Approximation problems on linear and nonlinear analysis

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Contents

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There is always one truth. El. Psy. Kongroo.

Summary (English)

In this dissertation, we study several classes of mappings that do or do not attain their naturally associated norm or numerical radius. In particular, we will discuss about operators and bilinear mappings between normed spaces, projective tensors, nuclear operators, and Lipschitz mappings.

The main contents of this document are organized in 5 chapters, where we cover the contents of the published papers [42, 43, 59, 84] and the submitted paper [49], and some background notes will be extracted from the published survey [40]. In this section, we will summarize in English the contents of each chapter (check the Resumen section from page 39 and the Resum section from page 65 for respective translations of this summary into Spanish and Valencian).

Summary of Chapter 1

Chapter 1 serves as an introduction. In Section 1.1, we make important remarks on how one could read this document. In Section 1.2, we establish the most basic notations and concepts that will be used throughout the text. In Section 1.3, we include the necessary historical background to motivate our work. Finally, in Section 1.4, we briefly explain the structure of the document and the contents of the upcoming chapters. We provide some historical facts to better motivate the other sections.

Inspired by the work of James 1957/1963 ([74, 75]), Bishop and Phelps 1961 ([17]), Lindenstrauss 1963 ([93]), Bollobás 1970 ([18]) and many others (see [2]) about the density of norm-attaining operators, in 2008, Acosta, Aron, García, and Maestre introduced and studied the Bishop-Phelps-Bollobás property, defined as follows (see [5]).

Definition 1 ([5])**.** A pair of Banach spaces (X, Y) has the *Bishop-Phelps-Bollobás property* (abbreviated *BPBp*) if given $\varepsilon \in (0, 1)$, there exists $\eta(\varepsilon) > 0$ such that whenever $T \in \mathcal{L}(X, Y)$ and $x \in S_X$ satisfy $\|T\| = 1$ and $\|T(x)\| > 1 - \eta(\varepsilon)$, there are $S \in \mathcal{L}(X, Y)$ and $y \in S_X$ such that $\|S\| = \|S(y)\| = 1$, $\|x - y\| < \varepsilon$, and $\|S - T\| < \varepsilon$.

Note that if the Banach spaces X and Y satisfy the BPBp, then in particular $\mathrm{NA}(X, Y)$ is dense in $\mathcal{L}(X, Y)$, although the converse is known to fail sometimes. The BPBp has been studied by many authors recently (see the surveys [3, 40] for a complete exposition of results about the BPBp up to 2022). Several interesting variations of the BPBp have also been introduced and studied lately by doing specific changes to Definition 1, such as the $\mathbf{L}_{o,o}$ (the BPBp but for each previously fixed T, you find an $\eta(\varepsilon, T)$ depending also on T, and also $S = T$).

This wide study of norm-attaining operators has also been extended to other kinds of mappings and norms. For instance, norm-attaining multilinear mappings, homogeneous polynomials, holomorphic functions, compact operators and Lipschitz mappings have been studied for a long time, and the same goes for operators that attain their numerical radius. Needless to say, BPBp properties have also been introduced and studied for these contexts. We refer again to Section 1.3 and to the survey [40] for more information on these and more properties. This setting is the starting point for this disertation.

Summary of Chapter 2

The contents of this chapter have been published in

[42] S. Dantas, M. Jung, and Ó. Roldán, Norm-attaining operators which satisfy a Bollobás type theorem, *Banach J. Math. Anal.* **15**(2) (2021), Paper No. 40, 26 pp.

Inspired by the $\mathbf{L}_{o,o}$ and its many applications, Chapter 2 is devoted to study a class $\mathcal{A}_{\|\cdot\|}(X,Y) \subset \mathrm{NA}(X,Y)$ of operators that satisfy some property like the $\mathbf{L}_{o,o}$, that is, such that whenever they almost attain their norms at some point x, they attain it at a nearby point x_0. The analogous class for numerical radius is also introduced and studied. The formal definition of these sets is the following.

Definition 2. Let X, Y be Banach spaces (over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$).

(i) $\mathcal{A}_{\|\cdot\|}(X,Y)$ stands for the set of all norm-attaining operators $T \in \mathcal{L}(X,Y)$ with $\|T\| = 1$ such that if $\varepsilon > 0$, then there is $\eta(\varepsilon,T) > 0$ such that whenever $x \in S_X$ satisfies $\|T(x)\| > 1 - \eta(\varepsilon,T)$, there is $x_0 \in S_X$ such that $\|T(x_0)\| = 1$ and $\|x_0 - x\| < \varepsilon$.

(ii) $\mathcal{A}_{\mathrm{nu}}(X)$ stands for the set of all numerical radius attaining operators $T \in \mathcal{L}(X,X)$ with $\nu(T) = 1$ such that if $\varepsilon > 0$, then there is $\eta(\varepsilon,T) > 0$ such that whenever $(x,x^*) \in \Pi(X)$ satisfies $|x^*(T(x))| > 1 - \eta(\varepsilon,T)$, there is $(x_0,x_0^*) \in \Pi(X)$ such that $|x_0^*(T(x_0))| = 1$, $\|x_0 - x\| < \varepsilon$, and $\|x_0^* - x^*\| < \varepsilon$.

In Section 2.2, a selection of results and examples about the classes $\mathcal{A}_{\|\cdot\|}$ and $\mathcal{A}_{\mathrm{nu}}$ are presented. For finite-dimensional Banach spaces, using the compactness of the unit ball and the fact that every operator attains its norm and numerical radius, we get the following positive characterization.

Theorem 3. *Let X be a finite-dimensional Banach space. Then*

(i) $\mathcal{A}_{\|\cdot\|}(X,Y) = \{T \in \mathcal{L}(X,Y) : \|T\| = 1\}$ *for any Banach space Y,*

(ii) $\mathcal{A}_{\text{nu}}(X) = \{T \in \mathcal{L}(X,X) : \nu(T) = 1\}$.

For functionals, we get positive results for a wide class of spaces, but also negative results for some other spaces. We summarize them.

Theorem 4. *Let X be a Banach space over $\mathbb{K}$.*

(i) $\mathrm{NA}(c_0,\mathbb{K}) \cap S_{\ell_1} = \mathcal{A}_{\|\cdot\|}(c_0,\mathbb{K})$.

(ii) *If X is uniformly convex, then $S_{X^*} = \mathcal{A}_{\|\cdot\|}(X,\mathbb{K})$.*

(iii) *There is $x^* \in \mathrm{NA}(\ell_1,\mathbb{K}) \cap S_{\ell_\infty}$ such that $x^* \notin \mathcal{A}_{\|\cdot\|}(\ell_1,\mathbb{K})$.*

(iv) *There is $x^* \in \mathrm{NA}(\ell_\infty,\mathbb{K}) \cap S_{\ell_\infty^*}$ such that $x^* \notin \mathcal{A}_{\|\cdot\|}(\ell_\infty,\mathbb{K})$.*

About general operators in a Banach space X, note that every isometry is in $\mathcal{A}_{\|\cdot\|}(X,X)$, but this is not always the case for $\mathcal{A}_{\text{nu}}(X)$. In fact, even in the setting of Hilbert spaces like $X = \ell_2$, there exist (see Example 2.2.5) operators in $\mathcal{A}_{\|\cdot\|}(X,X) \cap \mathcal{A}_{\text{nu}}(X)$, in $\mathcal{A}_{\|\cdot\|}(X,X) \setminus \mathcal{A}_{\text{nu}}(X)$, in $\mathcal{A}_{\text{nu}}(X) \setminus \mathcal{A}_{\|\cdot\|}(X,X)$, and operators that are not in $\mathcal{A}_{\|\cdot\|}(X,X) \cup \mathcal{A}_{\text{nu}}(X)$ despite being in $\{T \in \mathrm{NA}(X,X) \cap \mathrm{NRA}(X) \colon \nu(T) = \|T\| = 1\}$. This all adds some complexity to our question.

An important class of operators for which we are able to get a positive result is the class of compact operators. The following result shows that under some hypothesis on the involved spaces, every compact operator with norm 1 (and numerical radius 1) is in $\mathcal{A}_{\|\cdot\|}(X,Y)$ (and in $\mathcal{A}_{\text{nu}}(X)$).

Theorem 5. *Let X be a reflexive space which satisfies the Kadec-Klee property. Then,*

(i) $S_{\mathcal{K}(X,Y)} \subset \mathcal{A}_{\|\cdot\|}(X,Y)$ *for every Banach space* Y.

(ii) $\{T \in \mathcal{K}(X,X) \colon \nu(T) = \|T\| = 1\} \subset \mathcal{A}_{\text{nu}}(X)$ *whenever* X *is Fréchet differentiable.*

In particular, we show that under certain hypothesis on the involved Banach space X, every compact operator $T \in \mathcal{K}(X,X)$ with $\nu(T) = \|T\| = 1$ attains its numerical radius. Note that if X is an infinite-dimensional Banach space, the inclusion in (ii) must be strict, as the identity is always in $\mathcal{A}_{\text{nu}}(X)$, but it is never compact. We also get the following immediate consequence from the previous result.

Corollary 6. *Let* X *be a reflexive Banach space with the Kadec-Klee property and let* H *be a Hilbert space.*

(i) *If* Y *has the Schur property, then* $\mathcal{A}_{\|\cdot\|}(X,Y) = S_{\mathcal{L}(X,Y)}$.

(ii) $\{T \in \mathcal{K}(H,H) \colon \nu(T) = \|T\| = 1\} \subset \mathcal{A}_{\text{nu}}(H)$.

If we remove some of the hypothesis on the spaces from Theorem 5, none of the items holds true in general (see the operators from (2.2.3) and (2.2.5)). Moreover, Theorem 5 and Corollary 6 are no longer true for non-compact operators in general, as we will see in Section 2.3.

The proof of the following result (inspired by [1, Example 1.9]) provides us a wide class of compact operators $T \in \mathcal{A}_{\text{nu}}(H)$ such that $1 = \nu(T) < \|T\|$ and so, in particular, examples of operators which belong to $\mathcal{A}_{\text{nu}}(H)$ but not to $\mathcal{A}_{\|\cdot\|}(H,H)$ (see the proof of Proposition 2.2.9 for the details). Note that in this case we get $\mathcal{A}_{\text{nu}}$ in an uniform sense, where η only depends on ε.

Proposition 7. *Let* H *be a separable infinite-dimensional real Hilbert space. Then, there is* $T \in \mathcal{L}(H,H)$ *such that*

(i) *T is a compact operator.*

(ii) $1 = \nu(T) < \|T\|$ *and* T *attains its numerical radius.*

(iii) *given* $\varepsilon > 0$, *there is* $\eta(\varepsilon) > 0$ *such that whenever* $x_0 \in S_H$ *satisfies*

$$|\langle T(x_0), x_0\rangle| > 1 - \eta(\varepsilon),$$

there is $x_1 \in S_H$ *with* $\nu(T) = \langle T(x_1), x_1\rangle = 1$ *and* $\|x_1 - x_0\| < \varepsilon$.

In particular, $T \in \mathcal{A}_{\mathrm{nu}}(H)$ *and* $T \notin \mathcal{A}_{\|\cdot\|}(H, H)$.

The operators from (2.2.2), (2.2.3), and (2.2.5) show that in general there is no relation between the claims $T \in \mathcal{A}_{\|\cdot\|}(X, Y)$ and $T^* \in \mathcal{A}_{\|\cdot\|}(Y^*, X^*)$. However, if we put some extra assumptions on the spaces X and Y, then we can obtain the following result.

Proposition 8. *Let* X, Y *be Banach spaces and* $T \in \mathcal{L}(X, Y)$.

(i) *Suppose that* Y *is uniformly smooth. If* $T \in \mathcal{A}_{\|\cdot\|}(X, Y)$, *then* $T^* \in \mathcal{A}_{\|\cdot\|}(Y^*, X^*)$.

(ii) *Suppose that* X *is uniformly convex. If* $T^* \in \mathcal{A}_{\|\cdot\|}(Y^*, X^*)$, *then* $T \in \mathcal{A}_{\|\cdot\|}(X, Y)$.

(iii) *Suppose that* X *is reflexive. Then,* $T \in \mathcal{A}_{\mathrm{nu}}(X)$ *if and only if* $T^* \in \mathcal{A}_{\mathrm{nu}}(X^*)$.

Note that we cannot remove the uniform smoothness or the uniform convexity from items (i) and (ii) (see again (2.2.2), (2.2.3), and (2.2.5)). On c_0 we can prove the following result related to item (iii) above.

