|x_n - \overline{x}\| - \|x_m - \overline{x}\| \,| \leqslant \|x_n - x_m\| < \epsilon \quad \forall\, n > m > N_0,$$

so $\{\|x_n - \overline{x}\|\}_{n=1}^{+\infty} \subseteq \mathbb{R}$ is a Cauchy sequence, hence it is convergent and then f is well defined. Now we are going to see that f is Lipschitz-continuous. Indeed, given $x, y \in X$ and $n \in \mathbb{N}$ we know that

$$|\, \|x_n - \overline{x}\| - \|x_n - \overline{y}\| \,| \leqslant \|\overline{x} - \overline{y}\| = \left\| \int_G (g(x) - g(y)) d\mu(g) \right\| \leqslant$$

$$\leqslant \int_G \|g(x) - g(y)\| d\mu(g) = \int_G \|x - y\| d\mu(g) = \|x - y\|,$$

where we have used here the definition of the G-symmetrization and the fact that the norm is G-invariant. Then taking limits when $n \to +\infty$, we deduce that

$$|f(x) - f(y)| \leqslant \|x - y\| \quad \forall x, y \in X,$$

i.e., f is 1-Lipschitz-continuous. Finally, let's see that f is G-invariant and convex with respect to G. Pick $g \in G$, then

$$f(g(x)) = \lim_{n\to+\infty} \|x_n - g(\overline{x})\| = \lim_{n\to+\infty} \|x_n - \overline{x}\| = f(x),$$

so f is G-invariant.

Notice that

$$f(\overline{x}) = \lim_{n\to+\infty} \|x_n - \overline{\overline{x}}\| = \lim_{n\to+\infty} \|x_n - \overline{x}\| = f(x),$$

where we have used that the sequence is made of G-invariant points, and the G-invariance of the norm. Moreover, since $\{x_n\}_{n=1}^{+\infty}$ is a Cauchy sequence, we have that

$$f(x_m) = \lim_{n\to+\infty} \|x_n - x_m\| < \epsilon \quad \forall\, m \geqslant N_0.$$

Now, we can apply our hypothesis to obtain a G-invariant point, say $x_0 \in X$, such that

1. $f(x_0) \leqslant \epsilon$,
2. $f(x_0) \leqslant f(x) + \epsilon\|x - x_0\|$ for all $x_0 \neq x \in X$.

By using 2 with $x = x_m$, we have that

$$f(x_0) \leqslant f(x_m) + \epsilon\|x_m - x_0\| = f(x_m) + \epsilon\|x_m - \overline{x_0}\| \quad \text{for every } m \in \mathbb{N} \tag{4.2.1}$$

Notice now that

$$f(x_m) \to 0 \text{ when } m \to +\infty,$$

hence

$$\|x_m - \overline{x_0}\| \to \lim_{m\to+\infty} \|x_m - \overline{x_0}\| = f(x_0).$$

So, taking limits over m in (4.2.1) we have that

$$f(x_0) \leqslant \epsilon f(x_0),$$

which means that $(1-\epsilon)f(x_0) \leqslant 0$, but from the start we know that $\epsilon < 1$ and that $\inf_{x\in X} f(x) = 0$, hence, $f(x_0) = 0$, which means that $\|x_n - \overline{x_0}\| \to 0$ when $n \to +\infty$. And finally it is clear now, since x_0 is G-invariant, that $x_0 = \overline{x_0}$, $x_n \to x_0$ when $n \to +\infty$ in X_G. □

Remark 4.2.7. Note that the fact that X_G is a Banach space does not guarantee that X is a Banach space. Indeed, if we consider for instance the space c_{00} consisting of all real sequences of finite support and the group of permutations of the coordinates bigger than N an arbitrary but fixed natural number, it is clear that G has a natural action on X by

$$\varphi(x_1, x_2, \ldots, x_N, x_{N+1}, \ldots) = (x_1, \ldots, x_n, x_{\varphi(N)}, x_{\varphi(N+1)}, \ldots),$$

for every $\varphi \in G$ and every $(x_1, x_2, \ldots) \in c_{00}$. Then c_{00} is not a Banach space, but $(c_{00})_G = \{(x) \in c_{00} \mid x_n = 0, \ \forall n \geqslant N\}$ is clearly a Banach space.

4.2.1 Bishop-Phelps theorem

Here we are going to present a proof of the group invariant Bishop-Phelps theorem, by using the group invariant Ekeland's variational principle. This result was first proved in [30] using the original ideas of Bishop and Phelps. These alternative proof presented here is based on the proof of the Bishop-Phelps theorem presented in [51, Chapter 3].

Theorem 4.2.8 (G-Bishop-Phelps)**.** *Let X be a real Banach space and G a compact topological group acting on X. If $C \subseteq X$ is a convex, closed, bounded and G-invariant subset, then the norm-attaining functionals in C that are G-invariant are dense in X_G^*.*

Proof. Choose $f \in X_G^*$, $\epsilon > 0$, and define $\tilde{f}\colon X \to \mathbb{R} \cup \{+\infty\}$

$$\tilde{f}(x) = \begin{cases} -f(x) & \text{if } x \in C, \\ +\infty & \text{if } x \in X \backslash C. \end{cases}$$

Notice that $\tilde{f}$ is proper, lower semicontinuous (because f is continuous), and bounded below. We have to show that $\tilde{f}$ is G-invariant. Indeed, when $x \in C$ we can apply the G-invariance of f to obtain that

$$\tilde{f}(g(x)) = -f(g(x)) = -f(x) = \tilde{f}(x) \quad \forall\, g \in G,$$

and when $x \in X \backslash C$, we have that $g(x) \in X \backslash C$, by the G-invariance of C, hence

$$\tilde{f}(x) = +\infty = \tilde{f}(g(x)).$$

Let us continue by showing that $\tilde{f}$ is convex with respect to G. Indeed, for $x \in C$, we know that $\overline{x} \in C$ by the convexity of C, therefore, by the linearity and continuity of f, we can use Proposition 3.1.8, and by G-invariance of f we obtain that

$$\tilde{f}(\overline{x}) = -f(\overline{x}) = -f\left(\int_G g(x)d\mu(g)\right) = -f(x) = \tilde{f}(x).$$

Now, fix $x \in X \backslash C$. Then

$$\tilde{f}(\overline{x}) \leqslant +\infty = \tilde{f}(x).$$

Hence, $\tilde{f}$ is convex with respect to G.

Applying now Theorem 4.1.4 we obtain a G-invariant point $x_0 \in C$ such that

$$\tilde{f}(x_0) \leqslant \tilde{f}(x) + \epsilon\|x - x_0\| \quad \forall\, x \in X.$$

By definition of $\tilde{f}$ we have that

$$f(x_0) + \epsilon\|x - x_0\| \geqslant f(x) \quad \forall\, x \in C.$$

Define the sets

$$K_1 = \{(x,t) \;\mid\; x \in C,\, t \leqslant f(x)\} \subseteq X \times \mathbb{R},$$

$$K_2 = \{(x,t) \;\mid\; t \geqslant f(x_0) + \epsilon\|x - x_0\|\}\,,$$

and observe that both of them are (G, Id)-invariant since the first one is the epigraph of the function f, and the second one is G-invariant by the G-invariance of the norm and x_0. Moreover, the interior of K_2 is nonempty and disjoint from K_1. Therefore, by Theorem 3.3.4, there exists a (G, Id)-invariant functional in $X^* \times \mathbb{R}$ that separates K_1 and K_2, i.e., there exists $h \in X^* \backslash \{0\}$ and $\beta \in \mathbb{R}$ such that

$$\langle h, x\rangle + \alpha t \leqslant \beta \quad \text{if } (x,t) \in K_1,$$

and

$$\langle h, x\rangle + \alpha t \geqslant \beta \quad \text{if } (x,t) \in K_2.$$

Observe that α cannot be negative, otherwise $\langle h, x\rangle + \alpha t < \beta$ if $(x,t) \in K_2$ and t is large enough. Similarly, α cannot be equal to 0, otherwise $\langle h, x\rangle \geqslant \beta$ for all $x \in X$ that meaning that $h = 0$. Thus, $\alpha > 0$, and we can normalize it to be $\alpha = 1$.

Notice now that, since $(x_0, f(x_0)) \in K_1 \cap K_2$ we have that $\langle h, x_0\rangle + f(x_0) = \beta$. Therefore

$$\langle h, x\rangle + f(x) \leqslant \langle h, x_0\rangle + f(x_0) \quad \forall\, x \in C.$$

From where we deduce that $h + f$ attains its maximum on C at x_0. On the other hand, if $x \in X$ and $t = f(x_0) + \epsilon\|x - x_0\|$, then

$$\langle h, x_0\rangle + f(x_0) = \beta \leqslant \langle h, x\rangle + f(x_0) + \epsilon\|x - x_0\| \quad \forall\, x \in X,$$

since by definition $(x, t) \in K_2$. Hence

$$\langle h, x_0 - x\rangle \leqslant \epsilon\|x - x_0\|,$$

so

$$|\langle h, z\rangle| \leqslant \epsilon\|z\| \quad \forall\, z \in X,$$

and then $\|h\| \leqslant \epsilon$.

Defining now $\tilde{h} = f + h$ is clear that $\tilde{h}$ attains its maximum on C, is G-invariant and

$$\|\tilde{h} - f\| \leqslant \epsilon.$$

□

Remark 4.2.9. The previous result in the complex case holds if C is the unit ball.

A closer look at the previous proof shows that, indeed, the following equivalence of the Bishop-Phelps theorem holds.

Corollary 4.2.10. *Let X be a real Banach space, G a compact topological group acting on X, and C a nonempty, closed, convex and G-invariant subset. For $\epsilon > 0$, suppose that $f\colon C \to \mathbb{R}$ is lower semicontinuous, convex, bounded below and G-invariant. Then, there exists $h \in X_G^*$ such that $\|h\| \leqslant \epsilon$, and $f + h$ attains its maximum at some G-invariant point $x_0 \in C$.*

4.2.2 Brønsted-Rockafellar theorem

In this section we want to give a group invariant version of the Brønsted-Rockafellar theorem, see [51, Theorem 3.17]. But first, let us present, without proof, two results that will be crucial for the proof of the Brønsted-Rockafellar theorem. The first one is the so called Moreau-Rockafellar theorem, which says, roughly-speaking, that the sum of the subdifferentials is the subdifferential of the sum. For the proof of this result one can see [51, Theorem 3.16].

Theorem 4.2.11 (Moreau-Rockafellar)**.** *Let X be a Banach space and let $f, g\colon X \to \mathbb{R} \cup \{+\infty\}$ be two proper, convex and lower semiconitnuous functions. Suppose that $Dom(f) \cap Dom(g) \neq \emptyset$, then*

$$\partial f(x) + \partial g(x) \subseteq \partial(f+g)(x) \quad \forall\, x \in Dom(f+g).$$

Moreover, if there exists one point in $Dom(f) \cap Dom(g)$ for which one of the two functions is continuous, then we have that

$$\partial f(x) + \partial g(x) = \partial(f+g)(x) \quad \forall\, x \in Dom(f+g).$$

And as an easy Corollary of this result, we have the following.

Corollary 4.2.12. *Let X be a Banach space, $f\colon X \to \mathbb{R} \cup \{+\infty\}$ be proper, convex and lower semiconitnuous, and $h \in X^*$. For $x_0 \in Dom(f)$:*

$$\partial f(x_0) + h = \partial(f+h)(x_0).$$

With this brief introduction we can move now to see the G-invariant version of the Brønsted-Rockafellar theorem.

Theorem 4.2.13 (G-Brønsted-Rockafellar)**.** *Let X be a Banach space and G a compact topological group acting on X. Suppose that $f\colon X \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex, lower semicontinuous and G-invariant* *functional. Then for any G-invariant point x_0 and any functional x_0^*, with $x_0 \in Dom(f)$ and $x_0^* \in \partial_\epsilon f(x_0)$, and for all $\epsilon, \lambda > 0$, there exists a G-invariant point $z \in Dom(f)$ and a functional $x^* \in X_G^*$ such that*

$$x^* \in \partial f(x), \quad \|z - x_0\| \leqslant \frac{\epsilon}{\lambda}, \quad \|x^* - x_0^*\| \leqslant \lambda.$$

Proof. Fix a G-invariant point $x_0 \in \text{Dom}(f)$, $\epsilon > 0$ and $\lambda > 0$. Then, we know that $\partial_\epsilon f(x_0)$ is nonempty and G^*-invariant. Consider a G-invariant functional $x_0^* \in \partial_\epsilon f(x_0)$ and define

$$\begin{array}{rccc} \phi\colon & X & \to & \mathbb{R} \cup \{+\infty\} \\ & x & \mapsto & f(x) - \langle x_0^*, x \rangle. \end{array}$$

It is clear that ϕ is proper, and that $\text{Dom}(\phi) = \text{Dom}(f)$. Moreover ϕ is lower semicontinuous, since it is a composition of a lower semicontinuous function with a continuous one. We want to see now that ϕ is bounded below. Indeed, for all $\epsilon > 0$, since $x_0^* \in \partial_\epsilon f(x_0)$ we have that

$$\langle x_0^*, x - x_0 \rangle \leqslant f(x) - f(x_0) + \epsilon \quad \forall\, x \in X.$$

Therefore

$$f(x_0) - \langle x_0^*, x_0 \rangle \leqslant f(x) - \langle x_0^*, x \rangle + \epsilon \quad \forall\, \epsilon > 0,\ x \in X,$$

which means that

$$\phi(x_0) \leqslant \phi(x) + \epsilon \quad \forall\, \epsilon > 0,\ x \in X.$$

From here we deduce that ϕ is bounded below and in particular

$$\phi(x_0) \leqslant \inf_{x \in X} \phi(x) + \epsilon.$$

It is also clear that ϕ is G-invariant since it is a composition of G-invariant functions, and the only thing that remains to check is that ϕ is convex with respect to G. We want to see that

$$\phi(\overline{x}) \leqslant \int_G \phi(g(x))d\mu(g) = \phi(x),$$

which is equivalent, by definition of ϕ, to

$$f\left(\int_G g(x)d\mu(g)\right) - \left\langle x_0^*, \int_G g(x)d\mu(g)\right\rangle \leqslant \int_G f(g(x)) - \langle x_0^*, g(x)\rangle d\mu(g).$$

But this is clear, since f is convex with respect to G and G-invariant, and x_0^* is linear and G-invariant.

So, we can apply Theorem 4.1.4 to obtain that there exists a point $z \in \text{Dom}(\phi)$ such that

1. z is G-invariant,
2. $\phi(z) < \phi(x) + \lambda\|x - z\|$, for all $z \neq x \in X$,
3. $\|z - x_0\| \leqslant \frac{\epsilon}{\lambda}$.