Proposition 9. *Let* $T \in \mathcal{A}_{nu}(c_0)$ *be such that the range of* $T^* \in \mathcal{L}(\ell_1, \ell_1)$ *is in* $\operatorname{span}\{e_1^*, \ldots, e_N^*\}$ *for some* $N \in \mathbb{N}$. *Then,* $T^* \in \mathcal{A}_{nu}(\ell_1)$.

In Section 2.3, a complete characterization will be given for all diagonal operators that belong to $\mathcal{A}_{\|\cdot\|}(X,X)$ ($X = c_0$ or ℓ_p, $1 \leqslant p \leqslant \infty$), to $\mathcal{A}_{\mathrm{nu}}(X)$ ($X = c_0$ or ℓ_p, $1 \leqslant p < \infty$), to $\mathcal{A}_{\|\cdot\|}(c_0, \ell_p)$ ($1 < p < \infty$) and to $\mathcal{A}_{\|\cdot\|}(\ell_p, c_0)$ ($1 < p < \infty$). We can summarize those results as follows.

Theorem 10. *Let (X, Y) be (c_0, c_0), (ℓ_p, ℓ_p) ($1 \leqslant p \leqslant \infty$) or (ℓ_p, c_0) ($1 \leqslant p < \infty$). Let $T : X \to Y$ be the norm one diagonal operator associated to the bounded sequence of complex numbers $\{\alpha_n\}_{n=1}^{\infty}$. Then, $T \in \mathcal{A}_{\|\cdot\|}(X, Y)$ if and only if both of the following conditions are satisfied:*

1. *There exists some $n_0 \in \mathbb{N}$ such that $|\alpha_{n_0}| = 1$.*
2. *If $J = \{n \in \mathbb{N} : |\alpha_n| = 1\}$, then either $J = \mathbb{N}$ or $\sup_{n \in \mathbb{N} \setminus J} |\alpha_n| < 1$.*

Theorem 11. *Given $1 \leqslant p < \infty$, let $T : c_0 \to \ell_p$ be the norm one diagonal operator associated to the bounded sequence of complex numbers $\{\alpha_n\}_{n=1}^{\infty}$. Then, $T \in \mathcal{A}_{\|\cdot\|}(c_0, \ell_p)$ if and only if there is some $N \in \mathbb{N}$ such that $\alpha_n = 0$ for all $n > N$.*

Theorem 12. *Let $X = c_0$ or ℓ_p, $1 \leqslant p < \infty$. Let $T : X \to X$ be the numerical radius one diagonal operator associated to the bounded sequence of complex numbers $\{\alpha_n\}_{n=1}^{\infty}$. Then, $T \in \mathcal{A}_{nu}(X)$ if and only if both of the following conditions hold:*

1. *There exists some $n_0 \in \mathbb{N}$ such that $|\alpha_{n_0}| = 1$.*
2. *If $J = \{n \in \mathbb{N} : |\alpha_n| = 1\}$, then the cardinality of $\{\alpha_n : n \in J\}$ is finite and $\sup_{n \in \mathbb{N} \setminus J} |\alpha_n| < 1$ when $J \neq \mathbb{N}$.*

In particular, if $\{\alpha_n\}_{n=1}^{\infty} \subset \mathbb{R}$, $T \in \mathcal{A}_{nu}(X)$ if and only if $T \in \mathcal{A}_{\|\cdot\|}(X, X)$.

As a consequence, for the canonical projections on the classical sequence spaces, $P_N(x) := (x(1), x(2), \ldots, x(N), 0, \ldots)$, we get the following.

Corollary 13. *Let $N \in \mathbb{N}$ be given. If $X = c_0$ or ℓ_p, $1 \leqslant p \leqslant \infty$, then $P_N \in \mathcal{A}_{\|\cdot\|}(X, X) \cap \mathcal{A}_{\mathrm{nu}}(X)$.*

Given two Banach spaces X_1 and X_2, consider the mappings $P_i \in \mathcal{L}(X_1 \oplus X_2, X_i)$ such that $P_i(x_1, x_2) := x_i$, $i = 1, 2$, and $\iota_j \in \mathcal{L}(X_j, X_1 \oplus X_2)$ such that $\iota_i(x) := xe_i$, where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. For Banach spaces W and Z, if we have an operator $T \in \mathcal{L}(W, Z)$, then there is a simple way to define $\widetilde{T} \in \mathcal{L}(W \oplus Z, W \oplus Z)$: consider $\widetilde{T} := \iota_2 \circ T \circ P_1$, that is, $\widetilde{T}(w, z) = (0, T(w))$ for every $(w, z) \in W \oplus Z$. Conversely, we can define a pseudo-inverse process as follows: if we have an operator $S \in \mathcal{L}(W \oplus Z, W \oplus Z)$, then we can consider $\check{S} \in \mathcal{L}(W, Z)$ defined as $\check{S} := P_2 \circ S \circ \iota_1$, that is, $\check{S}(w) = (P_2 \circ S)(w, 0)$ for every $w \in W$. We get the following results.

Proposition 14. *Let W and Z be two Banach spaces, and let $T \in S_{\mathcal{L}(W,Z)}$. Then,*

(i) *If $\widetilde{T} \in \mathcal{A}_{\mathrm{nu}}(W \oplus_s Z)$, then $T \in \mathcal{A}_{\|\cdot\|}(W, Z)$, where $s = 1$ or $s = \infty$.*

(ii) *Suppose that W and Z are uniformly smooth Banach spaces. If $T \in \mathcal{A}_{\|\cdot\|}(W, Z)$, then $\widetilde{T} \in \mathcal{A}_{\mathrm{nu}}(W \oplus_1 Z)$.*

(iii) *Suppose that Z is uniformly convex and W is uniformly smooth. If $T \in \mathcal{A}_{\|\cdot\|}(W, Z)$, then $\widetilde{T} \in \mathcal{A}_{\mathrm{nu}}(W \oplus_\infty Z)$.*

Finally, note that items (ii) and (iii) are no longer true in general for arbitrary Banach spaces or for p-sums if $1 < p < \infty$, and there exists $S \in \mathcal{L}(W \oplus_s Z, W \oplus_s Z)$, with W and Z uniformly smooth and uniformly convex, such that $S \in \mathcal{A}_{\mathrm{nu}}(W \oplus_s Z)$ but $\check{S} \notin \mathcal{A}_{\|\cdot\|}(W, Z)$, $s = 1$ or $s = \infty$ (see the remarks after Propositions 2.4.1 and 2.4.4).

Summary of Chapter 3

The contents of this chapter have been published in

> [59] D. García, M. Maestre, M. Martín, and Ó. Roldán, On the compact operators case of the Bishop-Phelps-Bollobás property for numerical radius, *Results Math.* **76**(3) (2021), Paper No. 122, 23 pp.

In 2013, Guirao and Kozhushkina introduced and studied a version of the BPBp for numerical radius (see [69]). We include the definition as follows.

Definition 15 (Combining [69, Definition 1.2] and [87, Definition 5])**.** A Banach space X has the weak Bishop-Phelps-Bollobás property for the numerical radius (abbreviated weak BPBp-nu) if given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that, whenever $T \in \mathcal{L}(X, X)$ with $\nu(T) = 1$ and $(x, x^*) \in \Pi(X)$ satisfy $|x^*(T(x))| > 1 - \eta(\varepsilon)$, there exist $S \in \mathcal{L}(X, X)$ and $(y, y^*) \in \Pi(X)$ such that

$$\nu(S) = |y^*(S(y))|, \quad \|x - y\| < \varepsilon, \quad \|x^* - y^*\| < \varepsilon, \quad \text{and} \quad \|T - S\| < \varepsilon,$$

where $\Pi(X) := \{(x, x^*) \in S_X \times S_{X^*} \colon\ x^*(x) = 1\}$. If, moreover, S can be chosen so that $\nu(S) = 1$, we say that X has the Bishop-Phelps-Bollobás property for the numerical radius (abbreviated BPBp-nu).

Ever since the work [69], many results have been obtained about the BPBp-nu by many authors (we refer to [3, Section 6] and [40, Section 2.7] for expositions of the main results that have been obtained in this direction). In 2018, Dantas, García, Maestre, and Martín, introduced and studied the BPBp adapted to the setting of compact operators (see [39]). The BPBp-nu and the BPBp for compact operators motivated us to introduce and study the BPBp-nu for compact operators (by considering Definition 15 with $T \in \mathcal{K}(X,X)$ and $S \in \mathcal{K}(X,X)$). By exploring the existing proofs on the BPBp-nu and making small adaptations, we get a first list of spaces satisfying the BPBp-nu for compact operators. This is the aim of Section 3.2.

Examples 16. The following spaces have the BPBp-nu for compact operators: finite-dimensional spaces ([87, Proposition 2]), $c_0(\Gamma)$ and $\ell_1(\Gamma)$ for every index set Γ ([69, Corollaries 3.3 and 4.2]), and $L_1(\mu)$ for every measure μ ([7, Corollary 2.1] and [87, Theorem 9]).

Next, by adapting the notions of numerical index and second numerical index to the setting of compact operators, n_K and n'_K, respectively, and adapting the results from [87] and [89], we show that if a Banach space X is uniformly convex and uniformly smooth, then it has the weak BPBp-nu for compact operators, and if $n_K(X) > 0$ or $n'_K(X) > 0$, then the weak BPBp-nu for compact operators is equivalent to the BPBp-nu for compact operators. In particular, we show that for every measure μ and every $1 < p < \infty$, $L_p(\mu)$ has the BPBp-nu for compact operators.

In [34, Proposition 4.3] it was shown that if a Banach space X has the BPBp-nu for compact operators, then every absolute summand of X of type 1 or ∞ also has this property, and with the same mapping η. This allows us to carry the property from some spaces to some projections of those spaces. Now, it is natural to wonder if something can be said in the opposite direction. In [39, Lemma 2.1] a tool was presented that, in particular, allows us to carry the BPBp for compact operators from some projections of a space to the space itself. In order to get a somewhat similar result for the numerical radius, one needs to control things both in the space and in its dual. The most general result we obtained in this direction is the following lemma.

Lemma 17. *Let X be a Banach space satisfying that $n_K(X) > 0$. Suppose that there is a mapping $\eta\colon (0,1) \longrightarrow (0,1)$ such that given $\delta > 0$, $x_1^*, \ldots, x_n^* \in B_{X^*}$ and $x_1, \ldots, x_\ell \in B_X$, we can find norm one operators $\widetilde{P}\colon X \longrightarrow \widetilde{P}(X)$, $i\colon \widetilde{P}(X) \longrightarrow X$ such that for $P := i \circ \widetilde{P}\colon X \longrightarrow X$, the following conditions are satisfied:*

(i) $\|P^*(x_j^*) - x_j^*\| < \delta$, *for* $j = 1, \ldots, n$.

(ii) $\|P(x_j) - x_j\| < \delta$, *for* $j = 1, \ldots, \ell$.

(iii) $\widetilde{P} \circ i = \mathrm{Id}_{\widetilde{P}(X)}$.

(iv) $\widetilde{P}(X)$ *satisfies the Bishop-Phelps-Bollobás property for numerical radius for compact operators with the mapping* η.

(v) *Either P is an absolute projection and i is the natural inclusion, or $n_K(\widetilde{P}(X)) = n_K(X) = 1$.*

Then, X satisfies the BPBp-nu for compact operators.

Throughout Section 3.3, Lemma 17 is used to show that if a Banach space X with $n_K(X) > 0$ can be suitably projected into some net of spaces that have the BPBp-nu for compact operators with a common mapping η, then sometimes it is possible to show that X also has that property (see Proposition 3.3.2). This is used to show the following two results.

Corollary 18. *Let X be a Banach space with $n_K(X) > 0$. Then, the following statements are equivalent.*

(i) *The space $c_0(X)$ has the BPBp-nu for compact operators.*

(ii) *There is a function $\eta : (0,1) \longrightarrow (0,1)$ such that all the spaces $\ell_\infty^n(X)$, with $n \in \mathbb{N}$, have the BPBp-nu for compact operators with the function η.*

Moreover, if X is finite-dimensional, these properties hold whenever $c_0(X)$ or $\ell_\infty(X)$ have the BPBp-nu.

Corollary 19. *Let X be a Banach space such that X^* is isometrically isomorphic to ℓ_1. Then X has the BPBp-nu for compact operators.*

In Section 3.4, we present a series of topological tools that allow to conveniently cover a locally compact Hausdorff space L with smaller sets and find a suitable partition of the unit subordinated to those sets. This allows us to project the space $C_0(L)$ into some ℓ_∞^p space ($p \in \mathbb{N}$) in such a way that we can use Lemma 17. This strong approximation property we get on $C_0(L)$ and its dual is summarized in the following result.

Theorem 20. *Let L be a locally compact space. Given $\{f_1, \ldots, f_\ell\} \subset C_0(L)$ such that $\|f_j\| \leqslant 1$ for $j = 1, \ldots, \ell$, and given $\{\mu_1, \ldots, \mu_n\} \subset C_0(L)^*$ with $\|\mu_j\| \leqslant 1$ for $j = 1, \ldots, n$, for each $\varepsilon > 0$ there exists a norm one projection $P\colon C_0(L) \longrightarrow C_0(L)$ satisfying:*

(1) $\|P^*(\mu_j) - \mu_j\| < \varepsilon$, *for* $j = 1, \ldots, n$,

(2) $\|P(f_j) - f_j\| < \varepsilon$, *for* $j = 1, \ldots, \ell$,

(3) $P(C_0(L))$ *is isometrically isomorphic to* ℓ_∞^p *for some* $p \in \mathbb{N}$.

Finally, as a consequence, we get the following result.

Theorem 21. *If L is a locally compact Hausdorff space, then $C_0(L)$ has the BPBp-nu for compact operators.*

In particular, every $C(K)$ space (K compact Hausdorff) and every $L_\infty(\mu)$ space (μ any measure) has the BPBp-nu for compact operators. Note that it remains an open problem whether all the $C(K)$ spaces have the BPBp-nu, and only particular cases have been solved in the real case so far (see [13]), but for compact operators we get a definitive answer for these spaces.

Summary of Chapter 4

The contents of this chapter have been published in

[43] S. Dantas, M. Jung, Ó. Roldán, and A. Rueda Zoca, Norm-attaining tensors and nuclear operators, *Mediterr. J. Math.* **19**(1) (2022), Paper No. 38, 27 pp.

In Chapter 4, norm-attainment notions are introduced and studied for projective tensors in $X\widehat{\otimes}_\pi Y$ and nuclear operators in $\mathcal{N}(X,Y)$, for Banach spaces X and Y. In order to motivate why such questions may be interesting, recall that two of the main historical questions on norm-attaining operators are the following:

1. Is $\mathcal{K}(X,Y) \subset \overline{\mathrm{NA}(X,Y)}$ in general?
2. Is $\mathcal{F}(X,Y) \subset \overline{\mathrm{NA}(X,Y)}$ in general?

The first question was answered in the negative by Miguel Martín in 2014 (see [97]). The second question remains open, and is currently considered by many as the main open question in the theory of norm-attaining operators. Note that nuclear operators are in between finite-rank operators and compact operators, and projective tensors are closely related to them and have many applications to several fields within Functional Analysis. Another important factor to motivate this study is the fact that if it were true that for every finite-dimensional Banach space X every nuclear operator in $\mathcal{N}(X,Y)$ attains its nuclear norm, then we would get an definitive affirmative answer to the second question above. Nevertheless, the assumption turned out to be false, as we will see later.

In this chapter, the isometric identifications $(X\widehat{\otimes}_\pi Y)^* = \mathcal{L}(X,Y^*) = \mathcal{L}(Y,X^*) = \mathcal{B}(X\times Y,\mathbb{K})$ are used implicitly. Note also that if X^* or

Y has the approximation property, then $X^* \widehat{\otimes}_\pi Y = \mathcal{N}(X,Y)$ (see, for instance, [107, Corollary 4.8]). We introduce now norm-attainment notions for projective tensors and nuclear operators.

Definition 22. Let X, Y be Banach spaces. We say that

(i) $z \in X \widehat{\otimes}_\pi Y$ *attains its projective norm* if there is a bounded sequence $(x_n, y_n) \subset X \times Y$ with $\sum_{n=1}^\infty \|x_n\|\|y_n\| < \infty$ with $z = \sum_{n=1}^\infty x_n \otimes y_n$ and $\|z\|_\pi = \sum_{n=1}^\infty \|x_n\|\|y_n\|$. In this case, we say that z is a *norm-attaining tensor*, or $z \in \mathrm{NA}_\pi(X \widehat{\otimes}_\pi Y)$.

(ii) $T \in \mathcal{N}(X,Y)$ *attains its nuclear norm* if there is a bounded sequence $(x_n^*, y_n) \subset X^* \times Y$ with $\sum_{n=1}^\infty \|x_n^*\|\|y_n\| < \infty$ such that $T = \sum_{n=1}^\infty x_n^* \otimes y_n$ and that $\|T\|_{\mathcal{N}} = \sum_{n=1}^\infty \|x_n^*\|\|y_n\|$. In this case, we say that T is a *norm-attaining nuclear operator*, or $T \in \mathrm{NA}_{\mathcal{N}}(X,Y)$.

In Section 4.2, the first norm-attainment results in this setting are obtained. We start by finding two technical characterizations that allow us to claim that a tensor or nuclear operator attains its respective norm provided that there are many bilinear forms that attain their norms at many points in a specific way (see Theorems 4.2.1 and 4.2.2). With these results in mind and the fact that finite-dimensional spaces, c_0, ℓ_1, and Hilbert spaces all have the approximation property, we get our first collection of positive results, which we summarize as follows.

Proposition 23. *Every projective tensor in $X^* \widehat{\otimes}_\pi Y$ and every nuclear operator in $\mathcal{N}(X,Y)$ attain their respective norms if X and Y are finite-dimensional, if $X = Y$ is a complex Hilbert space, or if $X = c_0$.*

It is interesting to compare the last example from above with the classical theory of norm-attaining operators: if $\mathrm{NA}(X,Y) = \mathcal{L}(X,Y)$ for every Banach space Y, then in particular X must be reflexive by

James' theorem. The previous proposition motivates us to wonder if $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y) = X\widehat{\otimes}_\pi Y$ and $\mathrm{NA}_{\mathcal{N}}(X,Y) = \mathcal{N}(X,Y)$ hold in general for any Banach spaces X and Y. However, this is not the case, as the following results show.

Lemma 24. *Let X, Y be Banach spaces. If $B \in \mathcal{B}(X \times Y, \mathbb{K}) = (X\widehat{\otimes}_\pi Y)^*$ attains its functional norm at an element of $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)$, then $B \in \mathrm{NA}_{\mathcal{B}}(X \times Y, \mathbb{K})$.*

Proposition 25. *Let X, Y be Banach spaces. If $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y) = X\widehat{\otimes}_\pi Y$, then $\overline{\mathrm{NA}_{\mathcal{B}}(X \times Y, \mathbb{K})}^{\|\cdot\|_{\mathcal{B}}} = \mathcal{B}(X \times Y, \mathbb{K})$ (and $\overline{\mathrm{NA}(X, Y^*)}^{\|\cdot\|} = \mathcal{L}(X, Y^*)$).*

There are several known examples of Banach spaces X and Y for which $\overline{\mathrm{NA}(X, Y^*)}^{\|\cdot\|} \neq \mathcal{L}(X, Y^*)$, and so, there exist projective tensors that do not attain their projective norm. Using the approximation property, we also get nuclear operators that do not attain their nuclear norm. The following example is of particular interest, as it shows that not every projective tensor or nuclear operator attains their respective norms if only one of the Banach spaces is assumed to be finite-dimensional, providing a negative answer to one of the factors we used to motivate this study.

Example 26. Let $X = L_1(\mathbb{T})$, where the unit circle $\mathbb{T}$ is equipped with the Haar measure m, and let Y be the two-dimensional Hilbert space. It is shown in [65, Remark 5.7.(2)] that there is $T \in \mathcal{B}(X \times Y, \mathbb{K})$ which attains its norm as a linear functional on $X\widehat{\otimes}_\pi Y$ but not as an operator from X into Y^* (nor the more as a bilinear form on $X \times Y$). By Lemma 24, it follows that $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y) \neq X\widehat{\otimes}_\pi Y$, and so, $\mathrm{NA}_{\mathcal{N}}(Y, X) \neq \mathcal{N}(Y, X)$.

Since not every projective tensor or nuclear operator attains its norm, it is natural to wonder now if we have density results. In Section 4.3, we provide some positive density results. In order to get them, two approaches are used. First, note that by the first two characterizations, in order to get many norm-attaining projective tensors and nuclear operators, we want to have many norm-attaining bilinear forms that attain their norms at many points. The $\mathbf{L}_{o,o}$ ensures the existence of many norm-attaining operators that attain their norms at many points, and it can be adapted to bilinear mappings as follows.

Definition 27. We say that $(X \times Y, Z)$ satisfies the $\mathbf{L}_{o,o}$ for bilinear mappings (or just $\mathbf{L}_{o,o,\mathcal{B}}$) if given $\varepsilon > 0$ and $B \in \mathcal{B}(X \times Y, Z)$ with $\|B\|_{\mathcal{B}} = 1$, there exists $\eta(\varepsilon, B) > 0$ such that whenever $(x, y) \in S_X \times S_Y$ satisfies $\|B(x, y)\| > 1 - \eta(\varepsilon, B)$, there is $(x_0, y_0) \in S_X \times S_Y$ such that $\|B(x_0, y_0)\| = 1$, $\|x - x_0\| < \varepsilon$, and $\|y - y_0\| < \varepsilon$.

We get the following positive result.

Proposition 28. *Let X, Y be Banach spaces. If $(X^* \times Y, \mathbb{K})$ has the $\mathbf{L}_{o,o,\mathcal{B}}$, then, $\overline{\mathrm{NA}_{\mathcal{N}}(X, Y)}^{\|\cdot\|_{\mathcal{N}}} = \mathcal{N}(X, Y)$. If $(X \times Y, \mathbb{K})$ has the $\mathbf{L}_{o,o,\mathcal{B}}$, then, $\overline{\mathrm{NA}_{\pi}(X \widehat{\otimes}_{\pi} Y)}^{\|\cdot\|_{\pi}} = X \widehat{\otimes}_{\pi} Y$.*

In particular, note that the following relations are known, and so, we get some positive results for density.

Examples 29 ([48])**.** Let X, Y be Banach spaces.

(i) If $\dim(X), \dim(Y) < \infty$, then $(X \times Y, Z)$ has the $\mathbf{L}_{o,o,\mathcal{B}}$ for every Banach space Z.

(ii) If Y is uniformly convex, then $(X \times Y, \mathbb{K})$ has the $\mathbf{L}_{o,o,\mathcal{B}}$ if and only if (X, Y^*) has the $\mathbf{L}_{o,o}$.

(iii) If $1 < p, q < \infty$, then $(\ell_p \times \ell_q, \mathbb{K})$ has the $\mathbf{L}_{o,o,\mathcal{B}}$ if and only if $p > q'$.

However, note that the $\mathbf{L}_{o,o,\mathcal{B}}$ for bilinear forms is a very restrictive property, as it asks for both spaces to be reflexive to start with, and there are also pairs of reflexive spaces without the property, as we just saw. Therefore, we need a different approach to get more positive results. To use what we know about the finite-dimensional scenario, it would be convenient to have good subspaces of our spaces. Projective norms do not respect subspaces in general, but they behave well with 1-complemented subspaces, and so, we are interested to have a property that ensures the existence of many suitable 1-complemented subspaces of our spaces. We consider therefore the metric π-property.

Definition 30. Let X be a Banach space. We say that X has the *metric π-property* if given $\varepsilon > 0$ and $\{x_1, \ldots, x_n\} \subset S_X$ a finite collection in the sphere, then we can find a finite dimensional 1-complemented subspace $M \subset X$ such that for each $i \in \{1, \ldots, n\}$ there exists $x_i' \in M$ with $\|x_i - x_i'\| < \varepsilon$.

The previous concept is actually equivalent to the metric π approximation property (an approximation property where the approximating operators are all projections of norm 1), and this allows us to find many more examples of spaces for which we have density (we refer to [23], [76], and [94] for more information on the π-property). The following properties hold.

Theorem 31. *Let Y be a uniformly convex space or a space with the metric π-property. If X (respectively X^*) has the metric π-property, then $X\widehat{\otimes}_\pi Y = \overline{\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)}^{\|\cdot\|_\pi}$ (respectively, $\mathcal{N}(X,Y) = \overline{\mathrm{NA}_{\mathcal{N}}(X,Y)}^{\|\cdot\|_{\mathcal{N}}}$).*

Example 32. The following spaces have the metric π-property.