By 3, it is clear that $\|z - x_0\| \leqslant \frac{\epsilon}{\lambda}$, so we only need to show that there exists a functional $x^* \in \partial f(z)$, $\|x^* - x_0^*\| \leqslant \lambda$. We define now the function

$$\begin{array}{rccc} h\colon & X & \to & \mathbb{R} \\ & x & \mapsto & \lambda\|x - z\|, \end{array}$$

which is proper, convex, lower semicontinuous and satisfies that $h(z) = 0$. Observe that by (2) we have that

$$(\phi + h)(x) - (\phi + h)(z) > 0 \quad \forall\, x \in X,$$

in particular $0 \in \partial(\phi+h)(z)$. But we know that h has a point of continuity in $\text{Dom}(h) \cap \text{Dom}(\phi) = \text{Dom}(\phi)$, so applying now Theorem 4.2.11 we have that

$$\partial(\phi + h)(z) = \partial\phi(z) + \partial h(z).$$

Hence, $0 \in \partial\phi(z) + \partial h(z)$, therefore there exists $-z^* \in \partial\phi(z)$ such that $z^* \in \partial h(z)$, note that the symmetrized point $\overline{z^*}$ belongs to $\partial h(z)$ because the subdifferential is a convex set. Moreover, we know by Corollary 4.2.12 that $\partial\phi(z) = \partial f(z) + x_0^*$, so there exists $x^* \in \partial f(z)$ such that $x^* = -\overline{z^*} + x_0^*$. Notice that x^* is G-invariant because so are x_0^* and $\overline{z^*}$, and notice also that

$$\begin{aligned}\partial h(z) &= \{z^* \in X^* \mid \langle z^*, x - z\rangle \leqslant h(x) - h(z) \quad \forall\, x \in X\} \\ &= \{z^* \in X^* \mid \langle z^*, x - z\rangle \leqslant \lambda\|x - z\| \quad \forall\, x \in X\} \\ &= \{z^* \in X^* \mid \|z^*\| \leqslant \lambda\}\,.\end{aligned}$$

Therefore

$$\|x^* - x_0^*\| = \|\overline{-z^*}\| = \|\overline{z^*}\| \leqslant \lambda.$$

□

4.3 Drop theorem, Petal theorem, and EVP

As we said before, the Ekeland's variational principle has numerous applications and equivalences in functional analysis, and in particular in geometry of Banach spaces. This latter application of the EVP is relevant in many ways, since Ekeland's statement is purely analytic. During this section we are going to see the role that plays the EVP in order to obtain equivalences between geometric results such as the Drop theorem and the Petal theorem. Most of the results to which the EVP is equivalent have their own proof, one interesting fact here is that the Petal theorem is presented for the first time in [50, Theorem F] and it has, to our knowledge, no independent proof. The drop theorem, on the other hand, was showed independently by Danes in [17]. This result of Danes proved to be an important result in geometry of Banach spaces, and after its publication, there were strengthenings of the original statement. For example Guillerme in [43] showed, under some additional assumptions, that the drop theorem also holds in the nonconvex case, and when the set that we are working with is convex and unbounded. And in view of the importance of the drop theorem, Rolewicz in [54] introduced the spaces that have the drop property and started to study them.

For more equivalences of EVP to geometric statements we refer the reader to [18] and [44, Section 11.7]

The purpose of this section is to present a proof of the equivalence of the group invariant version of the Drop theorem (Theorem 4.3.10), the Petal theorem (Theorem 4.3.9), and the Ekeland's variational principle (Theorem 4.3.1). To prove these equivalences, we are going to use the following version of the Ekeland's variational principle, based on [33, Theorem 12].

Observe that [33, Theorem 12] is only stated for normed spaces, but a slight revision of the proof shows us the following metric version of the Theorem.

Theorem 4.3.1. *Let (M, d) be a complete metric space and $G \subseteq \mathcal{L}(M)$ be a compact topological group of isometries acting on M. Suppose d is G-invariant, and let $f\colon M \to \mathbb{R} \cup \{+\infty\}$ be a mapping that is lower semiconinuous, bounded below, proper, G-invariant and convex with respect to G. Then for all $\gamma > 0$ and every G-invariant point $x_0 \in M$, there exists a G-invariant point $a \in M$ such that*

$$f(a) < f(x) + \gamma d(x, a) \quad \forall\, x \in M,\ x \neq a, \tag{4.3.1}$$

$$f(a) \leqslant f(x_0) - \gamma d(a, x_0). \tag{4.3.2}$$

Proof. Consider the subspace $S = \{x \in M \mid f(x) + \gamma d(x, x_0) \leqslant f(x_0)\}$, which is closed, and in [33, Theorem 12] take $f_{|S}$ which satisfies all the conditions of the Theorem. Then, we know that there exists a G-invariant point $a \in S$ such that

$$f(a) < f(x) + \gamma d(a, x) \quad \forall\, x \in M,\ x \neq a.$$

But, since $a \in S$ we have that

$$f(a) \leqslant f(x_0) - \gamma d(a, x_0).$$

□

Let us present here these two objects, and study some interesting G-invariant properties related to them. Let (M, d) be a metric space, $a, b \in M$ and $\gamma > 0$, the petal of a and b is defined as

$$P_\gamma(a, b) = \{x \in M \mid \gamma d(a, x) + d(b, x) \leqslant d(a, b)\}\,.$$

Let $(X, \|\cdot\|)$ be a normed vector space, $a \in X$ and $B \subset X$ be a convex subset. The drop of a and B is defined as

$$D(a, B) = \{a + t(b - a) \mid t \in [0, 1],\, b \in B\}\,.$$

Let us observe the following properties of the drop and the petal when we consider a group G. The proof of these results is a direct consequence of the G-invariance of the distance, but we include them here for the sake of completeness.

Proposition 4.3.2. *Let (M, d) be a metric space and $G \subseteq \mathcal{L}(M)$ be a topological group of isometries acting on M. Let $\gamma > 0$, then*

$$g(P_\gamma(a, b)) = P_\gamma(g(a), g(b)) \quad \forall\, a, b \in M,\ \text{and}\ \forall\, g \in G.$$

Proof. Let $x \in P_\gamma(a,b)$. Observe that

$$d(g(a), g(b)) \geqslant \gamma d(g(a), g(x)) + d(g(x), g(b)) \quad \forall\, g \in G.$$

Since G is compact, and d is G-invariant, this is equivalent to

$$d(g^{-1}(g(a)), g^{-1}(g(b))) \geqslant$$

$$\geqslant \gamma d(g^{-1}(g(a)), g^{-1}(g(x))) + d(g^{-1}(g(x)), g^{-1}(g(b))).$$

But observe that this is the same as

$$d(a,b) \geqslant \gamma d(a,x) + d(x,b),$$

so $g(x) \in P_\gamma(g(a), g(b))$. This shows that $g(P_\gamma(a,b)) \subseteq P_\gamma(g(a), g(b))$.

Take now $x \in P_\gamma(g(a), g(b))$, we want to see that $g^{-1}(x) \in P_\gamma(a,b)$. Notice that

$$d(a,b) \geqslant \gamma d(a, g^{-1}(x)) + d(b, g^{-1}(x)),$$

if, and only if,

$$d(g(a), g(b)) \geqslant \gamma d(g(a), g(g^{-1}(x))) + d(g(b), g(g^{-1}(x))),$$

by the G-invariance of d. But this last inequality reads as follows:

$$d(g(a), g(b)) \geqslant \gamma d(g(a), x) + d(g(b), x).$$

This shows the other inclusion and concludes the proof. □

Proposition 4.3.3. *Let $(X, \|\cdot\|)$ be a normed vector space, $G \subseteq \mathcal{L}(X)$ be a topological group of isometries acting on X, $a \in X$ and $B \subset X$ be a convex subset. Then:*

$$g(D(a,B)) = D(g(a), g(B)).$$

Proof. Let $x \in D(x_0, B) = \text{conv}(x_0, B)$. Then, for some $b \in B$, and $t > 0$:

$$x = tx_0 + (1-t)b.$$

Now, by the linearity of g we know that

$$g(x) = tg(x_0) + (1-t)g(b) \in \text{conv}(g(x_0), g(B)) = D(g(x_0), g(B)).$$

This shows that $g(D(a, B)) \subseteq D(g(a), g(B))$.

Now take $x \in D(g(x_0), g(B)) = \text{conv}(g(x_0), g(B))$, then for some $z \in g(B)$, and for $t > 0$ it is clear that

$$x = tg(x_0) + (1-t)z.$$

Since $z \in g(B)$, there exists $b \in B$ such that $z = g(b)$. Then, by linearity of g,

$$x = g(tx_0 + (1-t)b) \in g(\text{conv}(x_0, B)) = g(D(x_0, B)).$$

This concludes the proof. □

The following G-invariant properties answer some of the natural questions that rise in our scenario of G-invariant mappings. The proof of this results is straightforward.

Property 4.3.4. *Let (M, d) be a metric space and $G \subseteq \mathcal{L}(M)$ be a topological group of isometries acting on M. Then, $a, b \in M$ are G-invariant if and only if $P_\gamma(a, b)$ is G-invariant for every $\gamma > 0$.*

Proposition 4.3.5. *Let $(X, \|\cdot\|)$ be a metric space, $G \subseteq \mathcal{L}(X)$ be a topological group of isometries acting on X, and $A, B \subseteq X$ be two G-invariant subsets. Then conv(A, B) is G-invariant.*

As a consequence of this result, we obtain the following.

Corollary 4.3.6. *Let $(X, \|\cdot\|)$ be a normed space and $G \subseteq \mathcal{L}(X)$ be a topological group of isometries acting on X. Assume that $x_0 \in X$, $B \subset X$ is a convex subset, and that both are G-invariant. Then $D(x_0, B)$ is G-invariant.*

We are going to show now that the previous conditions are indeed required. Through the following examples we will be working with the group $G = \{Id, \sigma\} \subseteq \mathbb{R}^2$, where $\sigma\colon \mathbb{R}^2 \to \mathbb{R}^2$ is defined as $\sigma(x, y) = (y, x)$.

Example 4.3.7. In $\mathbb{R}^2$ take $b = (0, 0)$, which is clearly G-invariant, and $a = (2, 0)$, which is not G-invariant. Then, the petal $P_\gamma(a, b)$ is not G-invariant. Also, take $B = B((0, 3), 1)$ the open ball centered at $(0, 3)$ with radius 1, which is clearly not G-invariant. Then, the drop $D(b, B)$ is not G-invariant.

However if we consider a non G-invariant point a, we cannot say anything about the drop.

Example 4.3.8. In $\mathbb{R}^2$ suppose $x = (0, 1)$ which is not G-invariant, and take $B = B((0, 0), 2)$ the ball centered at $(0, 0)$ with radius 2, which is clearly G-invariant, by G-invariance of the norm. Then $D(x_0, B) = B = B((0, 0), 2)$ which is G-invariant as before. Now, suppose $x = (5, 0)$ which is not G-invariant, and take $B = B((0, 0), 2)$ the ball centered at $(0, 0)$ with radius 2, which is clearly G-invariant. Since $g(x_0) = (0, 5)$, we can see that $x_0 \in D(x_0, B)$, but $x_0 \notin D(g(x_0), B)$. So, $D(x_0, B)$ is not G-invariant.

To continue, let us present the G-invariant version of the Petal theorem and Drop theorem.

Theorem 4.3.9 (G-invariant Petal theorem)**.** *Let (M, d) be a metric space, $G \subseteq \mathcal{L}(M)$ be a compact topological group of isometries acting on M, and $C \subset M$ a complete G-invariant subset. Assume that d is G-invariant and convex with respect to G. Let $x_0 \in C_G$, $b \in (M\backslash C)_G$,*

$r \leqslant d(b, C)$ and $s = d(b, x_0)$. Then, for all $\gamma > 0$, there exists a G-invariant point $a \in C \cap P_\gamma(x_0, b)$ such that $C \cap P_\gamma(a, b) = \{a\}$.

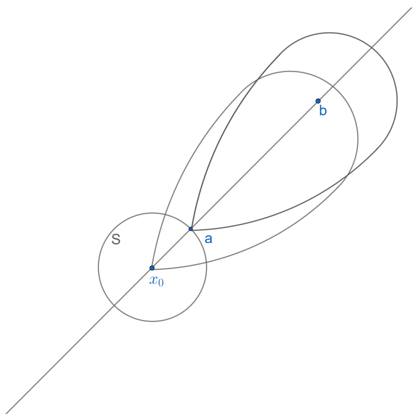


Figure 4.2. Statement of the Petal's Theorem

Theorem 4.3.10 (G-invariant Drop theorem)**.** *Let $(X, \|\cdot\|)$ be a normed space, $G \subseteq \mathcal{L}(X)$ be a compact topological group of isometries acting on X, and $C \subset X$ a complete G-invariant subset. Assume that $x_0 \in C_G$, and $B = \overline{B(b, r)}$, where $b \in X_G$ and $r < d(b, C)$. Then there exists a G-invariant point $a \in C \cap D(x_0, B)$ such that $C \cap D(a, B) = \{a\}$.*

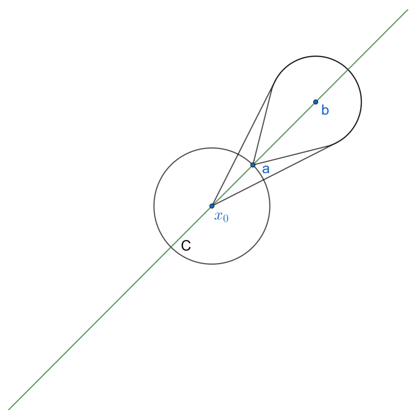


Figure 4.3. Statement of the Drop's Theorem

Let us now present the main goal of this section, that is, to show that the previous three theorems are equivalent.

Theorem 4.3.1 $\Rightarrow$ *Theorem 4.3.9.* Define the function

$$\begin{array}{rccc} f\colon & M & \to & \mathbb{R} \\ & x & \mapsto & d(x,b), \end{array}$$

which is continuous, bounded below by r and G-invariant. Note that since d is convex with respect to G, so is f.

Applying now Ekeland's variational principle we know that there exists $a \in S_G$ such that

$$f(a) < f(x) + \gamma d(a,x) \quad \forall\, x \in M,\ x \neq a \tag{4.3.3}$$

$$f(a) < f(x_0) - \gamma d(a,x_0). \tag{4.3.4}$$

By (4.3.3) we know that for every $x \in M\setminus\{a\}$ we have that $x \notin P_\gamma(a,b)$, this meaning that

$$M \cap P_\gamma(a,b) = \{a\}\,.$$

This concludes the proof. Observe also, that by (4.3.4),

$$d(a,b) < d(x_0,b) - \gamma d(x_0,a),$$

hence

$$\gamma d(x_0,a) < -d(a,b) + d(x_0,b) = s - d(a,b).$$

By hypothesis we know that $r < d(b,M) < d(b,a)$, thus

$$d(x_0,a) < \frac{s-r}{\gamma}.$$

□

Theorem 4.3.9 ⇒ *Theorem 4.3.10.* Let $B = B(b,r)$, and consider $X = C \cap D(x_0;B)$ which is a complete and G-invariant subspace. Define $d = d(b,C)$ and $\gamma = \frac{d-r}{d+r}$. By the Petal's theorem there exists a G-invariant point, say a, such that $\{a\} = X \cap P_\gamma(a,b)$.

Observe now that, since $t = d(a,b) \geqslant d > r$, it is clear that

$$\frac{d-r}{d+r} \leqslant \frac{t-r}{t+r},$$

so $D(a,B) \subseteq P_\gamma(a,b)$ for $t > r$. Moreover, since $a \in D(x_0,B)$, then $D(a,B) \subseteq D(x_0,B)$. Therefore

$$D(a,B) \cap C \subseteq D(a,B) \cap (D(x_0,B) \cap C) \subseteq P_\gamma(a,b) \cap X = \{a\}.$$

□

Remark 4.3.11. Note that to prove the last implication, it is not required the condition of d being convex with respect to G.

In order to prove the last implication we will use the following lemma, whose proof can be found in [50, Lemma 2.3].

Lemma 4.3.12. *Let X be a normed vector space, $B = \overline{B((0,h),r)} \subseteq X \times \mathbb{R}$ with radius $r \in]0,h[$ and the norm $\|(x,r)\| = \max(\|x\|, r)$. Then, the cone $K = \mathbb{R}_+ B$ generated by B is given by*

$$K = \left\{(x,t) \in X \times \mathbb{R} \mid t \geqslant r^{-1}(h-r)\|x\|\right\}.$$

Now we can move to the proof of the last implication.

Theorem 4.3.10 $\Rightarrow$ Theorem 4.3.1. We start replacing d by $d' = \min(\delta, d)$ where $\delta = \frac{1}{\gamma}(f(x_0) - \inf f(M) + 1)$. Observe that the two conditions of Ekeland's variational principle will still hold if we consider the distance d' instead of the distance d. Let F be the normed vector space of all continuous functions on M with the supremum norm. Then, (M,d) can be isometrically embedded in F via the mapping

$$\begin{array}{ccc} (M,d) & \rightarrow & F \\ x & \mapsto & d_x(y) = d(x,y), \end{array}$$

and $M \subseteq F$ is complete. Define $E = F \times \mathbb{R}$ with the norm $\|(x,t)\| = \max(\|x\|, |t|)$.