(i) Banach spaces with a finite dimensional decomposition with the decomposition constant 1 (consequently, every Banach space with Schauder basis can be renormed to have the metric π-property),

(ii) $L_p(\mu)$ $(1 \leqslant p < \infty$, μ any measure) and isometric preduals of L_1,

(iii) $X \oplus_a Y$, whenever X, Y satisfy the metric π-property and $|\cdot|_a$ is an absolute norm,

(iv) $X = [\bigoplus_{n\in\mathbb{N}} X_n]_{c_0}$ or $[\bigoplus_{n\in\mathbb{N}} X_n]_{\ell_p}$, for $1 \leqslant p < \infty$, whenever X_n has the metric π-property, for all n,

(v) $X\widehat{\otimes}_\pi Y$ and $X\widehat{\otimes}_\varepsilon Y$ whenever X, Y satisfy the metric π-property.

This shows that in many spaces, the density holds. We refer to the recent work [41, Section 4] for more positive density results involving the RNP, dual spaces, and the metric π-property (for instance, if X^* and Y^* RNP and one has approximation property, then $\overline{\mathrm{NA}_\pi(X^*\widehat{\otimes}_\pi Y^*)}^{\|\cdot\|_\pi} = X^*\widehat{\otimes}_\pi Y^*$, and if Z is any dual space, then $\overline{\mathrm{NA}_\pi(c_0\widehat{\otimes}_\pi Z)}^{\|\cdot\|_\pi} = c_0\widehat{\otimes}_\pi Z$). At this point, it is natural to wonder now if we always have density of norm-attaining projective tensors or nuclear operators. However, despite our wide collection of positive results, in Section 4.4, we get the following result for tensors.

Theorem 33. *Let $\mathcal{R}$ be Read's space. There exist subspaces X of c_0 and Y of $\mathcal{R}$ such that the set of tensors in $X\widehat{\otimes}_\pi Y^*$ which attain their projective norms is not dense in $X\widehat{\otimes}_\pi Y^*$.*

It is worth noting that the analogous question for nuclear operators remains open.

Finally, note that although it is not known whether every finite-rank operator can be approximated by norm-attaining operators, the analogous claim for tensors does not hold in general.

Proposition 34. *There are tensors of finite-rank which cannot be approximated by norm-attaining tensors.*

Summary of Chapter 5

The first half of this chapter has appeared in the published work

> [84] V. Kadets and Ó. Roldán, Closed linear spaces consisting of strongly norm attaining Lipschitz mappings, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **116** (2022), Paper No. 162, 12 pp.

and the second half, in the submitted work

> [49] S. Dantas, R. Medina, A. Quilis, and Ó. Roldán, On isometric embeddings into the set of strongly norm-attaining Lipschitz functions. Preprint available at arXiv:2208.02916.

In 2001, Godefroy asked if for every infinite-dimensional Banach space X, the set of norm-attaining functionals $\mathrm{NA}(X, \mathbb{K})$ always contains 2-dimensional linear spaces (see [64, Problem III]). This question was answered in the negative by Rmoutil in 2017: if $\mathcal{R}$ is the Read's renorming of c_0 from [103], then $\mathrm{NA}(\mathcal{R}, \mathbb{K})$ does not contain 2-dimensional linear spaces. In Chapter 5, we study the analogous spaciability question for the set of strongly norm-attaining (real) Lipschitz functions.

Let M be a pointed metric space, that is, a metric space with a distinguished point 0. $\mathrm{Lip}_0(M)$ is the Banach space of Lipschitz functions $f : M \to \mathbb{R}$ such that $f(0) = 0$ endowed with the Lipschitz norm

$$\|f\| := \sup\left\{\frac{|f(y) - f(x)|}{d(x, y)} : \ x, y \in M, \ x \neq y\right\}.$$

A Lipschitz function $f \in \mathrm{Lip}_0(M)$ is said to attain its norm strongly if there exist $x, y \in M$, $x \neq y$, such that $\|f\| = \frac{|f(y)-f(x)|}{d(x,y)}$. The set of strongly norm-attaining Lipschitz mappings on M is denoted $\mathrm{SNA}(M)$.

Strong norm-attainment of Lipschitz mappings, as well as other weaker norm-attainments, have been extensively studied for the last few years, ever since the first works on the topic ([66, 83]). It is known that strong norm-attainment is actually a very strict kind of norm-attainment (for instance, by [83, Lemma 2.2], if a mapping attains its norm strongly at a pair (x, y), it must attain it through the whole segment $[x, y]$ and be affine whenever defined). For this reason, in many metric spaces M, $\mathrm{SNA}(M)$ happens to not be dense in $\mathrm{Lip}_0(M)$, although positive results have also been obtained for some other metric spaces.

It is clear that if M has cardinal $n \in \mathbb{N}$, then $\mathrm{SNA}(M) = \mathrm{Lip}_0(M)$, and it is a Banach space. In Chapter 5 we tackle the following question: if M is infinite, does $\mathrm{SNA}(M)$ always contain linear spaces of dimension bigger than 1? By how strict this norm-attainment is, and keeping in mind Rmoutil's work for functionals, one may think that the answer to this question could be negative. However, we will see that this is far from being true. In order to do so, we rely on several techniques such as McShane's extension theorem (which allows us to extend Lipschitz mappings from a metric space M_1 to a larger metric space M_2 preserving its norm), Lipschitz-free spaces, and some other tools. In this chapter, we can and do implicitly assume all metric spaces to be complete (by McShane's extension theorem) and pointed, and all vector spaces to be real.

It is not true in general that if a Banach space X is in $\mathrm{SNA}(M)$ for some metric space M, then we can extend it with McShane and find the same space X in $\mathrm{SNA}(M_2)$ for every larger metric space M_2. However, with the $\|\cdot\|_1$ norm, we get the following result.

Lemma 35. *Let M be a pointed metric space such that for some subspace K of M,* $\mathrm{SNA}(K)$ *contains a linear subspace isometrically isomorphic* *to* ℓ_1^n *for some* $n \in \mathbb{N}$*. Then,* $\mathrm{SNA}(M)$ *also contains a linear subspace isometrically isomorphic to* ℓ_1^n*.*

Using this and some more tools, we are able to provide a definitive answer to our main question.

Theorem 36. *Let* $n > 1$ *be a natural number, and let* M *be a pointed metric space with at least* 2^n *distinct points. Then, there exists a linear subspace of* $\mathrm{SNA}(M)$ *which is isometrically isomorphic to* ℓ_1^n*.*

Corollary 37. *If* M *is an infinite pointed metric space, then for all* $n \in \mathbb{N}$*,* $\mathrm{SNA}(M)$ *contains an* n*-dimensional subspace isometric to* ℓ_1^n*.*

So if M is infinite, not only $\mathrm{SNA}(M)$ contains linear spaces of dimension at least 2, it actually contains all the ℓ_1^n spaces isometrically for $n \in \mathbb{N}$.

It is not hard to see that $\mathrm{SNA}([0,1])$ contains a copy of c_0. This motivates to ask what other Banach spaces can be formed. The answer, surprisingly, is that all of them, with right choices of the involved metric spaces.

Proposition 38. *The Banach space* Y *is a subspace of* $\mathrm{SNA}(B_{Y^*})$*.*

It is also interesting to wonder the inverse question: given a Banach space Y, how "small" can a metric space M be so that Y is a subspace of $\mathrm{SNA}(M)$? From the previous proposition, if Y has separable dual, M can be chosen to be separable, but what if not? The following result shows that this is actually a characterization.

Theorem 39. *For a Banach space* Y*, the following assertions are equivalent.*

(1) There is a separable metric space M *and a closed linear subspace* $Z \subset \mathrm{Lip}_0(M)$ *such that* Z *is isometric to* Y *and* $Z \subset \mathrm{SNA}(M)$*.*

(2) There is a separable Banach space X and a closed linear subspace $Z_1 \subset X^*$ such that Z_1 is isometric to Y and $Z_1 \subset \mathrm{NA}(X, \mathbb{R})$.

(3) Y^* is separable.

Therefore, for separable metric spaces such as $M = [0, 1]$, $\mathrm{SNA}(M)$ cannot contain spaces like ℓ_1 with non-separable dual. This adds some restriction to separable metric spaces. In fact, other restrictions appear for some small metric spaces, such as σ-precompact spaces, which include precompact spaces and all the $\mathbb{R}^n$ spaces (note that σ-precompact spaces are always separable).

Theorem 40. *If M is σ-precompact, then all Banach spaces isometrically contained in $\mathrm{SNA}(M)$ are separable and isomorphic to polyhedral spaces.*

As for positive results, we have mentioned that $\mathrm{SNA}([0, 1])$ contains c_0 isometrically. In fact, this can be extended to a wide class of metric spaces that includes all normed spaces.

Proposition 41. *If M is a metric space containing $[0, 1]$ isometrically, then $\mathrm{SNA}(M)$ contains c_0 isometrically.*

Actually, it is possible to see that for all metric spaces M with an infinite amount of non-isolated points, $\mathrm{SNA}(M)$ contains c_0 isometrically. But even in all the spaces M without this property that we were able to study, it seemed always possible to find c_0 in $\mathrm{SNA}(M)$ isomorphically. This motivated us to ask if this was always the case (see [84, Questions 1 and 2]), that is: if M is infinite, does $\mathrm{SNA}(M)$ always contain c_0 isomorphically? Recently, Avilés, Martínez-Cervantes, Rueda Zoca, and Tradacete answered this question in the positive by means of an elegant case distinction and with the help of Ramsey's theorem.

Theorem 42 ([15, Main Theorem])**.** *Let M be an infinite complete pointed metric space. Then $\mathrm{SNA}(M)$ contains c_0 isomorphically.*

As for the isometric embedding of c_0 in $\mathrm{SNA}(M)$, the authors showed in [15, Lemma 3.1] that if the involved metric space satisfies a certain geometrical property (this is satisfied, for instance, for metric spaces with an infinite amount of non-isolated points, and for discrete metric spaces that are not uniformly discrete), then $\mathrm{SNA}(M)$ contains c_0 isometrically. For the rest of metric spaces, they left the following as an open question (see [15, Remark 3.6]): if M is infinite, does $\mathrm{SNA}(M)$ contain c_0 isometrically? In [49], we provide a definitive answer to this question. To do so, we first find the following result, which slightly improves the conditions from [15, Lemma 3.1].

Lemma 43. *Let Γ be a nonempty index set. Let M be a pointed metric space such that there exist two sets $\{x_\gamma\}_{\gamma\in\Gamma}, \{y_\gamma\}_{\gamma\in\Gamma} \subset M$ with $x_\gamma \neq y_\gamma$, $x_\alpha \neq x_\beta$ for $\gamma, \alpha, \beta \in \Gamma$, $\alpha \neq \beta$. If $d(x_\alpha, x_\beta) \geqslant d(x_\alpha, y_\alpha) + d(x_\beta, y_\beta)$ for every $\alpha \neq \beta \in \Gamma$, then there is a linear subspace of* $\mathrm{SNA}(M)$ *isometric to* $c_0(\Gamma)$.

With the help of this result and some other technical lemmas, we are able to solve the question from [15, Remark 3.6] in the negative. We provide two different counterexamples with different behaviours. Given a point $x \in M$, we define its *separation radius* as the quantity $R(x) := \inf\{d(x,y)\colon\ y \in M\backslash\{x\}\}$, and we say that x *attains its separation radius* if that infimum is actually a minimum. We summarize our results.

Theorem 44. *There exist infinite metric spaces M_1 and M_2 such that* $\mathrm{SNA}(M_1)$ *and* $\mathrm{SNA}(M_2)$ *do not contain c_0 isometrically and such that M_1 is bounded with no points attaining their separation radii and M_2 is proper unbounded with all of its points attaining their separation radii.*

Interestingly enough, there also exist infinite uniformly discrete complete pointed metric spaces M with c_0 isometrically contained in $\mathrm{SNA}(M)$ in such a way that no point attains its separation radius, or that every point attains its separation radius. As for metric spaces which are not uniformly discrete, $\mathrm{SNA}(M)$ always contains c_0 isometrically.

Theorem 45. *Let M be an infinite non uniformly discrete metric space. Then, the set $\mathrm{SNA}(M)$ contains an isometric copy of c_0.*

Finally, for the non-separable setting, using Lemma 43 and a result inspired by [71, Proposition 3], we are able to get the following positive result. Recall that for a metric space M, M' denotes the set of cluster points of M, and $\mathrm{dens}(M)$ denotes the density character of M.

Theorem 46. *Let M be a pointed metric space such that $\mathrm{dens}(M') = \Gamma$ for some infinite cardinal Γ. Then there is a linear subspace of $\mathrm{SNA}(M)$ that is isometrically isomorphic to $c_0(\Gamma)$.*

We conclude the present dissertation with a Conclusions chapter (see page 277), which includes some remarks and open questions. At the end of the document, there is an extensive list of references (see page 287), followed by a glossary of concepts and notations.