Without loss of generality we may assume that $x_0 = 0$ and $f(x_0) = 0$. If this were not the case, we could achieve this by translating via the mapping $x \mapsto f(x) - f(x_0)$. We define now $\psi = -f$ which is G-invariant, and observe that $\psi(x_0) = f(x_0) = 0$. Take $m = \sup\{\psi(x) \mid x \in M\}$, $r > \frac{m}{\gamma}$, $h = \gamma r + r > m + r$, and define finally $B = B((0,h),r) = B(0,r) \times [h-r, h+r]$ and $K = \mathbb{R}_+ B$. For given $(x,t) \in B$ we have that

$$t \geqslant h - r > m,$$

therefore $(x,t) \notin C = \mathrm{Hipo}(\psi) = \{(x,t) \in M \times \mathbb{R} \mid t \leqslant \psi(x)\}$. Observe also that $(0,0) \in Hipo(\psi)$. Hence, by Theorem 4.3.10, there exists a

G-invariant point $(a, \alpha) \in C \cap D((0,0), B)$ such that

$$\{(a, \alpha)\} = C \cap D((a, \alpha), B).$$

Notice that

$$D((0,0), B) = \{(0,0) + t((0,0) + b) \mid t \in [0,1],\, b \in B\} =$$

$$= \{tb \mid t \in [0,1],\, b \in B\} = [0,1] \cdot B = [0,1] \cdot B((0,h), r).$$

Since $(a, \alpha) \in D((0,0), B)$, in particular $a \in B(0,r)$ and $(a,h) \in B \subseteq D((0,0), B)$. Then, by convexity

$$(a,t) = \beta(a,\alpha) + (1-\beta)(a,h) \quad \text{for } \beta \in [0,1],$$

thus $(a,t) \in D((0,0), B)$. Observe now that it can not happen that $\alpha < \psi(a)$, otherwise $(a, \alpha) \in B$ and $(a, \alpha) \in \text{int}(C)$, a contradiction. Therefore, $\alpha = \psi(a)$.

Applying now Lemma 4.3.12 we know that

$$K = \mathbb{R}_+ B = \left\{(x,t) \in F \times \mathbb{R} \mid t \geqslant r^{-1}(h-r)\|x\|\right\}.$$

Since $(a, \alpha) \in B$ and $\alpha = \psi(a)$:

$$\psi(a) \geqslant r^{-1}(h-r)\|a\| = \gamma\|a\| = \gamma d(a, x_0),$$

from where we deduce (4.3.2), taking into account that $\psi = -f$ and $f(x_0) = 0$, i.e.,

$$f(a) \leqslant f(x_0) - \gamma d(a, x_0).$$

Let now $(x,t) \in (a,\alpha)+K$, where $x \in M\setminus\{a\}$ and $t \leqslant m$. It is clear that $t-\alpha \geqslant \gamma\|x-a\| > 0$, so we can write

$$(x-a, t-\alpha) = s(z, h-r-\alpha)$$

where $z = s^{-1}(x-a)$, $s = \frac{t-\alpha}{h-r-\alpha}$. Observe that, since

$$t-\alpha \leqslant m-\alpha < \gamma r - \alpha = h-\alpha-r,$$

$$h-r-\alpha \geqslant h-r-m > 0,$$

it is clear that $s \in]0,1[$. Moreover, K is a convex cone, so

$$(a+z, h-r) = (a,\alpha) + (z, h-r-\alpha) \in K,$$

thus $(a+z, h-r) \in K \cap (E \times \{h-r\}) \subseteq B$, and by convexity of $D((a,\alpha),B)$ we have that

$$(x,t) = (a,\alpha) + s((a+z,h-r)-(a,\alpha)) \in D((a,\alpha),B).$$

In particular $(x,t) \notin C$. Since for all $x \in M$, $\psi(x) \leqslant m$, then $(x,\psi(x)) \notin (a,\alpha)+K$, therefore

$$\psi(x) - \psi(a) < \gamma\|x-a\|.$$

Thus

$$f(a) < f(x) + \gamma\|x-a\|.$$

□

We would like now to focus on a different version of group invariant Drop theorem and Petal's theorem, where instead of looking for group invariant points in the solution, we search for group invariant sets. Let us give a previous definition.

Definition 4.3.13. Let (M, d) be a metric space and G be a compact topological group of isometries acting on M. For a point $x \in E$ we define

$$s_G(x) := \inf\{d(x, g(x)) \mid g \in G \text{ and } g(x) \neq x\}.$$

The following result is a slight modification of the Petal theorem that allows us to extend the classical Petal theorem to what we call the flower theorem.

Proposition 4.3.14 (Flower theorem)**.** *Let (M, d) be a metric space, $G \subseteq \mathcal{L}(M)$ be a compact topological group of isometries acting on M, and $C \subset M$ a complete G-invariant subset of M. Let $x_0 \in C_G$, $b \in M \backslash C$. Then, for every $\gamma > 0$, there exists $a \in C \cap P_\gamma(x_0, b)$ such that*

$$C \cap P_\gamma(g(a), g(b)) = \{g(a)\} \text{ for every } g \in G.$$

Furthermore, for every $g, g' \in G$ with $d(g(b), g'(b)) > 2d(b, C)$ we have that

$$P_\gamma(g(a), g(b)) \cap P_\gamma(g'(a), g'(b)) = \varnothing.$$

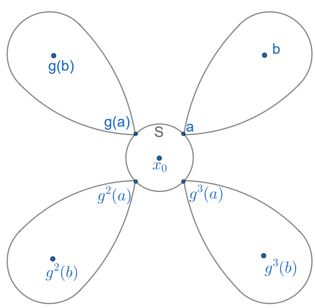


Figure 4.4. Flower theorem.

We can also obtain an analogous generalization for the Drop theorem.

Proposition 4.3.15 (Generalized drop theorem)**.** *Let $(E, \|\cdot\|)$ be a normed space, $G \subseteq \mathcal{L}(E)$ be a compact topological group of isometries acting on E, and $C \subset E$ be a complete G-invariant subset of E. Let $x_0 \in C_G$, $b \in E \backslash C$. Then, there exists $a \in C \cap D(x_0, b)$ such that*

$$C \cap D(g(a), g(b)) = \{g(a)\} \text{ for every } g \in G.$$

Furthermore, for every $g, g' \in G$ with $d(g(b), g'(b)) > 2d(b, C)$ we have that

$$D(g(a), g(b)) \cap D(g'(a), g'(b)) = \varnothing.$$

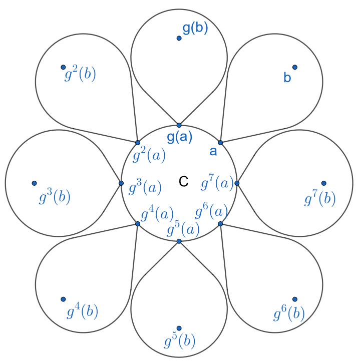


Figure 4.5. Generalized Drop theorem statement.

4.4 Caristi-Kirk, Ekeland and Takahasi's theorem

In 1993, Oettli and Théra showed the equivalence between the Caristi-Kirk fixed point theorem, Takahashi's theorem and EVP. The first result of these three to appear was EVP, [25]. After the publication of EVP, independently James Caristi and William Kirk presented in a congress in 1975 a series of results that Caristi obtained in his thesis supervised by Kirk in which they presented the so-called Caristi-Kirk fixed point theorem, see [13]. Many years after, Wataru Takahashi obtained a minimization result which generalized Caristi and Nadler fixed point theorems and EVP, see [57]. Finally, two years after the result of Takahashi, Oettli and Théra showed in [49] that Takahashi's minimization result, indeed, was equivalent to the EVP and Caristi fixed point theorem. However this equivalence was not clear at first glance. In order to show this equivalence, Oettli and Théra needed firstly to extend to a more general context the three results, and secondly they needed to add a fourth result that acted as a bridge between the three main theorems.

Recall that something similar happened with Penot's equivalence. He needed to add the Petal's theorem in order to obtain the equivalence between Ekeland's and the Drop theorem. Now, we are going to show the equivalence of this results but in the group invariant context. Let's first do some assumptions.

Let (M, d) be a complete metric space and $G \subseteq \mathcal{L}(M)$ be a compact topological group of isometries acting on M, so that d is G-invariant. Let $f\colon M \times M \to \mathbb{R} \cup \{+\infty\}$ be a function which is lower semicontinuous, G-invariant and convex with respect to G in the second coordinate that also satisfies

$$f(x, x) = 0 \quad \forall\, x \in M, \tag{4.4.1}$$

$$f(x, y) \leqslant f(x, z) + f(z, y) \quad \forall\, x, y, z \in M. \tag{4.4.2}$$

Assume that there exists $x_0 \in M_G$ such that

$$\inf_{x \in M} f(x_0, x) > -\infty, \tag{4.4.3}$$

and define the set

$$S_0 = \{x \in M \mid f(x_0, x) + d(x_0, x) \leqslant 0\}. \tag{4.4.4}$$

Note that by the G-invariance of f, d and x_0, the set S_0 is also G-invariant.

The assumptions on f and the existence of x_0 will hold through this section.

In order to show the previously mentioned equivalences we will use the following auxiliary result.

Theorem 4.4.1. *Let $U \subseteq M$ be G-invariant satisfying that*

$$\forall\, y \in S_0 \backslash U,\ \exists\, x \in M_G \text{ such that } x \neq \overline{y} \text{ and } f(\overline{y}, x) + d(\overline{y}, x) \leqslant 0. \tag{4.4.5}$$

Then, there exists $\hat{x} \in (S_0 \cap U)_G$.

Proof. Let us construct recursively a sequence of G-invariant points $x_n \in M$. Consider the initial point x_0 as the one given in assumption (4.4.3), so

$$\inf_{x \in M} f(x_0, x) > -\infty.$$

Given x_n define the set

$$S_n = \{x \in M \mid f(x_n, x) + d(x_n, x) \leqslant 0\},$$

and the number

$$\gamma_n = \inf_{x \in S_n} f(x_n, x).$$

Notice that for $n = 0$, the set S_0 is the one given in equation (4.4.4).

Clearly the set S_n is G-invariant since x_n is G-invariant, d is G-invariant and f is G-invariant with respect to the second coordinate. Also by assumption (4.4.1) it is clear that $S_n \neq \varnothing$, since $x_n \in S_n$, and $\gamma_n \leqslant 0$.

For $n \geqslant 1$, suppose x_{n-1} is known and G-invariant, and $\gamma_{n-1} > -\infty$. Set a G-invariant point $x_n \in S_{n-1}$ such that

$$f(x_{n-1}, x_n) \leqslant \gamma_{n-1} + \frac{1}{n}. \tag{4.4.6}$$

Under this assumptions we are going to show that $S_n \subseteq S_{n-1}$. Let $x \in S_n$, by assumption (4.4.1) and the fact that $x_n \in S_n$ and $x_n \in S_{n-1}$ it is clear that

$$f(x_{n-1}, x)+d(x_{n-1}, x) \leqslant f(x_{n-1}, x_n)+d(x_{n-1}, x_n)+f(x_n, x)+d(x_n, x) \leqslant 0,$$

so, indeed $x \in S_{n-1}$. Applying now (4.4.1) and (4.4.6) we see that $S_n \subseteq S_{n-1}$, and,

$$\begin{aligned}\gamma_n = \inf_{x \in S_n} f(x_n, x) &\geqslant \inf_{x \in S_n} (f(x_{n-1}, x) - f(x_{n-1}, x_n)) \\ &\geqslant \inf_{x \in S_{n-1}} (f(x_{n-1}, x) - f(x_{n-1}, x_n)) \\ &= \gamma_{n-1} - f(x_{n-1}, x_n) \geqslant -\frac{1}{n}.\end{aligned}$$

Then, if $x \in S_n$

$$d(x_n, x) \leqslant -f(x_n, x) \leqslant -\gamma_n \leqslant \frac{1}{n}.$$

Thus, $\operatorname{diam}(S_n) \to 0$. Moreover for every $k \geqslant n$ it is clear that $x_k \in S_k \subseteq S_n$. In particular,

$$d(x_k, x_n) \leqslant \frac{1}{n}.$$

Hence $\{x_n\}$ is a Cauchy sequence of group invariant points. Therefore, since M_G is closed, there exists a group invariant point, say $\hat{x}$, which is the limit of the sequence. Since $\mathrm{diam}(S_n) \to 0$, it is clear that

$$\bigcap_{n=0}^{+\infty} S_n = \{\hat{x}\}\,.$$

We claim that $\hat{x} \in U$. By contradiction, if $\hat{x} \notin U$, we know by hypothesis that there exists $x \in M$ such that $x \neq \hat{x}$ and

$$f(\hat{x}, x) + d(\hat{x}, x) \leqslant 0.$$

Also, since $\hat{x} \in \bigcap_{n=0}^{+\infty} S_n$

$$f(x_n, \hat{x}) + d(x_n, \hat{x}) \leqslant 0 \quad \forall\, n \geqslant 0.$$

Now, applying (4.4.1), we obtain

$$f(x_n, x) + d(x_n, x) \leqslant 0 \quad \forall\, n \geqslant 0,$$

this meaning that $x \in \bigcap_{n=0}^{+\infty} S_n$. But this would be a contradiction with the fact that $\bigcap_{n=0}^{+\infty} S_n = \{\hat{x}\}$. So $\hat{x} \in U$. □

To conclude this section we present the group invariant equivalences of Theorem 4.4.1, that are the group invariant generalizations of Ekeland's theorem, Takahashi's theorem, and Caristi-Kirk fixed point theorem respectively.

Theorem 4.4.2. *Let (M, d) be a complete metric space and $G \subseteq \mathcal{L}(M)$ be a compact topological group of isometries acting on M. Then, the following results are equivalent:*

(i) Let $U \subseteq M$ be G-invariant satisfying that

$$\forall\, y \in S_0 \backslash U,\ \exists\, x \in M_G \text{ with } x \neq \overline{y} \text{ and } f(\overline{y}, x) + d(\overline{y}, x) \leqslant 0.$$

Then, there exists $\hat{x} \in (S_0 \cap U)_G$.

(ii) *There exists $\hat{x} \in S_0$ such that $\hat{x}$ is G-invariant, and $f(\hat{x}, x) + d(\hat{x}, x) > 0$ for all $x \in M$, $x \neq \hat{x}$.*

(iii) *Suppose $\forall\, y \in S_0$ with $\inf_{x \in M} f(\overline{y}, x) < 0$, there exists*

$$x \in M_G \text{ with } x \neq \overline{y} \text{ and } f(\overline{y}, x) + d(\overline{y}, x) \leqslant 0.$$

Then, there exists $\hat{x} \in (S_0)_G$ such that $f(\hat{x}, x) \geqslant 0$ for all $x \in M_G$.

(iv) *Let $T \colon M \to M$ be a multivalued mapping such that for every $y \in S_0$ there exists*

$$x \in (T(y))_G \text{ with } f(\overline{y}, x) + d(\overline{y}, x) \leqslant 0.$$

Then, there exists $\hat{x} \in (S_0)_G$ such that $\hat{x} \in T(\hat{x})$.

Proof. $(ii) \Rightarrow (i)$

We know that there exists some G-invariant point $\hat{x} \in S_0$ such that

$$f(\hat{x}, x) + d(\hat{x}, x) > 0 \quad \forall\, x \neq \hat{x}.$$

In particular $\hat{x} \in U$, hence $\hat{x} \in S_0 \cap U$.

$(i) \Rightarrow (ii)$

Take $y \in M$ and define

$$\Gamma(y) = \{x \in M \mid x \neq \overline{y},\ \ f(\overline{y}, x) + d(\overline{y}, x) \leqslant 0\}.$$

Define now $U = \{y \in M \mid \Gamma(y) = \varnothing\}$. Then, if $y \notin U$, by definition, there exists some x such that $x \in \Gamma(y)$. Applying now (i) there exists some G-invariant point $\hat{x} \in S_0 \cap U$. So $\Gamma(\hat{x}) = \varnothing$, therefore

$$f(\hat{x}, x) + d(\hat{x}, x) > 0 \quad \forall\, x \neq \hat{x}.$$

$(iii) \Rightarrow (i)$

We proceed by contradiction. Assume that $y \notin U$ for every $y \in S_0$. Then, by hypothesis, there exists some $x \in M_G$ such that $x \neq \overline{y}$ and

$$f(\overline{y}, x) + d(\overline{y}, x) \leqslant 0.$$

From this we can deduce that $\inf \{f(\overline{y}, x) \mid x \in M\} < 0$. Therefore, applying (iii), there exists some G-invariant point $\hat{x} \in S_0$ such that $f(\hat{x}, x) \geqslant 0$ for every $x \in M_G$. This implies that

$$f(\hat{x}, x) + d(\hat{x}, x) \geqslant 0 \quad \forall\, \hat{x} \neq x \in M_G.$$

Thus a contradiction.

$(i) \Rightarrow (iii)$

Define

$$U = \left\{ y \in X \mid \inf_{x \in M} f(\overline{y}, x) \geqslant 0 \right\}.$$

Then the hypothesis of (i) follows from the hypothesis of (iii). Since (i) holds, there exists some G-invariant point $\hat{x} \in S_0 \cap U$. And since $\hat{x} \in U$, by definition,

$$\inf_{x \in M} f(\hat{x}, x) \geqslant 0.$$

In particular $f(\hat{x}, x) \geqslant 0$ for every $x \in M_G$.

$(iv) \Rightarrow (i)$

Define the multivalued map $T \colon M \to M$ as follows

$$T(y) = \{x \in M_G \mid x \neq \overline{y}\}.$$

We proceed by contradiction. Suppose $y \notin U$ for every $y \in S_0$. Then, the hypothesis of (iv) follows from (i). Hence, there exists some G-invariant point $\hat{x} \in S_0$. But this is a contradiction with the definition of T.

$(i) \Rightarrow (iv)$

Define

$$U = \{y \in M \mid y \in T(y) \text{ and is } G-\text{invariant}\}.$$

By the hypothesis of (iv) we obtain the hypothesis of (i). And since (i) holds, we obtain a G-invariant point $\hat{x} \in S_0 \cap U$. In particular $\hat{x} \in U$, and by definition $\hat{x} \in T(\hat{x})$. □

4.5 Applications of the Ekeland variational principle

We have just seen some of the equivalences of the EVP, which can also be seen as applications. Our next goal is to show more applications of this result. As we mentioned before, Ekeland's main motivation to prove the EVP was to use this result in control theory, and the EVP found a natural fit in the area of PDE. However we would like to start this section providing a geometric applciation.

It is worth pointing out that EVP is still being used nowadays in PDEs, see for instance [48].

4.5.1 Drop and generalized Drop theorem

We are going to present a geometrical application of the Drop theorem dealing with the notion of the contingent cone to a subset C of a Banach space X. Recall that the contingent cone to C in $a \in C$ is

$$K_C(a) = \limsup_{t\to 0^+} t^{-1}(C-a).$$

So, $v \in K_C(a)$ if, and only if, $\liminf_{t\to 0^+} t^{-1}d(a+tv, C) = 0$.

Theorem 4.5.1. *Let X be a normed space and $G \subseteq \mathcal{L}(X)$ be a compact topological group of isometries acting on X. Assume that $C \subseteq X$ is a complete G-invariant subset. Let $x_0 \in C$ and $y \in X$ be G-invariant and such that the segment $[x_0, y]$ is not contained in C. Then, for each $\rho > 0$, there exists a G-invariant point $a \in C$ such that*

$$\|x_0 - a\| \leqslant \|x_0 - y\| + \rho, \text{ and } y \notin a + K_C(a).$$

Proof. Without loss of generality we may assume that $x_0 = 0$. Let $\lambda \geqslant 1$ such that $y = \lambda z$ with $z \in (E \backslash C)_G$. Let $r \in]0, \rho]$ with $r < d(z, C)$ and define $B = B(z, r)$. Then, by Theorem 4.3.10, we know that there exists a G-invariant point $a \in C \cap D(x_0, B)$ such that $\{a\} = C \cap D(a, B)$.

Let us write now $a = \alpha(z + b)$ with $\alpha \in [0, 1[$ and $\|b\| \leqslant r$, and define $t = \frac{\lambda - 1}{\lambda - \alpha}$. It is clear that $t \in [0, 1[$. Define now $w = ta + (1 - t)y$. Since $t, \alpha \in [0, 1[$ and $\|b\| \leqslant r$, we obtain that

$$\|w - z\| = \|t\alpha b + (t\alpha + (1 - t)\lambda - 1)z\| = t\alpha\|b\| < r.$$

Hence $w \in B$ and

$$(y - a) = (1 - t)^{-1}(w - a) \in \mathbb{R}_+(B \backslash \{a\}) \subseteq E \backslash K_C(a),$$

as $(a +]0, 1[(B - a)) \cap C = \varnothing$. Finally

$$\|x_0 - a\| \leqslant \operatorname{diam}(x_0, B) \leqslant \|x_0 - z\| + r \leqslant \|x_0 - y\| + \rho.$$

□

Remark 4.5.2. Observe that the condition of y being G-invariant cannot be removed within this proof, since we define the point z so that $y = \lambda z$, and we use in this proof that z is G-invariant. If y were not G-invariant, then the only G-invariant point contained in the line $y = \lambda z$ would be the constant zero.

We can proceed similarly but, applying now Theorem 4.3.15 to obtain the following result.

Theorem 4.5.3. *Let X be a normed space and $G \subseteq \mathcal{L}(X)$ be a compact topological group of isometries acting on X. Assume that $C \subseteq X$ is a complete G-invariant set. Let $x_0 \in C$ be G-invariant, and choose $y \in X$ such that the segment $[x_0, y]$ is not contained in C. Then, for each $\rho > 0$ and for each $g \in G$, there exists $g(a) \in C$ such that*

$$\|x_0 - g(a)\| \leqslant \|x_0 - g(y)\| + \rho, \text{ and } g(y) \notin g(a) + K_C(g(a)).$$

Moreover, if $s_G > 2d(z, C)$, where z is such that $z \in (E \backslash C)_G$ and $y = \lambda z$, then for every $g, g' \in G$, we have that

$$g(a) + K_C(g(a)) \cap g'(a) + K_C(g'(a)) = \varnothing.$$

4.5.2 Partial differential equations

Finally, we want to show some applications of the group invariant Ekeland's variational principle to the area of partial differential equations. Let us start fixing some notation, and giving a brief explanation of Sobolev spaces. For more information on these spaces see [1] [11, Chapter 9], and [27, Chapter 5]. In this section $\Omega \subseteq \mathbb{R}^n$ is going to be an open-bounded subset of $\mathbb{R}^n$ with regular boundary[1]. Let $1 \leqslant p \leqslant +\infty$, the Sobolev space $W^{1,p}(\Omega)$ is defined by

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) \;\middle|\; \begin{array}{c} \exists g_1, \ldots, g_n \in L^p(\Omega) \quad \text{such that} \\ \int_\Omega u \frac{\partial \varphi}{\partial x_i} = -\int_\Omega g_i \varphi \;\; \forall \varphi \in C_c^\infty(\Omega), \;\; \forall 1 \leqslant i \leqslant n \end{array} \right\}.$$

[1]For the definition of regular boundary, see [11, Page 272]

This is, the space of functions in $L^p(\Omega)$ whose first derivatives are also in $L^p(\Omega)$. If $v \in W^{1,p}(\Omega)$, we will denote its weak derivative with respect to x_i by $\frac{\partial v}{\partial x_i}$, and ∇v is going to denote the gradient in the weak sense. If we do not say otherwise, all the derivatives that appear in this section should be understood in the sense of distributions.

Remark 4.5.4. Observe that if a function, say f, is differentiable, then the weak derivative and the directional derivative coincide. This is the reason why we denote the weak derivative in the same way as the partial derivative.

$W^{1,p}(\Omega)$ is equipped with the norm:

$$\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p .$$

Theorem 4.5.5. *$W^{1,p}(\Omega)$ is*

(i) A Banach space for $1 \leqslant p \leqslant +\infty$.

(ii) Reflexive for $1 < p < +\infty$.

(iii) Separable for $1 \leqslant p < +\infty$.

For $1 \leqslant p < +\infty$, $W_0^{1,p}(\Omega)$ denotes the closure of $C_c^1(\Omega)$ in $W^{1,p}(\Omega)$.

Theorem 4.5.6. *Suppose that Ω is of class C^1, and let $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ for $1 \leqslant p < +\infty$. Then, the following are equivalent:*

(i) $u = 0$ on $\partial\Omega$.

(ii) $u \in W_0^{1,p}(\Omega)$.

In view of this result, and roughly speaking, we understand the functions of $W_0^{1,p}(\Omega)$ to be those functions of $W^{1,p}(\Omega)$ that vanish on $\partial\Omega$. Also, Theorem 4.5.5 applies exactly in the same way for the space $W_0^{1,p}(\Omega)$.

$W^{-1,q}(\Omega)$ will be the dual set of $W_0^{1,p}(\Omega)$ for $1 \leqslant p \leqslant \infty$, where q denotes the conjugate exponent of p, i.e, $\frac{1}{p} + \frac{1}{q} = 1$. We have the elements of $W^{-1,q}(\Omega)$ completely described by the following result.

Theorem 4.5.7. *Let $F \in W^{-1,q}(\Omega)$. Then, there exist functions $f_0, \ldots, f_n \in L^q(\Omega)$ such that*

$$\langle F, v\rangle = \int_\Omega f_0 v + \sum_{i=1}^n \int_\Omega f_i \frac{\partial v}{\partial x_i} \quad \forall v \in W_0^{1,p}(\Omega),$$

and

$$\|F\| = \max_{1\leqslant i\leqslant n} \|f_i\|_q.$$

If Ω is bounded we can take $f_0 = 0$.

Let us recall here, without proof, three inequalities that are going to be very useful in what follows.

Theorem 4.5.8 (Jensen's inequality)**.** *Let $(X, \mathcal{A})$ be a measurable space, and let μ be a measure on $(X, \mathcal{A})$ such that $\mu(X) = 1$. Suppose that $\phi\colon \mathbb{R} \to \mathbb{R}$ is convex. If $f \in L^1(X, \mathcal{A}, \mu, \mathbb{R})$, then*

$$\phi\left(\int f d\mu\right) \leqslant \int \phi \circ f d\mu.$$

Theorem 4.5.9 (Hölder's inequality)**.** *Let $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ with $1 \leqslant p \leqslant +\infty$. Then, $fg \in L^1(\Omega)$ and*

$$\|fg\| \leqslant \|f\|_p \|g\|_q.$$

Theorem 4.5.10 (Poincaré's inequality)**.** *Suppose that* $1 \leqslant p < +\infty$ *and* Ω *is a bounded open set. Then, there exists a constant* C *such that*

$$\|u\|_p \leqslant C\|\nabla u\|_p \quad \forall u \in W_0^{1,p}(\Omega).$$

In all the applications we are going to study during this section, the group G will be a group of permutations on $\mathbb{R}^n$.

Plateau problem

Plateau problem is the problem of showing the existence of a minimal surface with a given boundary. The origins of this problem come from 1760 which was a problem raised by Joseph-Louis Lagrange, but the name is attributed to Joseph Plateau who stablished the Plateau's laws experimenting in soap films. The problem was solved independently by Jesse Douglas in 1930, [23], and by Tibor Radó in 1933, [52]. However, Plateau's problem in higher dimension, that is, k-dimensional surfaces in n-dimensional space turns out to be much more difficult and it has no solution in general, except if the domain of work is convex. In [25], Ekeland showed that you can perturb the Plateau's problem as you want, and obtain an optimal solution. Now, we are going to study this problem for the group-invariant case.

Let us recall here Corollary 4.2.2. This result is going to be very useful in Theorem 4.5.12.

Lemma 4.5.11. *Let* X *be a Banach space and* $G \subseteq \mathcal{L}(X)$ *be a compact topological group of isometries acting on* X*. Let* $f\colon X \to \mathbb{R} \cup \{+\infty\}$ *be a lower semicontinuous function, Gâteaux differentiable, bounded below,* G*-invariant, convex with respect to* G*, and so that there exists constants* $k, c > 0$ *with*

$$f(x) \geqslant k\|x\| + c \quad \forall\, x \in X. \tag{4.5.1}$$

Then, the range of $\delta f(x)$ is dense in kB_G, where B_G is the closed unit ball in X_G^*.

The first result of this section guarantees that the perturbed Plateau problem, when perturbing by a G-invariant function, has a unique G-invariant solution.

Theorem 4.5.12. *Suppose Ω, and $v_0 \in W_0^{1,1}(\Omega)$ are G-invariant. Then, there exists in $W^{-1,\infty}(\Omega)$ a neighbourhood of the origin, and a dense subset $\mathcal{T}$ in this neighbourhood, such that, for every G-invariant $T \in \mathcal{T}$, the perturbed minimal hypersurface equation*

$$T = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \frac{\nabla v}{(1 + |\nabla v|^2)^{\frac{1}{2}}},$$

$$v - v_0 \in W_0^{1,1}(\Omega),$$

and the perturbed Plateau's problem

$$\inf \left(\int_\Omega 1 + |\nabla v|^2 dx \right)^{\frac{1}{2}} - \langle T, v \rangle,$$

$$v - v_0 \in W_0^{1,1}(\Omega),$$

both have a unique G-invariant solution.

Proof. Define

$$F(v) = \left(\int_\Omega 1 + |\nabla v + \nabla v_0|^2 dx \right)^{\frac{1}{2}},$$

which is the function to be minimized on $W_0^{1,1}(\Omega)$. It is known that this function is convex, continuous, and Gâteaux differentiable, with derivative

$$F'(v) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \frac{\nabla v - \nabla v_0}{(1 + |\nabla v + \nabla v_0|^2)^{\frac{1}{2}}} \in W^{-1,\infty}(\Omega).$$

Observe that, since F is convex, in particular, is convex with respect to G.

Now we want to check that F is G-invariant. Observe that, since v_0 is G-invariant, so is ∇v_0. Therefore,

$$\begin{aligned} F(g(v)) &= \left(\int_\Omega 1 + |\nabla g \circ v(x) + \nabla g \circ v_0(x)|^2 dx \right)^{\frac{1}{2}} \\ &= \left(\int_\Omega 1 + |\nabla v(g(x)) + \nabla v_0(g(x))|^2 dx \right)^{\frac{1}{2}} \\ &= \left(\int_\Omega 1 + |\nabla v(x) + \nabla v_0(x)|^2 dx \right)^{\frac{1}{2}} = F(v), \end{aligned}$$

since Ω is G-invariant.

Finally, we want to see that F satisfies equation (4.5.1) so we can apply Lemma 4.5.11. Observe that

$$\begin{aligned} \int_\Omega |\nabla v| dx - \int_\Omega |\nabla v_0| dx \leqslant \int_\Omega |\nabla v + \nabla v_0| dx &= \|\nabla v + \nabla v_0\|_1 \\ &\leqslant \|\nabla v + \nabla v_0\|_2 = \left(\int_\Omega |\nabla v + \nabla v_0|^2 dx \right)^{\frac{1}{2}} \\ &\leqslant \left(\int_\Omega 1 + |\nabla v + \nabla v_0|^2 dx \right)^{\frac{1}{2}}, \end{aligned}$$

where we have used the triangle inequality and Hölder's inequality respectively. But now, by Poincaré's inequality, see [11, Corollary 9.19] we know that there exists a constant $C > 0$ such that

$$\|u\|_1 \leqslant C \|\nabla u\|_1 \quad \forall\, u \in W_0^{1,1}(\Omega).$$

In particular

$$\|\nabla u\|_1 \geqslant \frac{1}{C+1}\|u\|_{1,1} \quad \forall\, u \in W_0^{1,1}(\Omega).$$

Therefore

$$F(v) = \left(\int_\Omega 1 + |\nabla v + \nabla v_0|^2 dx\right)^{\frac{1}{2}} \geqslant \frac{1}{C+1}\|v\|_{1,1} - K,$$

where the constant $K = \|\nabla v_0\|_1$. Applying now Lemma 4.5.11 we deduce that there exists a dense subset $\mathcal{T}$ such that $\forall\, T \in \mathcal{T}$, $F'(v) = T$ has some solution $v \in W_0^{1,1}(\Omega)$. Finally, define for any $T \in W^{1,1}(\Omega)$

$$F_T(v) = F(v) - \langle T, v\rangle.$$

Then, for any $T \in \mathcal{T}$ there exists $v_T \in W_0^{1,1}(\Omega)$ such that $F'_T(v_T) = 0$. But since F_T is strictly convex, then v_T is the unique minimum of F_T in $W_0^{1,1}(\Omega)$. □

General partial differential equations

Let us continue by recalling the G-invariant version of the Palais-Smale minimizing sequences that will be needed for the proof of Theorem 4.5.14.