Resumen (Castellano)

En esta disertación, estudiamos varias clases de aplicaciones que alcanzan o no su norma o su radio numérico naturalmente asociado. En particular, hablaremos sobre operadores y aplicaciones bilineales entre espacios normados, tensores proyectivos, operadores nucleares y funciones Lipschitz.

El contenido principal de este documento está organizado en 5 capítulos, donde cubrimos el contenido de los trabajos publicados [42, 43, 59, 84] y el trabajo enviado [49], y se extraerán algunas notas de antecedentes del estudio publicado [40]. En esta sección, resumiremos en castellano el contenido de cada capítulo (consulte la sección Summary de la página 15 y la sección Resum de la página 65 para las respectivas traducciones de este resumen al inglés y al valenciano).

Resumen del Capítulo 1

El Capítulo 1 sirve como introducción. En la Sección 1.1, hacemos un comentario importante sobre cómo se puede leer este documento. En la Sección 1.2, establecemos notación y conceptos básicos que se usarán a lo largo del texto. En la Sección 1.3, incluimos los antecedentes históricos necesarios para motivar el trabajo. Finalmente, en la Sección 1.4, explicamos brevemente la estructura del documento y los contenidos de los próximos capítulos. Comentemos algunos hechos históricos para motivar mejor las otras secciones.

Inspirados por el trabajo de James 1957/1963 ([74, 75]), Bishop y Phelps 1961 ([17]), Lindenstrauss 1963 ([93]), Bollobás 1970 ([18]) y muchos otros (véase [2]) sobre la densidad de operadores que alcanzan su norma, en 2008, Acosta, Aron, García y Maestre introdujeron y estudiaron la propiedad de Bishop-Phelps-Bollobás (véase [5]).

Definición 1 ([5])**.** Un par de espacios de Banach (X,Y) tiene la *propiedad de Bishop-Phelps-Bollobás* (abreviada *BPBp*) si dado $\varepsilon \in (0,1)$, existe $\eta(\varepsilon) > 0$ tal que si $T \in \mathcal{L}(X, Y)$ y $x \in S_X$ satisfacen $\|T\| = 1$ y $\|T(x)\| > 1-\eta(\varepsilon)$, existe $S \in \mathcal{L}(X, Y)$ y $y \in S_X$ tal que $\|S\| = \|S(y)\| = 1$, $\|x - y\| < \varepsilon$ y $\|S - T\| < \varepsilon$.

Notemos que si los espacios de Banach X y Y satisfacen la BPBp, entonces, en particular, $\mathrm{NA}(X,Y)$ es denso en $\mathcal{L}(X,Y)$, aunque el recíproco no siempre es cierto. La BPBp ha sido estudiada por muchos autores recientemente (véanse los estudios [3, 40] para una exposición completa de resultados sobre la BPBp hasta 2022). Varias variaciones interesantes de la BPBp también se han introducido y estudiado últimamente a base de hacer cambios específicos a la definición 1, como la $\mathbf{L}_{o,o}$ (la BPBp pero para cada T fijado previamente, se encuentra un $\eta(\varepsilon, T)$ dependiendo también de T, y además $S = T$).

Este amplio estudio de los operadores que alcanzan sus normas también se ha extendido a otros tipos de aplicaciones y normas. Por ejemplo, las aplicaciones multilineales, polinomios homogéneos, funciones holomorfas, operadores compactos y aplicaciones Lipschitz que alcanzan sus normas se han estudiado durante mucho tiempo, y lo mismo ocurre con los operadores que alcanzan su radio numérico. Naturalmente, propiedades tipo BPBp también se han introducido y estudiado para estos contextos. Nos referimos nuevamente a la Sección 1.3 y al estudio [40] para más información sobre estas y más propiedades. Este escenario es el punto de partida de esta disertación.

Resumen del Capítulo 2

Los contenidos de este capítulo han sido publicados en

[42] S. Dantas, M. Jung, and Ó. Roldán, Norm-attaining operators which satisfy a Bollobás type theorem, *Banach J. Math. Anal.* **15**(2) (2021), Paper No. 40, 26 pp.

Inspirados por la $\mathbf{L}_{o,o}$ y sus muchas aplicaciones, el Capítulo 2 está dedicado a estudiar una clase $\mathcal{A}_{\|\cdot\|}(X,Y) \subset \mathrm{NA}(X,Y)$ de operadores que cumplen una propiedad como la $\mathbf{L}_{o,o}$, esto es, tales que si casi alcanzan su norma en x, la alcanzan en un punto cercano x_0. La clase análoga para el radio numérico también se introduce y estudia. La definición formal de estos conjuntos es la siguiente.

Definición 2. Sean X, Y espacios de Banach sobre $\mathbb{K} = \mathbb{R}$ o $\mathbb{C}$.

(i) $\mathcal{A}_{\|\cdot\|}(X,Y)$ representa el conjunto de todos los operadores norma-alcanzantes $T \in \mathcal{L}(X,Y)$ con $\|T\| = 1$ tales que si $\varepsilon > 0$, existe $\eta(\varepsilon, T) > 0$ de forma que siempre que $x \in S_X$ cumpla $\|T(x)\| > 1 - \eta(\varepsilon, T)$, existe $x_0 \in S_X$ tal que $\|T(x_0)\| = 1$ y $\|x_0 - x\| < \varepsilon$.

(ii) $\mathcal{A}_{\mathrm{nu}}(X)$ representa el conjunto de operadores que alcanzan su radio numérico, $T \in \mathcal{L}(X,X)$ con $\nu(T) = 1$ tales que si $\varepsilon > 0$, existe $\eta(\varepsilon, T) > 0$ tal que siempre que $(x, x^*) \in \Pi(X)$ cumpla que $|x^*(T(x))| > 1 - \eta(\varepsilon, T)$, existe $(x_0, x_0^*) \in \Pi(X)$ tal que $|x_0^*(T(x_0))| = 1$, $\|x_0 - x\| < \varepsilon$, y $\|x_0^* - x^*\| < \varepsilon$.

En la Sección 2.2, se presenta una selección de resultados y ejemplos sobre las clases $\mathcal{A}_{\|\cdot\|}$ y $\mathcal{A}_{\mathrm{nu}}$. Para espacios de Banach de dimensión finita, utilizando la compacidad de la bola unidad y el hecho de que cada operador alcanza su norma y radio numérico, obtenemos la siguiente caracterización positiva.

Teorema 3. *Sea X un espacio de Banach de dimensión finita.*

(i) $\mathcal{A}_{\|\cdot\|}(X,Y) = \{T \in \mathcal{L}(X,Y) : \|T\| = 1\}$ *para cualquier espacio de Banach Y,*

(ii) $\mathcal{A}_{\mathrm{nu}}(X) = \{T \in \mathcal{L}(X,X) : \nu(T) = 1\}$.

Para funcionales, obtenemos resultados positivos para una clase extensa de espacios, pero también negativos para otros.

Teorema 4. *Sea X un espacio de Banach sobre $\mathbb{K}$.*

(i) $\mathrm{NA}(c_0,\mathbb{K}) \cap S_{\ell_1} = \mathcal{A}_{\|\cdot\|}(c_0,\mathbb{K})$.

(ii) *Si X es uniformemente convexo, entonces* $S_{X^*} = \mathcal{A}_{\|\cdot\|}(X,\mathbb{K})$.

(iii) *Existe* $x^* \in \mathrm{NA}(\ell_1,\mathbb{K}) \cap S_{\ell_\infty}$ *tal que* $x^* \notin \mathcal{A}_{\|\cdot\|}(\ell_1,\mathbb{K})$.

(iv) *Existe* $x^* \in \mathrm{NA}(\ell_\infty,\mathbb{K}) \cap S_{\ell_\infty^*}$ *tal que* $x^* \notin \mathcal{A}_{\|\cdot\|}(\ell_\infty,\mathbb{K})$.

En cuanto a operadores generales sobre un espacio de Banach X, notemos que toda isometría está en $\mathcal{A}_{\|\cdot\|}(X,X)$, pero este no es siempre el caso para $\mathcal{A}_{\mathrm{nu}}(X)$. De hecho, incluso en el contexto de los espacios de Hilbert como $X = \ell_2$, existen (véase Ejemplo 2.2.5) operadores en $\mathcal{A}_{\|\cdot\|}(X,X) \cap \mathcal{A}_{\mathrm{nu}}(X)$, en $\mathcal{A}_{\|\cdot\|}(X,X) \backslash \mathcal{A}_{\mathrm{nu}}(X)$, en $\mathcal{A}_{\mathrm{nu}}(X) \backslash \mathcal{A}_{\|\cdot\|}(X,X)$, y operadores que no están en $\mathcal{A}_{\|\cdot\|}(X,X) \cup \mathcal{A}_{\mathrm{nu}}(X)$ a pesar de estar en $\{T \in \mathrm{NA}(X,X) \cap \mathrm{NRA}(X) \colon \nu(T) = \|T\| = 1\}$. Todo esto agrega algo de complejidad a nuestra pregunta.

Una clase importante de operadores para los que podemos obtener un resultado positivo son los operadores compactos. El siguiente resultado muestra que bajo algunas hipótesis sobre los espacios involucrados, todo operador compacto con norma 1 (y radio numérico 1) está en $\mathcal{A}_{\|\cdot\|}(X,Y)$ (y en $\mathcal{A}_{\mathrm{nu}}(X)$).

Teorema 5. *Sea X un espacio reflexivo con la propiedad de Kadec-Klee.*

(i) $S_{\mathcal{K}(X,Y)} \subset \mathcal{A}_{\|\cdot\|}(X,Y)$ *para todo espacio de Banach Y.*

(ii) $\{T \in \mathcal{K}(X,X) \colon \nu(T) = \|T\| = 1\} \subset \mathcal{A}_{\mathrm{nu}}(X)$ *si X es diferenciable Fréchet.*

En particular, mostramos que bajo ciertas hipótesis sobre el espacio de Banach X, todo operador compacto $T \in \mathcal{K}(X,X)$ con $\nu(T) = \|T\| = 1$ alcanza su radio numérico. Notemos que si X es un espacio de Banach de dimensión infinita, la inclusión en (ii) debe ser estricta, ya que la identidad siempre está en $\mathcal{A}_{\mathrm{nu}}(X)$, pero no es compacta. También obtenemos la siguiente consecuencia inmediata del resultado anterior.

Corolario 6. *Sea X un espacio de Banach reflexivo con la propiedad de Kadec-Klee y sea H un espacio de Hilbert.*

(i) *Si Y tiene la propiedad de Schur, entonces* $\mathcal{A}_{\|\cdot\|}(X,Y) = S_{\mathcal{L}(X,Y)}$.

(ii) $\{T \in \mathcal{K}(H,H) \colon \nu(T) = \|T\| = 1\} \subset \mathcal{A}_{\mathrm{nu}}(H)$.

Si eliminamos algunas de las hipótesis sobre los espacios en el Teorema 5, ambos enunciados dejan de ser ciertos en general (véanse los operadores de (2.2.3) y (2.2.5)). Además, el Teorema 5 y el Corolario 6 fallan en el contexto no compacto, como veremos en la Sección 2.3.

La demostración del siguiente resultado (inspirado en [1, Ejemplo 1.9]) nos proporciona una amplia clase de operadores compactos $T \in \mathcal{A}_{\mathrm{nu}}(H)$

tales que $1 = \nu(T) < \|T\|$ y, por tanto, ejemplos de operadores que pertenecen a $\mathcal{A}_{\mathrm{nu}}(H)$ pero no a $\mathcal{A}_{\|\cdot\|}(H,H)$ (véase la demostración de la Proposición 2.2.9 para más detalles). En este caso obtenemos $\mathcal{A}_{\mathrm{nu}}$ en un sentido uniforme, donde η solo depende de ε.

Proposición 7. *Sea H un espacio de Hilbert separable real de dimensión infinita. Entonces existe $T \in \mathcal{L}(H,H)$ tal que*

(i) *T es un operador compacto.*

(ii) *$1 = \nu(T) < \|T\|$ y T alcanza su radio numérico.*

(iii) *Dado $\varepsilon > 0$, existe $\eta(\varepsilon) > 0$ tal que si $x_0 \in S_H$ cumple que*

$$|\langle T(x_0), x_0\rangle| > 1 - \eta(\varepsilon),$$

existe $x_1 \in S_H$ tal que $\nu(T) = \langle T(x_1), x_1\rangle = 1$ y $\|x_1 - x_0\| < \varepsilon$.

En particular, $T \in \mathcal{A}_{\mathrm{nu}}(H)$ y $T \notin \mathcal{A}_{\|\cdot\|}(H,H)$.