Lemma 4.5.13. *Let X be a Banach space and $G \subseteq \mathcal{L}(X)$ be a compact topological group of isometries acting on X. Let $\varphi\colon X \to \mathbb{R}$ be Gâteaux differentiable, bounded below, G-invariant and convex with respect to the group. Then, there exists a sequence $\{x_n\}_{n=1}^{+\infty} \subseteq X$ such that*

(i) x_n *is G-invariant for all* $n \in \mathbb{N}$,

(ii) $\varphi(x_n) \to \inf\{\varphi(x) \mid x \in X\}$,

(iii) $\|\delta\varphi(x_n)\| \to 0$.

Using this lemma, we can obtain the following application.

Theorem 4.5.14. *Let $\Omega \subseteq \mathbb{R}^n$ be a G-invariant set, and let $p \in]1, +\infty[$. Suppose $f \colon \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a borelian function such that*

(i) $f(x, \xi) \geqslant 0$.

(ii) $\xi \mapsto f(x, \xi)$ *is a* C^1 *function.*

(iii) For given constants $a, b \geqslant 0$, f *satisfies a growth condition:* $|f'_\xi(x, \xi)| \leqslant a + b|\xi|^{p-1}$ *for all* $\xi \in \mathbb{R}^n$.

Suppose also that there exists a G-invariant function $v_0 \in W_0^{1,p}(\Omega)$ such that $\int_\Omega f(x, \nabla v_0(x))dx < +\infty$. Then, for all $\epsilon > 0$ there exists a G-invariant function $u_\epsilon \in W^{1,p}(\Omega)$ such that

1. $\left\| \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \frac{\partial f}{\partial \xi_i}(\cdot, \nabla u_\epsilon(\cdot)) \right\|_{-1,q} \leqslant \epsilon.$

2. $\int_\Omega \sum_{i=1}^{n} \left| \frac{\partial u_\epsilon}{\partial x_i} \right|^p dx = \alpha.$

Proof. Define the function

$$H(u) = \frac{1}{p} \int_\Omega \sum_{i=1}^{n} \left| \frac{\partial u}{\partial x_i} \right|^p dx - \alpha \quad \forall\, u \in W_0^{1,p}(\Omega).$$

It is known that this function is a C^1 function on $W_0^{1,p}(\Omega)$, finite everywhere, with derivative

$$H'(u) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right),$$

see [25, Lemma 4.2]. Observe now that H is G-invariant, since

$$\begin{aligned} H(g^*(u)) &= \frac{1}{p}\int_\Omega \sum_{i=1}^n \left|\frac{\partial g^*(u)}{\partial x_i}(x)\right|^p dx - \alpha \\ &= \frac{1}{p}\int_\Omega \sum_{i=1}^n \left|\frac{\partial u}{\partial x_i}(g(x))\right|^p dx - \alpha \\ &= \frac{1}{p}\int_\Omega \sum_{i=1}^n \left|\frac{\partial u}{\partial x_i}(x)\right|^p dx - \alpha = H(u), \end{aligned}$$

for all $u \in W_0^{1,p}(\Omega)$, and every $g \in G$. Now, it only remains to check that H is convex with respect to G. For given $u \in W_0^{1,p}(\Omega)$, observe that

$$H(\overline{u}) = \frac{1}{p}\int_\Omega \sum_{i=1}^n \left|\frac{\partial \overline{u}}{\partial x_i}(x)\right|^p dx - \alpha = \frac{1}{p}\int_\Omega \sum_{i=1}^n \left|\sum_{g\in G} \frac{1}{|G|}\frac{\partial g^*(u)}{\partial x_i}(x)\right|^p dx - \alpha.$$

Applying now the following inequality

$$|a+b|^p \leqslant (|a|+|b|)^p \quad \text{for } p \geqslant 1,$$

it follows that

$$\begin{aligned} \frac{1}{p}\int_\Omega \sum_{i=1}^n \left|\sum_{g\in G} \frac{1}{|G|}\frac{\partial g^*(u)}{\partial x_i}(x)\right|^p dx - \alpha &\leqslant \frac{1}{p}\int_\Omega \sum_{i=1}^n \left|\sum_{g\in G} \frac{1}{|G|}\left|\frac{\partial g^*(u)}{\partial x_i}(x)\right|\right|^p dx - \alpha \\ &\leqslant \frac{1}{p}\int_\Omega \sum_{i=1}^n \sum_{g\in G} \frac{1}{|G|}\left|\frac{\partial g^*(u)}{\partial x_i}(x)\right|^p dx - \alpha \\ &= \sum_{g\in G} \frac{1}{|G|}\frac{1}{p}\int_\Omega \sum_{i=1}^n \left|\frac{\partial g^*(u)}{\partial x_i}(x)\right|^p dx - \alpha \\ &= \sum_{g\in G} \frac{1}{|G|} H(g^*(u)) = H(u), \end{aligned}$$

where we have used in the second inequality that the power function x^p is convex on $\mathbb{R}^+$ and Jensen's inequality, see 4.5.8.

So we have shown that the inequality $H(\overline{u}) \leqslant H(u)$ holds. Therefore H is convex with respect to the group. Applying now Lemma 4.2.1, there exists a sequence of G-invariant points $\{u_n\} \subseteq W_0^{1,p}(\Omega)$ such that

$$H(u_n) \to \inf \left\{ H(v) \mid u \in W_0^{1,p}(\Omega) \right\}, \tag{4.5.2}$$

$$\|\partial H(u_n)\| \to 0. \tag{4.5.3}$$

From (4.5.3) one deduces directly (1). Observe that if we assume that H is bounded below by 0, then, from (4.5.2) we obtain (2). □

Control theory

Let us start making some assumptions. Let K be a G-invariant compact metrizable convex set, and consider the differential equation

$$\begin{cases} \frac{dx}{dt}(t) & = f(t, x(t), u(t)), \\ x(0) & = x_0, \end{cases} \tag{4.5.4}$$

where $x_0 \in \mathbb{R}^n$ and $f\colon [0,T] \times \mathbb{R}^n \times K \to \mathbb{R}^n$, for fixed $T \in \mathbb{R}$. Assume that the following properties are satisfied:

(i) f is continuous, G-invariant on the 3rd coordinate, and convex with respect to the group on the 3rd coordinate.

(ii) $\frac{\partial f}{\partial x_i}$ is well defined and is continuous for all $1 \leqslant i \leqslant n$.

(iii) There exists a $C > 0$ such that $\langle x, f(t,x,u)\rangle \leqslant C(1 + |x|^2)$.

We have the following result.

Theorem 4.5.15. *Suppose f satisfies the previous assumptions, and let $h\colon \mathbb{R}^n \to \mathbb{R}$ be a differentiable function. Then, for all $\epsilon > 0$, there exists* *a G-invariant measurable control v, whose trajectory is y, such that*

$$\begin{cases} h(y(T)) \leqslant \inf h(x(T)) + \epsilon, \\ \langle f(t, y(t), v(t)), p(t)\rangle \leqslant \min_{u\in K}\langle f(t, y(t), u(t)), p(t)\rangle + \epsilon, \end{cases}$$

where p is the solution of the differential system

$$\begin{cases} \frac{dp_i}{dt}(t) = -\sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i}(t, y(t), v(t))p_j(t) \quad \forall 1 \leqslant i \leqslant n, \\ p(T) = h'(y(T)). \end{cases}$$

Proof. Define V as the set of all G-invariant measurable controls $w\colon [0,T] \to K$, that is, all measurable controls $w\colon [0,T] \to K$ such that $g\big(w([0,T])\big) = w([0,T])$. Consider the following metric on V,

$$d(w_1, w_2) = \text{meas}\,\{t \in [0,T] \mid w_1(t) \neq w_2(t)\}\,,$$

where *meas* is the measure of the set. Then, V is a complete metric space, see [26, Lemma 10]. Now, consider the function

$$\begin{aligned} F\colon\ V &\to \quad \mathbb{R} \\ u &\mapsto h(x(T)), \end{aligned}$$

where x is the solution of the differential equation (4.5.4). It is well-known that F is continuous and bounded below, see [25, Lemma 7.3]. Let's see that F is G-invariant. Observe that we can express x as follows:

$$x(t) = x_0 + \int_0^t f(t, x(t), u(t))dt.$$

Hence, it is clear that x depends on u. To highlight it, we are going to denote x by x_u. Therefore, by the G-invariance of f, we have that

$$F(g^*(u)) = h(x_{g^*(u)}(t)) = h\left(x_0 + \int_0^t f(t, x(t), g(u(t)))dt\right) =$$

$$= h\left(x_0 + \int_0^t f(t, x(t), u(t))dt\right) = h(x_u(t)) = F(u).$$

Finally, we have to show that F is convex with respect to G. Again by convexity of f with respect to G, and using the convexity of K, it is clear that

$$\begin{aligned}
F\left(\frac{1}{|G|}\sum_{g\in G} g^*(u)\right) &= h\left(x_{\frac{1}{|G|}\sum_{g\in G} g^*(u)}(t)\right) \\
&= h\left(x_0 + \int_0^t f\left(t, x(t), \frac{1}{|G|}\sum_{g\in G} g(u(t))\right) dt\right) \\
&\leqslant h\left(x_0 + \frac{1}{|G|}\sum_{g\in G}\int_0^t f\left(t, x(t), g(u(t))\right) dt\right) \\
&= h\left(x_0 + \int_0^t f(t, x(t), u(t))dt\right) = h(x_u(t)) = F(u).
\end{aligned}$$

Applying Ekeland's variational principle, we know that there exists a G-invariant point $v \in V$ such that

$$F(v) \leqslant \inf_V F + \epsilon,$$

$$F(u) \geqslant F(v) - \epsilon d(u, v) \quad \forall\, u \in V.$$

From the first inequality it is clear that

$$g(v(T)) \leqslant g(x(T)) + \epsilon.$$

Taking now $t_0 > 0$ and $k_0 \in K$, define $u_\tau \in V$ for $\tau \geqslant 0$ as

$$u_\tau(t) = k_0 \quad \text{if } t \in [0, T]\cap]t_0 - \tau, t_0[,$$

$$u_\tau(t) = v(t) \quad \text{if } t \notin [0, T]\cap]t_0 - \tau, t_0[.$$

Clearly, if τ is sufficiently small, $d(u_\tau, v) = \tau$. Let us denote by x_τ the associated trajectory to u_τ, then $u_0 = v$ and $x_0 = y$. Taking u_τ in the second inequality of Ekeland, we obtain that

$$g(x_\tau(T)) - g(y(T)) \geqslant -\epsilon\tau \quad \forall\, \tau \geqslant 0.$$

Hence,

$$\frac{dg}{d\tau}(x_\tau(T))_{|\tau=0} \geqslant -\epsilon.$$

But it is known, see [26, Theorem 9], that

$$\frac{dg}{d\tau}(x_\tau(T))_{|\tau=0} = \langle f(t_0, y(t_0), k_0) - f(t_0, y(t_0), v(t_0)), p(t_0)\rangle.$$

Thus

$$\langle f(t_0, y(t_0), k_0) - f(t_0, y(t_0), v(t_0)), p(t_0)\rangle \geqslant -\epsilon,$$

which is the desired result, since k_0 is any G-invariant point of K and t_0 is almost every point of $[0, T]$. $\square$

Chapter 5

G-strong subdifferentiability and applications to norm attaining subspaces

Strongly subdifferentiability (SSD) is a concept that appeared in 1978 with the work of [41]. SSD is a natural non-smooth extension of Fréchet differentiability of the norm, basically a norm in a Banach space that is Gâteaux differentiable lacks strong subdifferentiability to be Fréchet differentiable. This concept has proved to be very useful in many different areas such as numerical ranges, in Banach algebras and Banach spaces,see for instance [2, 39, 38]. In [2] the authors obtain a result that states that a dual space with a strongly subdifferentiable norm must be reflexive. Generalizing this result to the context of group invariant functionals is our aim in this chapter. Moreover, Godefroy showed that every strongly subdifferentiable Banach space is an Asplund space [42].

Nowadays the notion of strongly subdifferentiable norms keeps being studied as we can see in [21], and even more recently, strongly subdifferentiable points in Lipschitz free spaces have been studied, see for instance [14].

It is worth pointing out here that there is a stronger result than the one that showed Godefroy in his paper, that states that the dual of a Banach space having strongly subdifferentiable norm is equivalent to the Banach space being reflexive and that every exposed face of the unit ball is strongly exposed. This result can be achieved as a consequence of Godefroy's result and an observation that every w^*-exposed face in the dual ball must be strongly w^*-exposed.

The structure of the Chapter is as follows, Section 5.1 is devoted to presenting all the previous, and necessary, concepts and results that apply directly in the proof of Theorem 5.2.7, or indirectly in the proof of some important lemmas of that same result. Also, all of the results presented in Section 5.1 are classical results that we are extending. Section 5.2 contains the main result of this chapter, Theorem 5.2.7, as well as some consequences of this result.

The contents of this chapter have appeared in the submitted work

[34] J. Falco, and D. Isert, G-strong subdifferentiability and applications to norm attaining subspaces, *arXiv preprint arXiv:2403.18467* (2024).

5.1 Extending classical results

5.1.1 Hahn-Banach extension theorem

Our first aim in this chapter is to present a version of the Hahn-Banach extension theorem. This result complements the results presented in [19]. To achieve this, let us first recall the following well known result, which plays a critical role in the extension process. The proof of this result can be found in [28, Lemma 3.9].

Lemma 5.1.1. *Let X be a vectorial space, and let $f, f_1, \ldots, f_n$ be linear functionals in X. If $\cap_{i=1}^n f_i^{-1}(0) \subseteq f^{-1}(0)$, then f is a linear combination of $f_1, \ldots, f_n$.*

Theorem 5.1.2. *Let X be a Banach space, G a compact topological group acting on X, and let A be w_G^*-closed, convex and G^*-invariant in X^*. If $f \in X^* \backslash A$ is G-invariant, then there exists a G-invariant point, x, such that $\sup_{h \in A} \langle h, x\rangle < \langle f, x\rangle$.*

Proof. Since A is w_G^*-closed, then, there exists U a w_G^*-neighbourhood of zero such that $(f + U) \cap A = \varnothing$. We can assume that U is a convex neighbourhood of zero of the form

$$U = \{y^* \in U^* \mid |\langle y^*, x_i\rangle| < \epsilon \ \ \forall\, 1 \leqslant i \leqslant n\}$$

for some $x_1, \ldots, x_n \in X_G$ and $\epsilon > 0$. By simmetry of U it is clear that $f \notin U + A$, and since $A + U$ is w_G^*-open, then $A + U$ is open and convex. Applying now the Hahn-Banach separation theorem to f and $A + U$, we know that there exists a G-invariant functional $F \in X^{**}$ such that

$$\langle F, f\rangle > \sup_{h \in A+U} \langle F, h\rangle \geqslant \sup_{h \in A} \langle F, h\rangle.$$

We claim that $F = \pi(x)$ for some $x \in X_G$. Fix $h_0 \in A_G$ and observe that

$$C = \sup_U (F) \leqslant \langle F, f\rangle - \langle F, h_0\rangle < +\infty.$$

Consider now the points x_i as G-invariant linear functionals in X^*. Let $y^* \in \cap_i x_i^{-1}(0)$, then $ty^* \in U$ for all $t > 0$. Therefore, $F(ty^*) \leqslant C$, in particular

$$F(y^*) \leqslant \frac{C}{t} \quad \text{and} \quad F(-y^*) \leqslant \frac{C}{t}.$$

Hence $F(y^*) = 0$, and we just obtained that $\cap_i x_i^{-1}(0) \subseteq F^{-1}(0)$. Applying now Lemma 5.1.1, we deduce that F is a linear combination of

$x_1, \ldots, x_n$ so $F \in X_G$ and

$$\langle f, F \rangle > \sup_{h \in A} \langle h, F \rangle.$$

□

Theorem 5.1.3. *Let X be a Banach space and G a compact topological group acting on X. For every $x_0 \in X_G$, there exists $f_0 \in X_G^*$ such that*

$$\|f_0\| = \|x_0\| \quad \text{and} \quad \langle f_0, x_0 \rangle = \|x_0\|^2$$

Proof. Define $H = \mathbb{R}x_0$, which is a G-invariant subspace, and

$$\begin{array}{rccc} h\colon & H & \to & \mathbb{R} \\ & tx_0 & \mapsto & \|x_0\|^2. \end{array}$$

Take $p(x) = \|h\|_{H^*}\|x\|$. Then by the G-invariant version of the Hahn-Banach extension theorem ([30, Proposition 1]), we know that there exists a G-invariant functional, say f_0, such that f_0 extends h to X^* and $\|f_0\|_{X^*} = \|h\|_{H^*}$. Therefore:

(i) $\langle f_0, x_0 \rangle = h(x_0) = \|x_0\|^2$.

(ii) $\|f_0\|_{X^*} = \|h\|_{H^*} = \|x_0\|$.

□

5.1.2 James' theorem

Our next aim is to show James' theorem for group invariant functionals. We know that James' original result was showed first for separable Banach spaces and then for any Banach spaces, see [46, 45]. But, thanks to the group invariant version of James' theorem for the space X_G that Falcó proved in [30, Theorem 6] we will prove James' theorem for both separable and non-separable Banach spaces. Also, our result is a bit different than the one that Falcó showed previously. Observe that, thanks to Theorem 3.4.12, we can obtain Theorem 5.1.4 that provides a key characterization of G-reflexivity in terms of the space of G-invariant points. Specifically, it asserts that the space X is G-reflexive if, and only if, the space of G-invariant points, X_G, is reflexive. This result emphasizes the relationship between the group action and the reflexive properties of the subspace of G-invariant points. So, in view of Theorem 5.1.4, we can obtain an alternative version of the G-invariant James' theorem, that is, Theorem 5.1.5.

Theorem 5.1.4. *Let X be a Banach space and G a compact topological group acting on X. Then, X is G-reflexive if, and only if, X_G is reflexive.*

Proof. By Theorem 3.4.16 it is enough to show that B_X is $w_G(X, X^*)$-compact if, and only if, B_{X_G} is $w(X_G, X_G^*)$-compact.

Note that the mapping

$$\begin{array}{rccc} \varphi\colon & X_G^* & \to & (X_G)^* \\ & f & \mapsto & f_{|X_G} \end{array}$$

is onto by the G-invariant Hahn-Banach theorem. And is injective because, if $f \neq h$, then there exists a point $x \in X$ such that $f(x) \neq h(x)$. Therefore, $f(\overline{x}) \neq h(\overline{x})$. From where we conclude that $f_{|X_G} \neq h_{|X_G}$.

Note that the bijection given by φ induces a bijection between the $w_G(X, X^*)$ topology in X and the $w(X_G, X_G^*)$ topology in X_G. By this bijection, any covering of B_X in the $w_G(X, X^*)$-topology is uniquelly associated with a covering of B_X in the $w_G(X, X^*)$-topology, and the other way around.

Thus, B_{X_G} is compact in the $w(X_G, X_G^*)$-topology, if and only if B_{X_G} is $w(X_G, X_G^*)$-compact. □

Now the proof of James' theorem is straightforward by using the previous Theorem and [30, Theorem 6].

Theorem 5.1.5 (G-invariant James)**.** *Let X be a Banach space and G a compact topological group acting on X. Then X is G-reflexive if, and only if, every G-invariant functional is norm-attaining.*

5.1.3 Moreau's maximum formula

Now we want to give a generalization of Moreau's maximum formula, the result that we obtain here is a feeble one since it is just a restriction of the original result to the space of X_G. However, we will give an example where we see that we cannot obtain a more general result in this direction.

Let us start by presenting some definitions. We will come again to these concepts later on this chapter and study them a bit more profoundly.

Definition 5.1.6. Let $(X, \|\cdot\|)$ be a Banach space and u an element of S_X. We define the G-invariant duality mapping as follows

$$J_G(u) = \left\{f \in X_G^* \mid \|f\| = \|u\| \text{ and } \langle f, u\rangle = \|u\|^2\right\}.$$

Note that Theorem 5.1.3 ensures that $J_G(u) \neq \varnothing$ for every $u \in S_{X_G}$.

Definition 5.1.7. Let $(X, \|\cdot\|)$ be a Banach space. We say that $\|\cdot\|$ is G-strongly subdifferentiable (G-SSD) at $u \in S_X$ if the limit

$$\tau(u, x) = \lim_{t \to 0^+} \frac{\|u + tx\| - 1}{t}$$

exists uniformly for all $x \in B_{X_G}$. If this happens for all $u \in S_{X_G}$, we say that X is G-strongly subdifferentiable.

From these definitions, we can deduce the following properties.

Proposition 5.1.8. *Let X be a Banach space and G a compact topological group acting on X. Then, $J_G(u)$ is G^*-invariant for all $u \in S_{X_G}$, that is, $g^*(J_G(u)) = J_G(u)$ for all $g \in G$ and all $u \in S_{X_G}$.*

Proof. Fix $u \in S_{X_G}$ $f \in J_G(u)$ and $g \in G$. Then, since f and u are G-invariant,

$$\langle g^*(f), u\rangle = \langle f, g(u)\rangle = \langle f, u\rangle = \|u\|^2.$$

Also,

$$\|g^* f\| = \|f\| = \|u\|.$$

And it is clear that $g^*\left(J_G(u)\right) \subseteq J_G(u)$ for every $g^* \in G^*$. Applying inverse mappings we obtain the other inclusion. □

A direct application of Moreau's theorem to the Banach space X_G gives us the following result.

Proposition 5.1.9 (Moreau G-invariant)**.** *Let X be a Banach space and G a compact topological group acting on X. Let $f\colon X \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex, G-invariant function that is continuous at $x_0 \in Dom(f) \cap X_G$. Then,*

$$d^+ f(x_0)(x) = \sup\{\langle x^*, x\rangle \ \mid \ x^* \in \partial_G f(x_0)\} \quad \forall\, x \in X_G.$$

Moreover, this supremum is attained at some point $x^* \in \partial_G f(x_0)$.

The proof of this result is based on the fact that

$$\sup\{\langle x^*, x\rangle \ \mid \ x^* \in \partial f|_{X_G}(x_0)\} = \sup\{\langle x^*, x\rangle \ \mid \ x^* \in \partial_G f(x_0)\}$$

which can be easily obtained by using the G-invariant Hahn-Banach extension theorem.

Note that, in general, the previous result cannot be improved in the sense that we cannot remove the condition of $x \in X_G$ as can be seen with the following example.

Example 5.1.10. Let $G = \{Id, -Id\} \subseteq \mathbb{R}$, and $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$. If $x_0 = 0$, observe that x_0 is G-invariant, in fact, it is the only G-invariant point. On the one hand, notice that

$$d^+ f(0)(1) = \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{|h|}{h} = 1.$$

On the other hand, observe that

$$\partial_G |\cdot|(0) = \{h \in \mathbb{R}_G \mid h(x) \leqslant |x|, \forall\, x \in \mathbb{R}\} = \{0\}\,.$$

Thus,

$$\sup\{h(x) \mid h \in \partial_G |\cdot|(0)\} = 0.$$

Therefore, in general, the G-invariant Moreau's maximum formula is not true if we drop the condition of $x \in X_G$.

Remark 5.1.11. By Moreau's maximum formula we know that for any $x_0, x \in X_G$

$$\tau(x_0, x) = \lim_{t\to 0^+} \frac{\|x_0 + tx\| - 1}{t} = d^+\|x_0\|(x) = \max\{\langle x^*, x\rangle \mid x^* \in \partial_G \|\cdot\|(x_0)\}\,.$$

Corollary 5.1.12. *Let X be a Banach space and G a compact topological group acting on X. Then, for all $x_0 \in X_G$ and for all $x \in X_G$, there exists a functional $x^* \in \partial_G \|x_0\|$ such that*

$$d^+\|x_0\|(x) = \lim_{t\to 0^+} \frac{\|x_0 + tx\| - \|x_0\|}{t} = \langle x^*, x\rangle.$$

Corollary 5.1.13. *Let X be a Banach space, G a compact topological group acting on X and $f\colon X \to Y$ be a G-invariant and a Gâteaux* *differentiable mapping. Then, for all $x_0, x \in X_G$ the function*

$$\begin{array}{rccl} \varphi\colon & [0,1] & \to & \mathbb{R} \\ & t & \mapsto & \|f(x_0+tx)\| \end{array}$$

admits a right derivative at every point, and there exists a G-invariant functional $x^* \in \partial\|\cdot\|(f(x_0+tx))$ such that

$$\lim_{h\to 0^+} \frac{\varphi(t+h)-\varphi(t)}{h} = \langle x^*, df(x_0+tx)(x)\rangle.$$

Proof. Fix $x_0, x \in X_G$ and $0 < t < 1$. Take $h > 0$ such that $t+h \in [0,1]$ and observe that

$$\frac{\varphi(t+h)-\varphi(t)}{h} = \frac{\|f(x_0+(t+h)x)\| - \|f(x_0+tx)\|}{h}.$$

Define now the function

$$\begin{array}{rccl} \phi\colon & [0,1] & \to & Y \\ & h & \mapsto & \frac{f(x_0+(t+h)x)-f(x_0+tx)}{h}, \end{array}$$

which is well-defined, and by being f Gâteaux differentiable at x_0+tx:

$$\lim_{h\to 0^+} \phi(h) = \partial f(x_0+tx)(x) = \phi(0).$$

This later meaning that, for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|\phi(h)-\phi(0)\| < \frac{\epsilon}{\delta} \quad \forall\, |h| < \delta,$$

in particular

$$\|f(x_0+tx)+h\phi(h)\| - \|f(x_0+tx)+h\phi(0)\| \leqslant h\|\phi(h)-\phi(0)\| < \epsilon \quad \forall\, |h| < \delta.$$

Therefore

$$\lim_{h\to o^+} \frac{\varphi(t+h)-\varphi(t)}{h} = \lim_{h\to 0^+} \frac{\|f(x_0+tx)+h\phi(h)\| - \|f(x_0+tx)\|}{h} =$$
$$= \lim_{h\to 0^+} \frac{\|f(x_0+tx)+h\phi(0)\| - \|f(x_0+tx)\|}{h}.$$

Applying now Corollary 5.1.12 this later limit exists, hence φ admits a right directional derivative in every direction, and there exists a G-invariant functional $x^* \in \partial\|\cdot\|(f(x_0+tx))$ such that

$$\lim_{h\to o^+} \frac{\varphi(t+h)-\varphi(t)}{h} = \langle x^*, df(x_0+tx)(x)\rangle.$$

□

5.1.4 Krein-Smulian theorem

Our last goal of this section is to present a G-invariant version of the Krein-Smulian theorem. This result establishes the conditions under which a convex and G^*-invariant set in the dual of a Banach space X can be w^*_G-closed. To this end, we will introduce the following definitions and lemmas, which lay the groundwork for the result.

Definition 5.1.14. Let X be a Banach space, G a compact topological group acting on X, and A a subset of X. We define the G-polar set of A as follows

$$A^{\circ_G} = \{x^* \in X^*_G \mid |\langle x^*, x\rangle| \leqslant 1, \ \forall\, x \in A\}.$$

We have the following properties.

Proposition 5.1.15. *Let X be a Banach space, G be a compact topological group acting on X and A be a subset of X. Then, the following holds* *(i)* A°_G} *is* G^**-invariant.*

(ii) *If* $A_G \neq \varnothing$*, then* $A^{\circ_G} \subseteq A^\circ \subseteq (A_G)^\circ$*. Moreover, if* A *is convex and* G*-invariant we have that* $A^{\circ_G} \subseteq (A_G)^{\circ_G} \subseteq A^\circ$*.*

(iii) *If* A *is* G*-invariant, then* $A^\circ \subseteq (\overline{A})^\circ$*, where* $\overline{A} = \{\overline{x} \mid x \in A\}$*.*

(iv) A°_G} *is* w^*_G*-close in* X^**.*

Proof. (i) Let $x^* \in A^{\circ_G}$ and $g \in G$. By G-invariance of x^*

$$|\langle g^*(x^*), x\rangle| = |\langle x^*, g(x)\rangle| = |\langle x^*, x\rangle| \leqslant 1.$$

Thus, $g^*(A^{\circ_G}) \subseteq A^{\circ_G}$ for all $g^* \in G^*$. By taking inverse mappings, we obtain the other inclusion.

(ii) The first inclusion is clear since we have less functionals in A°_G} than in A°. And the second inclusion is clear since $A_G \subseteq A$, which means that in the set $(A_G)^\circ$ there are less conditions than in the set A°, thus $A^\circ \subseteq (A_G)^\circ$. Observe that if A is convex and G-invariant, inclusion $A^{\circ_G} \subseteq (A_G)^{\circ_G}$ follows similarly. To show that $(A_G)^{\circ_G} \subseteq A^\circ$, take $x^* \in (A_G)^{\circ_G}$, then $x^* \in X^*_G \subseteq X^*$. Moreover, for any $x \in A$ we have that

$$|\langle x^*, x\rangle| = |\langle x^*, \overline{x}\rangle| \leqslant 1,$$

where the first equality follows by linearity and continuity of x^*. Hence, it follows that $(A_G)^{\circ_G} \subseteq A^\circ$.

(iii) Observe that $(\overline{A})^\circ = \{x^* \in X^* \mid |\langle x^*, \overline{x}\rangle| \leqslant 1 \;\; \forall\, x \in A\}$. Take $x^* \in A^\circ$, then,

$$|\langle x^*, \overline{x}\rangle| = \left|\langle x^*, \int_G g(x)d\mu(g)\rangle\right| \leqslant \int_G |\langle x^*, g(x)\rangle| d\mu(g) \leqslant 1,$$

since $g(x) \in A$ for being A G-invariant, and $x^* \in A^\circ$.

(iv) We are going to show that $X^* \backslash A^{\circ_G}$ is w_G^*-open. Recall that for all $\phi \in X_G^*$

$$\mathcal{B} = \{f \in X^* \mid \langle f - \phi, x_i \rangle < \epsilon \ \forall\, 1 \leqslant i \leqslant n\}$$

for any choices of $\epsilon > 0$, $x_1, \ldots, x_n \in X_G$ is a basis of neighbourhoods of ϕ in the w_G^* topology.

If $\phi \in X^* \backslash A^{\circ_G}$ is a G-invariant function, by definition we know that

$$|\langle \phi, x_0 \rangle| > 1 \quad \text{for some } x_0 \in A.$$

Fix $\epsilon = (\langle \phi, x_0 \rangle - 1)/2$, define $\mathcal{E} = \{f \in X^* \mid \langle f - \phi, x_0 \rangle < \epsilon\}$. Then, for every $f \in \mathcal{E}$,

$$|\langle f, x_0 \rangle| \geqslant |\langle \phi, x_0 \rangle| - |\langle f - \phi, x_0 \rangle| \geqslant |\langle \phi, x_0 \rangle| - \epsilon > 1.$$

Hence, $X^* \backslash A^{\circ_G}$ is w_G^*-open and the conclusion holds.

If $\phi \in X^* \backslash A^{\circ_G}$ is not G-invariant, then, there exists a point $x \in X$ and an element of the group $g \in G$ such that $\phi(x) \neq \phi(g(x))$. Take $\epsilon < \|f(x) - f(g(x))\|$. Then, we have that for all $f \in X^*$ such that $\|f - h\| < \epsilon$, h is not G-invariant. Therefore, $h \in X^* \backslash A^{\circ_G}$ even though it might satisfy that $|\langle h, x \rangle| \leqslant 1$ for all $x \in A$.

□

Now we are going to show two fundamental lemmas needed to prove the Krein-Smulian theorem.