Los operadores de (2.2.2), (2.2.3), y (2.2.5) muestran que, en general, no hay relación entre que $T \in \mathcal{A}_{\|\cdot\|}(X,Y)$ y $T^* \in \mathcal{A}_{\|\cdot\|}(Y^*,X^*)$. Sin embargo, si añadimos condiciones extra a los espacios X e Y, obtenemos el siguiente resultado.

Proposición 8. *Sean X, Y espacios de Banach y $T \in \mathcal{L}(X,Y)$.*

(i) *Si Y es uniformemente suave, si $T \in \mathcal{A}_{\|\cdot\|}(X,Y)$, entonces $T^* \in \mathcal{A}_{\|\cdot\|}(Y^*,X^*)$.*

(ii) *Si X es uniformemente convexo, si $T^* \in \mathcal{A}_{\|\cdot\|}(Y^*,X^*)$, entonces $T \in \mathcal{A}_{\|\cdot\|}(X,Y)$.*

(iii) *Supongamos que X es reflexivo. Entonces, $T \in \mathcal{A}_{\mathrm{nu}}(X)$ si y sólo si $T^* \in \mathcal{A}_{\mathrm{nu}}(X^*)$.*

Notemos que no podemos eliminar la suavidad y convexidad uniformes en (i) y (ii) (véanse de nuevo (2.2.2), (2.2.3), y (2.2.5)). En c_0, obtenemos el siguiente resultado relacionado con (iii).

Proposición 9. *Sea $T \in \mathcal{A}_{nu}(c_0)$ tal que el rango de $T^* \in \mathcal{L}(\ell_1, \ell_1)$ está en* $\operatorname{span}\{e_1^*, \ldots, e_N^*\}$ *para algún $N \in \mathbb{N}$. Entonces, $T^* \in \mathcal{A}_{nu}(\ell_1)$.*

En la Sección 2.3, se dará una caracterización completa de todos los operadores diagonales que pertenecen a $\mathcal{A}_{\|\cdot\|}(X, X)$ ($X = c_0$ o ℓ_p, $1 \leqslant p \leqslant \infty$), a $\mathcal{A}_{\mathrm{nu}}(X)$ ($X = c_0$ o ℓ_p, $1 \leqslant p < \infty$), a $\mathcal{A}_{\|\cdot\|}(c_0, \ell_p)$ ($1 < p < \infty$) y a $\mathcal{A}_{\|\cdot\|}(\ell_p, c_0)$ ($1 < p < \infty$). Podemos resumir esos resultados como sigue.

Teorema 10. *Sea (X, Y) igual a (c_0, c_0), (ℓ_p, ℓ_p) ($1 \leqslant p \leqslant \infty$) o (ℓ_p, c_0) ($1 \leqslant p < \infty$). Sea $T : X \to Y$ el operador diagonal de norma 1 asociado a la sucesión acotada de complejos $\{\alpha_n\}_{n=1}^{\infty}$. Entonces, $T \in \mathcal{A}_{\|\cdot\|}(X, Y)$ si y sólo si se dan las dos condiciones siguientes:*

1. *Existe $n_0 \in \mathbb{N}$ tal que $|\alpha_{n_0}| = 1$.*
2. *Si $J = \{n \in \mathbb{N} : |\alpha_n| = 1\}$, entonces $J = \mathbb{N}$ o $\sup_{n \in \mathbb{N} \setminus J} |\alpha_n| < 1$.*

Teorema 11. *Dado $1 \leqslant p < \infty$, sea $T : c_0 \to \ell_p$ el operador diagonal de norma 1 asociado a la sucesión acotada de complejos $\{\alpha_n\}_{n=1}^{\infty}$. Entonces, $T \in \mathcal{A}_{\|\cdot\|}(c_0, \ell_p)$ si y sólo si existe $N \in \mathbb{N}$ tal que $\alpha_n = 0$ para todo $n > N$.*

Teorema 12. *Sea $X = c_0$ o ℓ_p, $1 \leqslant p < \infty$. Sea $T : X \to X$ el operador diagonal de radio numérico 1 asociado a la sucesión acotada de complejos $\{\alpha_n\}_{n=1}^{\infty}$. Entonces, $T \in \mathcal{A}_{nu}(X)$ si y sólo si se dan las dos condiciones siguientes:*

1. *Existe $n_0 \in \mathbb{N}$ tal que $|\alpha_{n_0}| = 1$.*
2. *Si $J = \{n \in \mathbb{N} : |\alpha_n| = 1\}$, entonces el cardinal de $\{\alpha_n : n \in J\}$ es finito y $\sup_{n \in \mathbb{N} \setminus J} |\alpha_n| < 1$ cuando $J \neq \mathbb{N}$.*

En particular, si $\{\alpha_n\}_{n=1}^{\infty} \subset \mathbb{R}$*,* $T \in \mathcal{A}_{nu}(X)$ *si y sólo si* $T \in \mathcal{A}_{\|\cdot\|}(X, X)$*.*

Como consecuencia, para las proyecciones canónicas sobre los espacios de sucesiones clásicos, $P_N(x) := (x(1), x(2), \ldots, x(N), 0, \ldots)$, obtenemos lo siguiente.

Corolario 13. *Sea* $N \in \mathbb{N}$ *dado. Si* $X = c_0$ *o* ℓ_p*,* $1 \leqslant p \leqslant \infty$*, entonces* $P_N \in \mathcal{A}_{\|\cdot\|}(X, X) \cap \mathcal{A}_{\mathrm{nu}}(X)$*.*

Finalmente, en la Sección 2.4, estudiamos la relación entre $\mathcal{A}_{\|\cdot\|}(W, Z)$ y $\mathcal{A}_{\mathrm{nu}}(W \oplus Z)$ para algunas sumas directas de los espacios de Banach W y Z. Dados dos espacios de Banach X_1 y X_2, consideremos las aplicaciones $P_i \in \mathcal{L}(X_1 \oplus X_2, X_i)$ tales que $P_i(x_1, x_2) := x_i$, $i = 1, 2$, y $\iota_j \in \mathcal{L}(X_j, X_1 \oplus X_2)$ tales que $\iota_i(x) := xe_i$, donde $e_1 = (1, 0)$ y $e_2 = (0, 1)$. Para los espacios de Banach W y Z, si tenemos un operador $T \in \mathcal{L}(W, Z)$, entonces existe una forma sencilla de definir $\widetilde{T} \in \mathcal{L}(W \oplus Z)$: consideremos $\widetilde{T} := \iota_2 \circ T \circ P_1$, es decir, $\widetilde{T}(w, z) = (0, T(w))$ para cada $(w, z) \in W \oplus Z$. Por el contrario, podemos definir un proceso pseudo-inverso de la siguiente manera: si tenemos un operador $S \in \mathcal{L}(W \oplus Z, W \oplus Z)$, entonces podemos considerar $\breve{S} \in \mathcal{L}(W, Z)$ definido como $\breve{S} := P_2 \circ S \circ \iota_1$, es decir, $\breve{S}(w) = (P_2 \circ S)(w, 0)$ para cada $w \in W$. Obtenemos los siguientes resultados.

Proposición 14. *Sean* W *y* Z *dos espacios de Banach, y sea* $T \in S_{\mathcal{L}(W,Z)}$*. Entonces,*

(i) *Si* $\widetilde{T} \in \mathcal{A}_{\mathrm{nu}}(W \oplus_s Z)$*, entonces* $T \in \mathcal{A}_{\|\cdot\|}(W, Z)$*, donde* $s = 1$ *o* $s = \infty$*.*

(ii) *Supongamos que* W *y* Z *son uniformemente suaves. Si* $T \in \mathcal{A}_{\|\cdot\|}(W, Z)$*, entonces* $\widetilde{T} \in \mathcal{A}_{\mathrm{nu}}(W \oplus_1 Z)$*.*

(iii) *Supongamos que* Z *es uniformemente convexo y* W *es uniformemente suave. Si* $T \in \mathcal{A}_{\|\cdot\|}(W, Z)$*, entonces* $\widetilde{T} \in \mathcal{A}_{\mathrm{nu}}(W \oplus_\infty Z)$*.*

Finalmente, notemos que (ii) y (iii) ya no se cumplen en general para espacios de Banach arbitrarios o para p-sumas si $1 < p < \infty$, y existe $S \in \mathcal{L}(W \oplus_s Z, W \oplus_s Z)$, con W y Z uniformemente suaves y uniformemente convexos, tal que $S \in \mathcal{A}_{\text{nu}}(W \oplus_s Z)$ pero con $\check{S} \notin \mathcal{A}_{\|\cdot\|}(W, Z)$, $s = 1$ o $s = \infty$ (véanse los comentarios tras las Proposiciones 2.4.1 y 2.4.4).

Resumen del Capítulo 3

Los contenidos de este capítulo han sido publicados en

[59] D. García, M. Maestre, M. Martín, and Ó. Roldán, On the compact operators case of the Bishop-Phelps-Bollobás property for numerical radius, *Results Math.* **76**(3) (2021), Paper No. 122, 23 pp.

En 2013, Guirao y Kozhushkina introdujeron y estudiaron en [69] una versión de la BPBp para el radio numérico. La definimos como sigue.

Definición 15 (Combinando [69, Definition 1.2] y [87, Definition 5])**.** Un espacio de Banach X tiene la propiedad débil de Bishop-Phelps-Bollobás para el radio numérico (abreviada weak BPBp-nu) si dado $\varepsilon > 0$, existe $\eta(\varepsilon) > 0$ tal que si $T \in \mathcal{L}(X, X)$ con $\nu(T) = 1$ y $(x, x^*) \in \Pi(X)$ cumplen que $|x^*(T(x))| > 1 - \eta(\varepsilon)$, existen $S \in \mathcal{L}(X, X)$ y $(y, y^*) \in \Pi(X)$ tales que

$$\nu(S) = |y^*(S(y))|, \quad \|x - y\| < \varepsilon, \quad \|x^* - y^*\| < \varepsilon, \quad \text{y} \quad \|T - S\| < \varepsilon.$$

Si, además, S se puede elegir siempre con $\nu(S) = 1$, decimos que X tiene la propiedad de Bishop-Phelps-Bollobás para el radio numérico (abreviada BPBp-nu).

Desde ese trabajo ([69]), muchos autores han obtenido múltiples resultados sobre la BPBp-nu (véanse [3, Section 6] y [40, Section 2.7] para exposiciones de los principales resultados obtenidos sobre en esta dirección). En 2018, Dantas, García, Maestre y Martín, introdujeron y estudiaron la BPBp adaptada a los operadores compactos (véase [39]). La BPBp-nu y la BPBp para operadores compactos motivaron a introducir y profundizar sobre la BPBp-nu para operadores compactos (considérese la Definición 15, pero con $T \in \mathcal{K}(X, X)$ y $S \in \mathcal{K}(X, X)$). Al explorar las pruebas existentes sobre BPBp-nu y hacer pequeñas adaptaciones, obtenemos una primera lista de espacios que satisfacen la BPBp-nu para operadores compactos. Éste es el objetivo de la Sección 3.2

Ejemplos 16. Los siguientes espacios tienen la BPBp-nu para operadores compactos: espacios de dimensión finita ([87, Proposition 2]), $c_0(\Gamma)$ y $\ell_1(\Gamma)$ para cualquier conjunto índice Γ ([69, Corollaries 3.3 and 4.2]), y $L_1(\mu)$ para cualquier medida μ ([7, Corollary 2.1] y [87, Theorem 9]).

Ahora, adaptando las nociones de índice numérico y segundo índice numérico al contexto de los operadores compactos, n_K y n'_K, respectivamente, y adaptando los resultados de [87] y [89], mostramos que si un espacio de Banach X es uniformemente convexo y uniformemente suave, entonces tiene la weak BPBp-nu para operadores compactos, y si $n_K(X) > 0$ o $n'_K(X) > 0$, entonces la weak BPBp-nu para operadores compactos es equivalente a la BPBp-nu para operadores compactos. En particular, mostramos que para cada medida μ y cada $1 < p < \infty$, $L_p(\mu)$ tiene la BPBp-nu para operadores compactos.

En [34, Proposition 4.3] se mostró que si un espacio de Banach X tiene la BPBp-nu para operadores compactos, entonces todo sumando absoluto de X de tipo 1 e ∞ también tiene esta propiedad, y con la misma función η. Esto nos permite llevar la propiedad de algunos espacios a algunas proyecciones de esos espacios. Es natural preguntarse si se puede decir algo en sentido opuesto. En [39, Lemma 2.1] se presentó una herramienta que, en particular, nos permite llevar la BPBp para operadores compactos desde unas proyecciones de un espacio al propio espacio. Para obtener un resultado análogo para el radio numérico, es necesario controlar todo tanto en el espacio como en su dual. El resultado más general obtenido esta dirección es el siguiente lema.