Lemma 5.1.16. *X be a Banach space and G a compact topological group acting on X. Let $C \subseteq X^*$ be a G^*-invariant and a convex set such that $\delta B_{X^*} \cap C$ is w_G^*-closed for every $\delta > 0$, and, moreover, $C \cap B_{X^*} = \varnothing$.*

Then, there exists a sequence $\{F_n\}_{n=0}^{+\infty} \subseteq X$ *of* G*-invariant finite subsets that satisfies*

(i) $F_n \subseteq \frac{1}{n} B_X$ *for all* $n \in \mathbb{N}$.

(ii) $C \cap nB_{X^*} \cap F_0^{\circ_G} \cap \cdots \cap F_{n-1}^{\circ_G} = \varnothing$ *for all* $n \in \mathbb{N}$.

Moreover, there exists a sequence $\{x_n\}_{n=1}^{+\infty} \subseteq X_G$ *such that:*

(iii) $\lim_{n\to+\infty} \|x_n\| = 0$,

(iv) for all $x^* \in C$, *there exists* $n \in \mathbb{N}$ *such that* $|\langle x^*, x_n\rangle| > 1$.

Proof. We construct $\{F_n\}_{n=1}^{+\infty}$ by induction. Define $F_0 = \{0\}$. It is clear that (i) and (ii) are satisfied and that F_0 is G-invariant. Suppose now that $F_1, \ldots, F_{n-1}$ are all G-invariant, finite, and they satisfy (i) and (ii). Suppose, by contradiction, that

$$C \cap (n+1)B_{X^*} \cap F_0^{\circ_G} \cap \cdots \cap F_{n-1}^{\circ_G} \cap F^{\circ_G} \neq \varnothing$$

for all G-invariant, finite subsets of $\frac{1}{n} B_X$, F. Define

$$K = C \cap (n+1)B_{X^*} \cap F_0^{\circ_G} \cap \cdots \cap F_{n-1}^{\circ_G}.$$

We know by hypothesis and by Proposition 5.1.15, (iv), that $C \cap (n+1)B_{X^*}$ and $F_i^{\circ_G}$ are w_G^*-closed, and it is clear that both of them are G-invariant. In particular, K is w_G^*-closed. Now, by Theorem 3.4.12, we have that B_{X^*} is w_G^*-compact, and since K is a w_G^*-closed subset of a compact subset, we deduce that K is w_G^*-compact. Moreover, it is G-invariant for being a finite intersection of G-invariant subsets.

Since $K \cap F^{\circ_G} \neq \varnothing$, by the finite intersection property we get that:

$$I = \bigcap \left\{ K \cap F^{\circ_G} \mid F \subseteq \frac{1}{n} B_X \text{ is any finite subset} \right\} \neq \varnothing.$$

Let $x^* \in I$, then $x^* \in F^{\circ_G}$, so

$$|\langle x^*, x\rangle| \leqslant 1 \quad \forall x \in F \subseteq \frac{1}{n}B_X.$$

So, $\|x^*\| \leqslant n$. Thus, we have obtained that

$$\begin{aligned} x^* \in K \cap nB_{X^*} &= C \cap (n+1)B_{X^*} \cap F_0^{\circ_G} \cap \cdots \cap F_{n-1}^{\circ_G} \cap nB_{X^*} \\ &= C \cap nB_{X^*} \cap F_0^{\circ_G} \cap \cdots \cap F_{n-1}^{\circ_G} = \varnothing, \end{aligned}$$

a contradiction. Then,

$$C \cap (n+1)B_{X^*} \cap F_0^{\circ_G} \cap \cdots \cap F_{n-1}^{\circ_G} \cap F^{\circ_G} = \varnothing.$$

This proves (i) and (ii), let us show now (iii). Define $F'_n = \{\overline{x} \mid x \in F_n\}$. For the sequence $\{x_n\}_{n=1}^{+\infty}$ we rearrange, if required, the elements of $\cup_{n=0}^{+\infty} F'_n$ and call them $\{x_n\}_{n=1}^{+\infty}$. Observe that every $x_n \in \frac{1}{n}B_X$, this implies (iii). The only thing that remains to show is that (i) and (ii) hold for F'_n. Define,

$$K' = C \cap nB_{X^*} \cap (F'_0)^{\circ_G} \cap \cdots \cap (F'_{n-1})^{\circ_G}.$$

We want to show that if K' is non-empty, then K is also non-empty. Take $x^* \in K'$, then it is clear, by convexity, that $\overline{x}^* \in nB_{X^*}$ and $\overline{x}^* \in C$. Moreover, $x^* \in (F'_{n-1})^{\circ}$, we want to see that $\overline{x^*} \in F_{n-1}^{\circ}$. This is clear by G-invariance of F:

$$|\langle \overline{x}^*, x\rangle| = \left|\langle \int_{G^*} g^*(x^*) d\mu(g), x\rangle\right| = \left|\int_{G^*} \langle g^*(x^*), x\rangle d\mu(g)\right| \leqslant$$

$$\leqslant \left|\int_G \langle x^*, g(x)\rangle d\mu(g)\right| = |\langle x^*, \overline{x}\rangle| \leqslant 1.$$

This shows that $\overline{x}^* \in F_{n-1}^{\circ}$, also observe that $\overline{x}^* \neq 0$ since C is a convex set which does not contain the zero, otherwise we would have a contradiction with the hypothesis $C \cap B_{X^*} = \varnothing$. And we have shown that for given $x^* \in K'$, then $\overline{x}^* \in \overline{K}$, i.e., if $K' \neq \varnothing$ then $K \neq \varnothing$. Thus, (i) and (ii) also hold when changing F_i by F_i'.

Finally, let's move to the proof of (iv). Let $x^* \in C$, there exists $n_0 \in \mathbb{N}$ such that $\|x^*\| \leqslant n_0$. So, there exists an index $j \in \{1, \ldots, n_0\}$ with $x^* \in F_j^\circ$. But, by (ii), there exists a $k \in \{0, \ldots, j\}$ such that

$$|\langle x^*, x_k \rangle| > 1.$$

□

With the help of the previous lemma, we can obtain the following one.

Lemma 5.1.17. *Let X be a Banach space, G a compact topological group acting on X, and $C \subseteq X^*$ a G^*-invariant convex subset. Suppose that $\delta B_{X^*} \cap C$ is w_G^*-closed for every $\delta > 0$, and $C \cap B_{X^*} = \varnothing$. Then, there exists $x \in X_G$ such that*

$$\langle x^*, x \rangle \geqslant 1 \quad \forall\, x^* \in C.$$

Proof. By Lemma 5.1.16 we know that there exists a sequence $\{x_n\}_{n=1}^{+\infty}$ of G-invariant points such that $\lim_{n\to+\infty} \|x_n\| = 0$. Then

$$\begin{array}{rccc} T\colon & X^* & \to & (c_0, \|\cdot\|_\infty) \\ & x^* & \mapsto & \{\langle x^*, x_n \rangle\}_{n=1}^{+\infty} \end{array}$$

is a linear, well-defined, and continuous operator. So, since C is convex, $T(C)$ is again convex. We want to show now that $C \cap nB_{X^*} = \varnothing$ implies that $T(C) \cap B_{c_0}^\circ = \varnothing$. Suppose, by contradiction, that there exists $z \in T(C) \cap B_{c_0}^\circ$. Then $z \in T(C)$, so there exists a point $x^* \in C$ such that $T(x^*) = z$. On the other hand, $z \in B_{c_0}^\circ = B_{l_1}$, therefore, $z = \{z_n\}_{n=1}^{+\infty}$ with $\sum_{n=1}^{+\infty} |z_n| < 1$. But $z_n = \langle x^*, x_n \rangle$, by definition of T, hence

$$\sum_{n=1}^{+\infty} |\langle x^*, x_n \rangle| < 1.$$

This is a contradiction with (iv) of Lemma 5.1.16. Applying now the classical Hahn-Banach separation theorem to $T(C)$ and $B^\circ_{c_0}$ we know that there exists a point $y = \{y_n\}_{n=1}^{+\infty} \in S_{l_1}$ such that

$$\sup \left\{ \langle y, x \rangle \mid x \in B^\circ_{c_0} \right\} \leqslant \langle y, T(x^*) \rangle \quad \forall\, x^* \in C. \tag{5.1.1}$$

It is clear that:

$$\sup \left\{ \langle y, x \rangle \mid x \in B^\circ_{c_0} \right\} = 1. \tag{5.1.2}$$

Moreover

$$\langle y, T(x^*) \rangle = \sum_{n=1}^{+\infty} y_n \langle x^*, x_n \rangle = \sum_{n=1}^{+\infty} \langle x^*, x_n y_n \rangle \quad \forall\, x^* \in C.$$

By condition (iii) of Lemma 5.1.16, we know that there exists $M > 0$ such that $\|x_n\| < M$ for all $n \in \mathbb{N}$. And, since $y \in S_{l_1}$:

$$\|x_n y_n\| \leqslant \|x_n\| \, |y_n| \leqslant M |y_n|.$$

Thus,

$$\sum_{n=1}^{+\infty} \|x_n y_n\| \leqslant M \sum_{n=1}^{+\infty} |y_n| < +\infty.$$

Now, since X is a Banach space, every absolutely convergent series is convergent, i.e, there exists a point $x \in X$ such that

$$x = \sum_{n=1}^{+\infty} x_n y_n.$$

Observe that x is G-invariant since for every $g \in G$:

$$g(x) = g\left(\sum_{i=1}^{+\infty} x_n y_n\right) = \sum_{i=1}^{+\infty} y_n g(x_n) = \sum_{i=1}^{+\infty} x_n y_n = x,$$

where we have applied that x_n is G-invariant for every $n \in \mathbb{N}$, and the linearity and continuity of g. Putting this now together with (5.1.1) and (5.1.2), we conclude the following

$$1 \leqslant \langle y, T(x^*)\rangle = \lim_{n\to+\infty} \sum_{k=1}^{n} \langle x^*, x_k y_k\rangle = \langle x^*, x\rangle.$$

□

The next result will help us know when a convex subset is norm-closed.

Proposition 5.1.18. *Let X be a Banach space, G a compact topological group acting on X, and C a convex subset of X^*. If $\delta B_{X^*} \cap C$ is w^*_G-closed for every $\delta > 0$, then C is norm-closed in X^*.*

Proof. Let $\{x_n^*\}_{n=1}^{+\infty}$ be a sequence in C that converges in norm to x^* in X^*. Then, there exists a $\delta > 0$ such that $\{x_n^*\}_{n=1}^{+\infty} \subseteq \delta B_{X^*}$, hence, $x_n^* \in C \cap \delta B_{X^*}$. Since $\{x_n^*\}$ converges in norm to x^*, then it converges w^*_G to x^*. Also, $C \cap \delta B_{X^*}$ is w^*_G-closed so $x^* \in C \cap \delta B_{X^*}$. In particular, $x^* \in C$. □

Finally, we will give a G-invariant version of the Krein-Smulian theorem.

Theorem 5.1.19 (G-Krein-Smulian)**.** *Let X be a Banach space, G a compact topological group acting on X, and $C \subseteq X^*$ a G^*-invariant convex set. If $\delta B_{X^*} \cap C$ is w^*_G-closed for every $\delta > 0$, then C is w^*_G-closed in X^*.*

Proof. Observe that C is w_G^*-closed if for all $x^* \in X^*\backslash C$ there exists a w_G^*-neighbourhood U of x^* such that $U \cap C = \emptyset$, i.e., x^* is not in the w_G^*-closure of C.

Fix $x_0^* \in X^*\backslash C$. We are going to show now that $\overline{x_0^*} \in X^*\backslash C$. Suppose, by contradiction, that $\overline{x_0^*} \in C$, then for δ big enough, $x_0^*, \overline{x_0^*} \in \delta B_{X^*}$. Note that $\overline{x_0^*} \in C \cap \delta B_{X^*}$ but $x_0^* \notin C \cap \delta B_{X^*}$. Now, for U a neighbourhood of $\overline{x_0^*}$ there exists $x_1, \ldots, x_n \in X_G$ and $\epsilon > 0$ such that, for $i = 1, \ldots, n$,

$$\langle x^* - \overline{x_0^*}, x_i \rangle \leqslant \epsilon.$$

But observe that

$$\begin{aligned} \langle x_0^* - \overline{x_0^*}, x_i \rangle &= \langle x_0^*, x_i \rangle - \left\langle \int_G g^*(x_0^*) d\mu(g), x_i \right\rangle \\ &= \langle x_0^*, x_i \rangle - \int_G \langle x_0^*, g(x_i) \rangle d\mu(g) \\ &= \langle x_0^*, x_i \rangle - \langle x_0^*, x_i \rangle = 0. \end{aligned}$$

Then, $x_0^* \in U$, and this for every U neighbourhood of $\overline{x_0^*} \in C$. Thus, $x_0^* \in \overline{C \cap \delta B_{X^*}}^{w_G^*} = C \cap \delta B_{X^*}$, since $C \cap \delta B_{X^*}$ is a w_G^*-closed set. This being a contradiction with the fact that $x_0^* \notin C$.

By Proposition 5.1.18 we know that C is norm-closed, so, since $\overline{x_0^*} \in X^*\backslash C$, there exists a $\delta > 0$ such that $(\overline{x_0^*} + \delta B_{X^*}) \cap C = \emptyset$, which is equivalent to say that

$$(\delta^{-1}(C - \overline{x^*})) \cap B_{X^*} = \emptyset.$$

Applying now Lemma 5.1.17, we know that there exists $x \in X_G$ such that $\langle y^*, x \rangle \geqslant 1$ for all $y^* \in \delta^{-1}(C - \overline{x_0^*})$. Therefore,

$$W = \{x^* \in X^* \mid \langle x^*, x \rangle < 1 \;\; \forall x \in X_G\}$$

is a w_G^*-neighbourhood of zero which does not intersect $\delta^{-1}(C - \overline{x_0^*})$, i.e., x^* is not in the w_G^*-closure of C. □

5.2 G-SSD as a sufficient condition for G-reflexivity

The purpose of this section is to prove a necessary condition for a space to be G-reflexive. In order to obtain such result we will first introduce the notion of dissipative set and study some properties of this set that will be crucial in Theorem 5.2.7.

Let us first recall that, in general, $\partial\|\cdot\|(x_0) = J(x_0)$. The following result, shows that the same equality holds in our scenario.

Proposition 5.2.1. *Let X be a Banach space, and G a compact topological group acting on X. Then, for fixed $x_0 \in S_X$, $\partial_G\|\cdot\|(x_0) = J_G(x_0)$.*

Proof. Recall that:

$$\partial_G\|\cdot\|(x_0) = \{x^* \in X_G^* \mid \langle x^*, x - x_0\rangle \leqslant \|x\| - \|x_0\| \ \ \forall\, x \in X\}$$

and

$$J_G(x_0) = \left\{x^* \in X_G^* \mid \|x^*\| = \|x_0\|, \ \ \langle x^*, x_0\rangle = \|x_0\|^2\right\}.$$

Take $x^* \in J_G(x_0)$, then applying the definition of $J_G(x_0)$ and taking into account that $x_0 \in S_X$, we obtain that

$$\langle x^*, x - x_0\rangle = \langle x^*, x\rangle - \langle x^*, x_0\rangle = \langle x^*, x\rangle - \|x_0\|^2 \leqslant \|x^*\|\,\|x\| - \|x_0\|^2 =$$

$$= \|x_0\|(\|x\| - \|x_0\|) = \|x\| - \|x_0\|.$$

Then, it is clear that $J_G(x_0) \subseteq \partial_G\|\cdot\|(x_0)$.

Now take $x^* \in \partial_G\|\cdot\|(x_0)$, by definition we know that

$$\langle x^*, x - x_0\rangle \leqslant \|x\| - \|x_0\| \leqslant \|x - x_0\|.$$

Hence $\|x^*\| \leqslant 1$. Now, for $x = 0$, it is clear that

$$\langle x^*, -x_0\rangle \leqslant -\|x_0\|.$$

Therefore

$$\|x^*\| \geqslant \langle x^*, x_0\rangle \geqslant \|x_0\| = 1.$$

From where we deduce that $\|x^*\| = 1$, and $\langle x^*, x_0\rangle = 1$. Thus, the other inclusion holds, that is, $\partial_G\|\cdot\|(x_0) \subseteq J_G(x_0)$. □

As a consequence, we have that, for $x_0 \in X$ and $x \in X_G$,

$$\tau(x_0, x) \geqslant \tau(\overline{x_0}, x) = \max\{\langle x^*, x\rangle \mid x^* \in J_G(\overline{x_0})\}$$

Now, we will present the concept of dissipative set, which plays a key role in the proof of Theorem 5.2.7.

Definition 5.2.2. Let $(X, \|\cdot\|)$ be a Banach space and u an element of S_{X_G}. We say that a point $x \in X$ is G-dissipative if $\langle x^*, x\rangle \leqslant 0$ for all $x^* \in J_G(u)$. We will denote the set of all G-dissipative elements as follows

$$Dis_G(X) = \{x \in X \mid \langle x^*, x\rangle \leqslant 0 \ \forall\, x^* \in J_G(u)\}.$$

Recall the following properties of the dissipative set.

Proposition 5.2.3. *Let $(X, \|\cdot\|)$ be a Banach space, G a compact topological group acting on X, and $u \in S_{X_G}$ be fixed. The set $Dis_G(X)$ has the following properties* *(i)* $Dis_G(X)$ *is* G*-invariant, that is,* $g(Dis_G(X))) = Dis_G(X)$ *for all* $g \in G$.

(ii) $Dis_G(X)$ *is a non-empty closed convex cone.*

(iii) $Dis_G(X) = \{x \in X \mid \tau(u, \overline{x}) \leqslant 0\}$.

Proof. For the proof of (i) take $x \in \mathrm{Dis}_G(X)$, it is clear that

$$\langle x^*, g(x)\rangle = \langle x^*, x\rangle \leqslant 0 \quad \forall\, x^* \in J_G(u).$$

Therefore, $g(Dis_G(X)) \subseteq Dis_G(X)$ for all $g \in G$. Applying inverse mappings, we obtain the other inclusion.

(ii) holds by definition. Let us move to the proof of (iii). Given $x \in \mathrm{Dis}_G(X)$, by definition we know that $\langle x^*, x\rangle \leqslant 0$ for all $x^* \in J_G(u)$, in particular $\langle x^*, \overline{x}\rangle \leqslant 0$ for all $x^* \in J_G(u)$. Therefore, by Moreau's maximum formula and Remark 5.1.11, there exists $x_0^* \in J_G(u)$ such that

$$0 \geqslant \langle x_0^*, \overline{x}\rangle = \max\{\langle x^*, \overline{x}\rangle \mid x^* \in J_G(u)\} = \tau(u, \overline{x}).$$

The other inclusion follows from the fact that for all $x^* \in J_G(u)$,

$$\langle x^*, \overline{x}\rangle \leqslant \max\{\langle x^*, \overline{x}\rangle \mid x^* \in J_G(u)\} = \tau(u, \overline{x}) \leqslant 0.$$

□

Proposition 5.2.4. *Let* X *be a Banach* G*-SSD space at* $u \in S_{X_G}$*, and let* G *be a compact topological group acting on* X*. Then,*

$$B_X \cap (Dis_G(X) \cap X_G) = B_{X_G} \cap \bigcap_{t>0} \left[-\frac{u}{t} + \left(\frac{1}{t} + \phi_u(t)\right) B_X\right],$$

where

$$\phi_u(t) := \sup\left\{\frac{\|u + tx\| - 1}{t} - \tau(u, x) \mid x \in B_{X_G}\right\}.$$

Proof. Given $x \in B_X \cap (\mathrm{Dis}_G(X) \cap X_G)$, by Proposition 5.2.3 we know that $\tau(u, x) \leqslant 0$. Then,

$$\frac{\|u + tx\| - 1}{t} \leqslant \phi_u(t) + \tau(u, x) \leqslant \phi_u(t) \quad \forall\, t > 0.$$

Hence, $\|t^{-1}u + x\| \leqslant t^{-1} + \phi_u(t)$, i.e.,

$$x \in B_{X_G} \cap \bigcap_{t>0} \left[-\frac{u}{t} + \left(\frac{1}{t} + \phi_u(t) \right) B_X \right].$$

On the other hand, choose x in the above intersection, then:

$$\left\| \frac{u}{t} + x \right\| \leqslant \frac{1}{t} + \phi_u(t).$$

So

$$\frac{\|u + tx\| - 1}{t} \leqslant \phi_u(t).$$

Taking limits now

$$\lim_{t \to 0^+} \frac{\|u + tx\| - 1}{t} \leqslant \lim_{t \to 0^+} \phi_u(t).$$

But observe that $\lim_{t \to 0^+} \frac{\|u+tx\|-1}{t} = \tau(u, x)$, and $\lim_{t \to 0^+} \phi_u(t) = 0$. Therefore, we have obtained that $0 \geqslant \tau(u, x)$. This means, by Proposition 5.2.3, that $x \in Dis_G(X)$. Since by hypothesis $x \in B_{X_G}$, then $x \in B_X \cap (\mathrm{Dis}_G(X) \cap X_G)$ □

Let us give now two previous results which will prove crucial in the main theorem of this section. In the following one, we give a condition to decide when the dissipative of a set is w^*_G-closed.

Proposition 5.2.5. *Let X be a Banach space such that X^* is G-SSD. Then, $Dis_G(X^*) \cap X^*_G$ is w^*_G-closed.*

Proof. Fix $\delta > 0$ and $u^* \in S_{X_G^*}$, since X^* is G-strongly subdifferentiable, by Proposition 5.2.4, we know that

$$\delta B_{X^*} \cap (\mathrm{Dis}_G(X^*) \cap X_G^*) = \delta B_{X_G^*} \cap \bigcap_{t>0} \left[-\frac{u^*}{t} + \left(\frac{1}{t} + \phi_{u^*}(t) \right) B_{X^*} \right].$$

Then, $\delta B_{X^*} \cap (\mathrm{Dis}_G(X^*) \cap X_G^*)$ is w_G^*-closed. Applying now Theorem 5.1.19 we deduce that $\mathrm{Dis}_G(X^*) \cap X_G^*$ is w_G^*-closed. □

Lemma 5.2.6. *Let X be a Banach space, G a compact topological group acting on X. Then, for $u, x \in X_G$, we have that $u - \eta x \notin \mathrm{Dis}_G(X)$ if, and only, if $\eta < \tau(u, x)$.*

Proof. Observe that, for fixed $x \in S_{X_G}$, $u - \eta x \notin \mathrm{Dis}_G(X)$ if, and only if, $\langle x^*, u - \eta x \rangle > 0$ for some $x^* \in J_G(u)$, by definition. And this condition is equivalent to show that $\langle x^*, u \rangle > \eta$. On the one hand, suppose that $\langle x^*, u \rangle > \eta$, then we know by Moreau's Maximum formula, Theorem 5.1.9, that

$$\tau(u, x) = \max \{ \langle x^*, x \rangle \mid x^* \in J_G(u) \}.$$

Then,

$$\tau(u, x) \geqslant \langle x^*, u \rangle > \eta.$$

On the other hand, suppose that $\eta < \tau(u, x)$, then there exists $x^* \in X^*$ such that $\tau(u, x) \geqslant \langle x^*, u \rangle > \eta$. □

Finally, the main result of this chapter.

Theorem 5.2.7. *Let X be a Banach space, G a compact topological group acting on X. If X^* is G-SSD, then X is G-reflexive.*

Proof. Suppose that $\| \cdot \|^*$ is G-SSD at $u^* \in S_{X_G^*}$. We want to show that every G-invariant direction, say $z^* \in X_G^*$, is norm-attaining, so we can apply Theorem 5.1.5 and deduce that X is reflexive.

We know by Proposition 5.2.5 that $\mathrm{Dis}_G(X^*) \cap X_G^*$ is w_G^*-closed, convex and G-invariant. By Lemma 5.2.6 we know that $u^* - \eta z^* \notin \mathrm{Dis}_G(X^*)$ if $\eta < \tau(u^*, z^*)$. Applying now Theorem 5.1.2, we know that there exists $x_0 \in S_{X_G}$ such that

$$\sup\{\langle x^*, x_0\rangle \mid x^* \in \mathrm{Dis}_G(X^*)\} < \langle u^*, x_0\rangle - \eta\langle z^*, x_0\rangle. \tag{5.2.1}$$

We assert that

$$\langle x^*, x_0\rangle \leqslant 0 \quad \forall\, x^* \in \mathrm{Dis}_G(X^*).$$

Otherwise, $\sup\{\langle x^*, x_0\rangle \mid x^* \in \mathrm{Dis}_G(X^*)\} > 0$. Hence, there would exist $x_0^* \in \mathrm{Dis}_G(X^*)$ such that $\langle x_0^*, x_0\rangle > 0$. Since $\mathrm{Dis}_G(X^*)$ is a convex cone, this would mean that $nx_0^* \in \mathrm{Dis}_G(X^*)$ for every $n \in \mathbb{N}$. Then

$$\lim_{n\to+\infty}\langle nx_0^*, x_0\rangle = +\infty.$$

This would be a contradiction with the upper bound obtained in (5.2.1).

Observe that, by Theorem 5.1.3, $J_G(x_0) \neq \varnothing$. Pick $x^* \in J_G(x_0)$. Then, again by Lemma 5.2.6, it is clear that

$$x^* - \tau(x^*, z^*)z^* \in \mathrm{Dis}_G(X^*).$$

By definition of the dissipative, we know that

$$\langle x^*, x_0\rangle - \tau(x^*, z^*)\langle z^*, x_0\rangle \leqslant 0.$$

Then

$$\langle x^*, x_0\rangle \leqslant \tau(x^*, z^*)\langle z^*, x_0\rangle \leqslant \|z^*\|\langle z^*, x_0\rangle \leqslant |\langle z^*, x_0\rangle| \leqslant \|z^*\|\,\|x_0\|.$$

Notice that $\|z^*\| = \|x_0\| = \langle x^*, x_0\rangle = 1$, therefore

$$|\langle z^*, x_0\rangle| = 1.$$

This means that z^* attains the norm, and the result follows. □

As a consequence of this, we obtain the following.

Corollary 5.2.8. *Let X be a Banach space, G a compact topological group acting on X. If X^* is G-SSD, then every G-invariant functional is norm-attaining.*

We would like to finish with the following result that ensures the existence of large vectorial spaces inside the set of norm-attaining functionals.

Corollary 5.2.9. *Let X be a Banach space, and G a compact topological group acting on X. If X^* is G-SSD, then the set of norm-attaining operators on X contains, at least, the vectorial space X^*_G.*

Let us conclude with the following result, which emphasizes the connection between the existence of vector spaces of norm-attaining functionals and the property of being G-reflexive for a given group G. This relation not only provides insight into the geometric structure of Banach spaces under group invariance but also allows us to obtain new results in the classical theory by leveraging G-invariant functionals. These findings enrich our understanding of G-invariant properties in functional analysis and their implications for traditional results.

Proposition 5.2.10. *Let X be a Banach space. If the set of norm-attaining functionals on X contains a finite dimensional Banach space E, then there exists a compact topological group G acting on X such that $E = X^*_G$. In particular, X is G-reflexive.*

Proof. Since E is a finite dimensional space, we can consider $f_1, \ldots, f_n$ a base of E all of them of norm one. Since the vectors are linearly independent, we can consider $x_1, \ldots, x_n$ so that they are linearly independent with $f_i(x_j) = \delta_{i,j}$ and

$$E = span\{\cap_{i=1}^n Ker(f_i), x_1, \ldots, x_n\}.$$

Consider now the operator $T : X \mapsto X$ given by $T(y + \sum_{i=1}^{n} \lambda x_i) = -y + \sum_{i=1}^{n} \lambda x_i$. It is easy to see that T is well defined and linear.

Since $T^{-1} = T$ we can define the compact group $G = \{Id, T\}$. It is clear that every function f_i is G-invariant so $E \subset X_G^*$. To see the other inclusion, consider $f \in X_G^*$. Then, if $x \in \cap_{i=1}^{n} Ker(f_i)$, by the definition of T and using that f is G-invariant, we have that,

$$f(T(x)) = f(-x)$$

hence $\cap_{i=1}^{n} Ker(f_i) \subset \ker(f)$. Thus, by Lemma 5.1.1 we have that f is a linear combination of $f_1, \ldots, f_n$. □

Corollary 5.2.11. *Let X be a Banach space. Then, the set of norm attaining functionals on X contains a finite dimensional space of dimension n if, and only if, there exists a compact topological group G with X being G-reflexive and X_G^* has dimension n.*

Chapter 6

Conclusions, open questions, and future work

In this chapter we will summarize the conclusions we got in this work, and we will also discuss some remarks and open questions that arose from our study.

6.1 Problems found

First let us start with some problems that we found during our research. Several of the problems mentioned here come from the fact that we need to do a symmetrization argument. But the main problem is that, usually, after symmetrizing, we need to take this symmetrized point back to the original Banach space. However, when you try to come back to the original Banach space, you find that there is no guarantee that you recover the same point that you previously symmetrized, because when you symmetrize you go to the quotient space, but when you try to “des-"symmetrize, then you do not know which point of the class of representation will you recover. You would love, in an ideal world, to recover the original point that you symmetrized, but you have no guarantee that you are going to recover that point.

1. In 2011, Ekeland showed a new inverse function theorem between Fréchet spaces. This result is quite interesting since the conditions on the function are more feeble than the ones that we have in the classical inverse function theorem [24]. For example, he do not ask the function to be $\mathcal{C}^2$ nor $\mathcal{C}^1$, and not even Fréchet differentiable. The proof of this result was obtained by means of Ekeland's variational principle. So, we thought it would be interesting to generalize this inverse function theorem to the group invariant case. And in the process of trying to generalize it, we found that it is not possible to generalize this result to the group invariant case.

2. In [9, P. 167] there is a problem about an alternative proof of the Banach fixed point theorem by means of the Ekeland's variational principle. We tried to do this proof for the G-invariant setting, but again we needed to do a symmetrization argument, and when trying to come back, there was no guarantee that the point we found was a fixed point. In contrast with the previous case. It is not clear that this result does not hold, since it is possible that this result can be obtained with an alternative approach.

6.2 Future work

1. In [10], Bourgain showed that the Radon-Nikodým property is equivalent to the Bishop-Phelps property. And in [19], the authors showed that the Radon-Nikodým property implies the G-Bishop-Phelps property, only the necessary condition was given. The authors could not prove the sufficient condition. So there are two problems to solve here, first, give an example to show that the

sufficient condition, in general, is not true. And secondly, try to show that the sufficient condition holds when, instead of working with RNp, you work with the G-RNp. Of course, this second problem is a more difficult one, since we will have to generalize to the group invariant setting the notion of dentability, for example, and many results. We started this research in Chapter 5, most of the work done there can be, and will be, used in order to open another way of tackling this problem, different from the one made by Dantas, Falco and Mingu, by means of Asplund spaces. We hope that with this other point of view we can complete the sufficient condition.

2. During this whole thesis, we worked with compact groups. This has two meanings, on the one hand the group invariant setting was not much studied in functional analysis, except for measure theory, see [37, Chapter 22], so starting to understand the G-invariance notion with compact groups would be the first natural step to take, since we can intuitively think about compact groups like a generalization of finite groups. On the other hand, with compact groups you have one advantage, you can naturally define the symmetrized point and do symmetrization arguments that help you solve some problems that arise in the proofs when trying to generalize. But with non-compat groups you cannot do this at least so easily. So in the future, another challenge to take would be direct this group invariant setting to more general groups that are not necessarily compact, trying to generalize this symmetrized point to those groups, and see what one can obtain. This would be also of high interest. Working with more general groups would rise more negative results and with them, some more difficult problems that would require a different approach.

3. Observe also, that we started making the assumption that $G \subseteq \mathcal{L}(X)$. One can also wonder what happens when, instead of seeing the elements of the group like linear and continuous operators, you see them as, for example, holomorphic mappings, i.e., $G \subseteq \mathcal{H}(X)$. This is also another future problem to tackle.

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