Lema 17. *Sea X un espacio de Banach con $n_K(X) > 0$. Supongamos que hay una función $\eta\colon (0,1) \longrightarrow (0,1)$ tal que dados $\delta > 0$, $x_1^*, \ldots, x_n^* \in B_{X^*}$ y $x_1, \ldots, x_\ell \in B_X$, podemos encontrar operadores de norma 1 $\widetilde{P}\colon X \longrightarrow \widetilde{P}(X)$, $i\colon \widetilde{P}(X) \longrightarrow X$ tales que para $P := i \circ \widetilde{P}\colon X \longrightarrow X$, se cumplen estas condiciones:*

(i) $\|P^*(x_j^*) - x_j^*\| < \delta$, *para* $j = 1, \ldots, n$.

(ii) $\|P(x_j) - x_j\| < \delta$, *para* $j = 1, \ldots, \ell$.

(iii) $\widetilde{P} \circ i = \mathrm{Id}_{\widetilde{P}(X)}$.

(iv) $\widetilde{P}(X)$ *tiene BPBp-nu para operadores compactos con la función* η.

(v) *O bien P es una proyección absoluta e i es la inclusión natural, o* $n_K(\widetilde{P}(X)) = n_K(X) = 1$.

Entonces, X tiene la BPBp-nu para operadores compactos.

A lo largo de la Sección 3.3, el Lema 17 se usa para mostrar que si un espacio de Banach X con $n_K(X) > 0$ puede ser adecuadamente proyectado en alguna red de espacios que tienen la BPBp-nu para operadores compactos con una función común η, entonces a veces es posible mostrar que X también tiene esa propiedad (véase Proposición 3.3.2). Esto se utiliza para obtener los siguientes dos resultados.

Corolario 18. *Sea X un espacio de Banach con $n_K(X) > 0$. Entonces las siguientes afirmaciones equivalen.*

(i) *El espacio $c_0(X)$ tiene la BPBp-nu para opeadores compactos.*

(ii) *Hay una función $\eta : (0,1) \longrightarrow (0,1)$ tal que todos los $\ell_\infty^n(X)$, con $n \in \mathbb{N}$, tienen la BPBp-nu para operadores compactos con η.*

Además, si X es de dimensión finita, estas propiedades se dan cuando $c_0(X)$ o $\ell_\infty(X)$ tienen la BPBp-nu.

Corolario 19. *Sea X un espacio de Banach tal que X^* es isométricamente isomorfo a ℓ_1. Entonces X tiene la BPBp-nu para operadores compactos.*

En la Sección 3.4, presentamos una serie de herramientas topológicas que permiten cubrir convenientemente un espacio Hausdorff localmente compacto L con conjuntos más pequeños y encontrar una partición de la unidad adecuada subordinada a esos conjuntos. Esto nos posibilita proyectar el espacio $C_0(L)$ en algún espacio ℓ_∞^p ($p \in \mathbb{N}$) de tal manera que nos permite usar Lemma 17. Esta propiedad de aproximación fuerte que obtenemos en $C_0(L)$ y su dual se resume en el siguiente resultado.

Teorema 20. *Sea L un espacio localmente compacto Hausdorff. Dados $\{f_1, \ldots, f_\ell\} \subset C_0(L)$ con $\|f_j\| \leqslant 1$ para $j = 1, \ldots, \ell$, y $\{\mu_1, \ldots, \mu_n\} \subset C_0(L)^*$ con $\|\mu_j\| \leqslant 1$ para $j = 1, \ldots, n$, para cada $\varepsilon > 0$ existe una proyección de norma 1 $P\colon C_0(L) \longrightarrow C_0(L)$ tal que:*

(1) $\|P^*(\mu_j) - \mu_j\| < \varepsilon$, *para* $j = 1, \ldots, n$,

(2) $\|P(f_j) - f_j\| < \varepsilon$, *para* $j = 1, \ldots, \ell$,

(3) $P(C_0(L))$ *es isométricamente isomorfo a* ℓ_∞^p *para algún* $p \in \mathbb{N}$.

Finalmente, como consecuencia, obtenemos lo siguiente.

Teorema 21. *Si L es un espacio localmente compacto Hausdorff, entonces $C_0(L)$ tiene la BPBp-nu para operadores compactos.*

En particular, todo espacio $C(K)$ (K compacto Hausdorff) y todo espacio $L_\infty(\mu)$ (μ cualquier medida) tiene la BPBp-nu para operadores compactos. Notemos que a día de hoy sigue siendo un problema abierto si todos los espacios $C(K)$ tienen la BPBp-nu, y hasta ahora solo se han resuelto casos particulares en el caso real (véase [13]), pero para operadores compactos obtenemos una respuesta definitiva para estos espacios.

Resumen del Capítulo 4

Los contenidos de este capítulo han sido publicados en

[43] S. Dantas, M. Jung, Ó. Roldán, and A. Rueda Zoca, Norm-attaining tensors and nuclear operators, *Mediterr. J. Math.* **19**(1) (2022), Paper No. 38, 27 pp.

En el Capítulo 4, se introducen y estudian las nociones de alcanzamiento de normas para tensores proyectivos en $X\widehat{\otimes}_\pi Y$ y operadores nucleares en $\mathcal{N}(X,Y)$, para espacios de Banach X y Y. Para motivar por qué tales preguntas pueden ser interesantes, recordemos que dos de las principales preguntas históricas sobre los operadores norma-alcanzantes son las siguientes:

1. ¿Es $\mathcal{K}(X,Y) \subset \overline{\mathrm{NA}(X,Y)}$ en general?

2. ¿Es $\mathcal{F}(X,Y) \subset \overline{\mathrm{NA}(X,Y)}$ en general?

La primera pregunta fue respondida negativamente por Miguel Martín en 2014 (véase [97]). La segunda pregunta permanece abierta, y muchos la consideran actualmente como la principal pregunta abierta en la teoría de los operadores que alcanzan normas. Notemos que los operadores nucleares se encuentran entre los operadores de rango finito y los operadores compactos, y los tensores proyectivos están estrechamente relacionados con ellos y tienen muchas aplicaciones en múltiples campos dentro del análisis funcional. Otro factor importante para motivar este estudio es el hecho de que si fuera cierto que para todo espacio de Banach de dimensión finita X todo operador nuclear en $\mathcal{N}(X, Y)$ alcanza su norma nuclear, entonces obtendríamos una respuesta afirmativa a la segunda pregunta de antes. Sin embargo, la suposición resulta ser falsa, como veremos más adelante.

En este capítulo, las identificaciones isométricas $(X \widehat{\otimes}_\pi Y)^* = \mathcal{L}(X, Y^*) = \mathcal{L}(Y, X^*) = \mathcal{B}(X \times Y, \mathbb{K})$ se usarán implícitamente. Notemos también que si X^* o Y tiene la propiedad de aproximación, entonces $X^* \widehat{\otimes}_\pi Y = \mathcal{N}(X, Y)$ (véase, por ejemplo, [107, Corollary 4.8]). Introducimos a continuación las nociones de alcanzamiento de norma en estos contextos.

Definición 22. Sean X, Y dos espacios de Banach. Decimos que

(i) $z \in X \widehat{\otimes}_\pi Y$ *alcanza su norma proyectiva* si existe una sucesión acotada $(x_n, y_n) \subset X \times Y$ con $\sum_{n=1}^\infty \|x_n\| \|y_n\| < \infty$ tal que $z = \sum_{n=1}^\infty x_n \otimes y_n$ y $\|z\|_\pi = \sum_{n=1}^\infty \|x_n\| \|y_n\|$. En este caso, decimos que z es un *tensor que alcanza su norma*, o $z \in \operatorname{NA}_\pi(X \widehat{\otimes}_\pi Y)$.

(ii) $T \in \mathcal{N}(X, Y)$ *alcanza su norma nuclear* si existe una sucesión acotada $(x_n^*, y_n) \subset X^* \times Y$ con $\sum_{n=1}^\infty \|x_n^*\| \|y_n\| < \infty$ tal que $T = \sum_{n=1}^\infty x_n^* \otimes y_n$ y $\|T\|_\mathcal{N} = \sum_{n=1}^\infty \|x_n^*\| \|y_n\|$. En este caso, decimos que T es *operador nuclear que alcanza su norma*, o $T \in \operatorname{NA}_\mathcal{N}(X, Y)$.

En la Sección 4.2, se obtienen los primeros resultados de alcanzamiento de norma en este contexto. Empezamos encontrando dos caracterizaciones técnicas que nos permitan afirmar que un tensor u operador nuclear alcanza su respectiva norma siempre que existan muchas formas bilineales que alcanzan sus normas en muchos puntos de una manera específica (véanse Teoremas 4.2.1 y 4.2.2). Con estos resultados en mente y el hecho de que los espacios de dimensión finita, c_0, ℓ_1 y los espacios de Hilbert tienen la propiedad de aproximación, obtenemos nuestra primera colección de resultados positivos.

Proposición 23. *Todo tensor proyectivo de* $X^*\widehat{\otimes}_\pi Y$ *y todo operador nuclear de* $\mathcal{N}(X,Y)$ *alcanza su norma respectiva si* X *e* Y *tienen dimensión finita, si* $X = Y$ *es un espacio de Hilbert complejo, o si* $X = c_0$*.*

Es interesante comparar el último ejemplo de arriba con la teoría clásica de los operadores que alcanzan normas: si $\mathrm{NA}(X,Y) = \mathcal{L}(X,Y)$ para cada espacio de Banach Y, entonces en particular X debe ser reflexivo por el Teorema de James. La proposición anterior nos motiva a preguntarnos si $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y) = X\widehat{\otimes}_\pi Y$ y $\mathrm{NA}_{\mathcal{N}}(X,Y) = \mathcal{N}(X,Y)$ se cumplen en general para cualesquiera espacios de Banach X y Y. Sin embargo, este no es el caso, como muestran los siguientes resultados.

Lema 24. *Sean* X, Y *espacios de Banach. Si* $B \in \mathcal{B}(X \times Y, \mathbb{K}) = (X\widehat{\otimes}_\pi Y)^*$ *alcanza su norma como funcional en un elemento del espacio* $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)$*, entonces* $B \in \mathrm{NA}_{\mathcal{B}}(X \times Y, \mathbb{K})$*.*

Proposición 25. *Sean* X, Y *espacios de Banach. Si* $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y) = X\widehat{\otimes}_\pi Y$*, entonces* $\overline{\mathrm{NA}_{\mathcal{B}}(X \times Y, \mathbb{K})}^{\|\cdot\|_{\mathcal{B}}} = \mathcal{B}(X \times Y, \mathbb{K})$ *(y por tanto, también se obtiene que* $\overline{\mathrm{NA}(X, Y^*)}^{\|\cdot\|} = \mathcal{L}(X, Y^*)$*).*

Hay muchos ejemplos conocidos de espacios de Banach X y Y que no cumplen que $\overline{\mathrm{NA}(X, Y^*)}^{\|\cdot\|} = \mathcal{L}(X, Y^*)$, por lo que existen tensores proyectivos que no alcanzan su norma proyectiva. Usando la propiedad de aproximación, también obtenemos operadores nucleares que no alcanzan su norma nuclear. El siguiente ejemplo es de particular interés, ya que muestra que no todos los tensores proyectivos u operadores nucleares alcanzan sus respectivas normas si se asume que sólo uno de los espacios de Banach es de dimensión finita, resolviendo negativamente uno de los factores que usamos para motivar este estudio.

Ejemplo 26. Sea $X = L_1(\mathbb{T})$, donde la circunferencia unidad $\mathbb{T}$ está equipada con la medida de Haar m, y sea Y el espacio de Hilbert de dimensión 2. Se muestra en [65, Remark 5.7.(2)] que existe $T \in \mathcal{B}(X \times Y, \mathbb{K})$ que alcanza su norma como funcional en $X \widehat{\otimes}_\pi Y$ pero no como operador de X en Y^* (y por tanto, tampoco como forma bilineal en $X \times Y$). Por el Lema 24, obtenemos que $\mathrm{NA}_\pi(X \widehat{\otimes}_\pi Y) \neq X \widehat{\otimes}_\pi Y$, y por tanto, $\mathrm{NA}_{\mathcal{N}}(Y, X) \neq \mathcal{N}(Y, X)$.

Dado que no todos los tensores proyectivos u operadores nucleares alcanzan su norma, es natural preguntarse ahora si tenemos resultados de densidad. En la Sección 4.3, proporcionamos algunos resultados de densidad positivos. Para conseguirlos, se utilizan dos enfoques. Primero, notemos que por las dos primeras caracterizaciones, para obtener muchos tensores proyectivos y operadores nucleares que alcancen sus normas, queremos tener muchas formas bilineales que alcancen sus normas en muchos puntos. La $\mathbf{L}_{o,o}$ asegura la existencia de muchos operadores que alcanzan sus normas en muchos puntos, y se puede adaptar a aplicaciones bilineales de la siguiente manera.

Definición 27. Decimos que $(X \times Y, Z)$ tiene la $\mathbf{L}_{o,o}$ para aplicaciones bilineales (o simplemente, $\mathbf{L}_{o,o,\mathcal{B}}$) si dados $\varepsilon > 0$ y $B \in \mathcal{B}(X \times Y, Z)$ con $\|B\|_{\mathcal{B}} = 1$, existe $\eta(\varepsilon, B) > 0$ tal que siempre que $(x, y) \in S_X \times S_Y$ cumple que $\|B(x, y)\| > 1 - \eta(\varepsilon, B)$, existe $(x_0, y_0) \in S_X \times S_Y$ tal que $\|B(x_0, y_0)\| = 1$, $\|x - x_0\| < \varepsilon$, y $\|y - y_0\| < \varepsilon$.

Obtenemos el siguiente resultado.

Proposición 28. *Sean X, Y espacios de Banach. Si $(X^* \times Y, \mathbb{K})$ tiene la $\mathbf{L}_{o,o,\mathcal{B}}$, entonces, $\overline{\mathrm{NA}_{\mathcal{N}}(X,Y)}^{\|\cdot\|_{\mathcal{N}}} = \mathcal{N}(X,Y)$. Si $(X \times Y, \mathbb{K})$ tiene la $\mathbf{L}_{o,o,\mathcal{B}}$, entonces, $\overline{\mathrm{NA}_{\pi}(X\widehat{\otimes}_{\pi}Y)}^{\|\cdot\|_{\pi}} = X\widehat{\otimes}_{\pi}Y$.*

En particular, notemos que las siguientes relaciones son conocidas, proporcionando resultados positivos de densidad.

Ejemplos 29 ([48])**.** Sean X, Y espacios de Banach.

(i) Si $\dim(X), \dim(Y) < \infty$, entonces $(X \times Y, Z)$ tiene la $\mathbf{L}_{o,o,\mathcal{B}}$ para todo espacio de Banach Z.

(ii) Si Y es uniformemente convexo, entonces $(X \times Y, \mathbb{K})$ tiene la $\mathbf{L}_{o,o,\mathcal{B}}$ si y sólo si (X, Y^*) tiene la $\mathbf{L}_{o,o}$.

(iii) Si $1 < p, q < \infty$, entonces $(\ell_p \times \ell_q, \mathbb{K})$ tiene la $\mathbf{L}_{o,o,\mathcal{B}}$ si y sólo si $p > q'$.

Sin embargo, la $\mathbf{L}_{o,o,\mathcal{B}}$ para formas bilineales es una propiedad muy restrictiva, ya que requiere que ambos espacios sean reflexivos para empezar, y también hay pares de espacios reflexivos sin la propiedad, como acabamos de ver. Por lo tanto, necesitamos un enfoque diferente para obtener más resultados positivos. Para usar lo que sabemos sobre el escenario de dimensión finita, sería conveniente tener buenos subespacios de nuestros espacios. Las normas proyectivas no respetan los subespacios en general, pero se comportan bien con los subespacios 1-complementados, por lo que nos interesa tener una propiedad que asegure la existencia de muchos subespacios 1-complementados adecuados de nuestros espacios. Por tanto, consideramos la propiedad π métrica.

Definición 30. Sea X un espacio de Banach. Decimos que X tiene la *propiedad π métrica* si dados $\varepsilon > 0$ y $\{x_1, \ldots, x_n\} \subset S_X$ una colección finita en la esfera, podemos encontrar un subespacio 1-complementado de dimensión finita $M \subset X$ tal que para cada $i \in \{1, \ldots, n\}$ existe $x_i' \in M$ con $\|x_i - x_i'\| < \varepsilon$.

El concepto anterior es en realidad equivalente a la propiedad de aproximación π métrica (una propiedad de aproximación donde los operadores aproximantes son todos proyecciones de norma 1), y esto nos permite encontrar muchos más ejemplos de espacios para los que tenemos densidad (véanse [23], [76] y [94] para más información sobre la propiedad π). Se cumplen las siguientes propiedades.

Teorema 31. *Sea Y un espacio uniformemente convexo o un espacio con la propiedad π métrica. Si X (respectivamente, X^*) tiene la propiedad π métrica, entonces $X \widehat{\otimes}_\pi Y = \overline{\mathrm{NA}_\pi(X \widehat{\otimes}_\pi Y)}^{\|\cdot\|_\pi}$ (respectivamente, $\mathcal{N}(X, Y) = \overline{\mathrm{NA}_{\mathcal{N}}(X, Y)}^{\|\cdot\|_{\mathcal{N}}}$).*

Ejemplo 32. Los siguientes espacios tienen la propiedad π métrica.

(i) Espacios de Banach con una descomposición finito dimensional con constante 1 (consecuentemente, todo espacio de Banach con base de Schauder se puede renormar para tener la propiedad π métrica),

(ii) $L_p(\mu)$ ($1 \leqslant p < \infty$, μ cualquier medida) y duales isométricos de L_1,

(iii) $X \oplus_a Y$, si X, Y tienen la propiedad π métrica y $|\cdot|_a$ es una norma absoluta,

(iv) $X = \left[\bigoplus_{n\in\mathbb{N}} X_n\right]_{c_0}$ o $\left[\bigoplus_{n\in\mathbb{N}} X_n\right]_{\ell_p}$, para $1 \leqslant p < \infty$, si X_n tiene la propiedad π métrica para todo n,

(v) $X \widehat{\otimes}_\pi Y$ y $X \widehat{\otimes}_\varepsilon Y$ cuando X, Y tienen la propiedad π métrica.

Esto muestra que en muchos espacios la densidad se cumple. Remitimos al trabajo reciente [41, Section 4] para más resultados de densidad positivos relacionados con la RNP, espacios duales y la propiedad π métrica (por ejemplo, si X^* y Y^* tienen la RNP y uno tiene la propiedad de aproximación, entonces $\overline{\mathrm{NA}_\pi(X^*\widehat{\otimes}_\pi Y^*)}^{\|\cdot\|_\pi} = X^*\widehat{\otimes}_\pi Y^*$, y si Z es cualquier espacio dual, entonces $\overline{\mathrm{NA}_\pi(c_0\widehat{\otimes}_\pi Z)}^{\|\cdot\|_\pi} = c_0\widehat{\otimes}_\pi Z$). En este punto, es natural preguntarse ahora si siempre tenemos densidad de tensores proyectivos u operadores nucleares que alcanzan normas. Sin embargo, a pesar de nuestra amplia colección de resultados positivos, en la Sección 4.4, obtenemos el siguiente resultado negativo para los tensores.

Teorema 33. *Sea $\mathcal{R}$ el espacio de Read. Existen subespacios X de c_0 y Y de $\mathcal{R}$ tales que el conjunto de tensores de $X\widehat{\otimes}_\pi Y^*$ que alcanzan sus normas proyectivas no es denso en $X\widehat{\otimes}_\pi Y^*$.*

Cabe señalar que la pregunta análoga para los operadores nucleares permanece abierta.

Finalmente, notemos que aunque no se sabe si todo operador de rango finito puede aproximarse por operadores que alcanzan su norma, la afirmación análoga para los tensores no se cumple en general.

Proposición 34. *Existen tensores de rango finito que no se pueden aproximar por tensores que alcanzan su norma proyectiva.*

Resumen del Capítulo 5

El Capítulo 5 tiene dos mitades bien diferenciadas.

La primera mitad de este capítulo ha sido publicada en

[84] V. Kadets and Ó. Roldán, Closed linear spaces consisting of strongly norm attaining Lipschitz mappings, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **116** (2022), Paper No. 162, 12 pp.

y la segunda mitad ha aparecido en el trabajo enviado

[49] S. Dantas, R. Medina, A. Quilis, and Ó. Roldán, On isometric embeddings into the set of strongly norm-attaining Lipschitz functions. Preprint available at arXiv:2208.02916.

En 2001, Godefroy preguntó si para cada espacio de Banach de dimensión infinita X, el conjunto de funcionales que alcanzan sus normas $\mathrm{NA}(X, \mathbb{K})$ siempre contiene espacios lineales de dimensión 2 (véase [64, Problem III]). Esta pregunta fue respondida negativamente por Rmoutil en 2017: si $\mathcal{R}$ es el renormamiento de c_0 de Read ([103]), entonces $\mathrm{NA}(\mathcal{R}, \mathbb{K})$ no contiene espacios lineales de dimensión 2. En el Capítulo 5, estudiamos la cuestión de la espacialidad análoga para el conjunto de funciones Lipschitz (reales) que alcanzan su norma fuertemente.

Sea M un espacio métrico "pointed", es decir, un espacio métrico con un punto distinguido 0. $\mathrm{Lip}_0(M)$ es el espacio de Banach de las funciones Lipschitz $f : M \to \mathbb{R}$ tales que $f(0) = 0$ dotado de la norma Lipschitz

$$\|f\| := \sup\left\{\frac{|f(y) - f(x)|}{d(x, y)} :\ x, y \in M,\ x \neq y\right\}.$$

Se dice que una función Lipschitz $f \in \mathrm{Lip}_0(M)$ alcanza su norma fuertemente si existen $x, y \in M$, $x \neq y$, tales que $\|f\| = \frac{|f(y)-f(x)|}{d(x,y)}$. El conjunto de funciones Lipschitz que alcanzan su norma fuertemente en M se denota $\mathrm{SNA}(M)$.

El alcanzamiento fuerte de norma de funciones Lipschitz, así como otros alcanzamientos de norma más débiles, se han estudiado ampliamente durante los últimos años, desde los primeros trabajos sobre el tema ([66, 83]). Se sabe que el alzancamiento fuerte de norma es en realidad bastante estricto (por ejemplo, según [83, Lemma 2.2], si una función alcanza fuertemente su norma en un par (x, y), debe alcanzarla a lo largo de todo el segmento $[x, y]$ y ser afín siempre que esté definida). Por esta razón, en muchos espacios métricos M, $\mathrm{SNA}(M)$ pasa a no ser denso en $\mathrm{Lip}_0(M)$, aunque también se han obtenido resultados positivos para algún otro espacio métrico.

Claramente, si M tiene cardinal $n \in \mathbb{N}$, entonces $\mathrm{SNA}(M) = \mathrm{Lip}_0(M)$, y es un espacio de Banach. En el Capítulo 5 abordamos la siguiente pregunta: si M es infinito, ¿$\mathrm{SNA}(M)$ siempre contiene espacios lineales de dimensión mayor que 1? Por lo estricto que es este alcanzamiento de norma, y teniendo en cuenta el trabajo de Rmoutil para funcionales, se puede pensar que la respuesta a esta pregunta puede ser negativa. Sin embargo, veremos que esto está lejos de ser cierto. Para hacerlo, nos basamos en varias técnicas, como el teorema de extensión de McShane (que nos permite extender funciones Lipschitz de un espacio métrico M_1 a un espacio métrico más grande M_2 conservando su norma), espacios Lipschitz-free, y algunas otras herramientas. En este capítulo, asumiremos implícitamente que todos los espacios métricos son completos (debido al teorema de extensión de McShane) y "pointed", y que todos los espacios vectoriales son reales.

En general, no es cierto que si un espacio de Banach X está en $\mathrm{SNA}(M)$ para algún espacio métrico M, entonces podemos extenderlo mediante McShane y encontrar el mismo espacio X en $\mathrm{SNA}(M_2)$ para cada espacio métrico mayor M_2. Sin embargo, con la norma $\|\cdot\|_1$, obtenemos el siguiente resultado.

Lema 35. *Sea M un espacio métrico "pointed" tal que para algún subespacio K de M,* $\mathrm{SNA}(K)$ *contiene un subespacio lineal isométrico a ℓ_1^n para algún $n \in \mathbb{N}$. Entonces,* $\mathrm{SNA}(M)$ *también contiene un subespacio isométrico a ℓ_1^n.*

Usando esto y alguna otra técnica, podemos dar una respuesta definitiva a nuestra pregunta.

Teorema 36. *Sea $n > 1$ un número natural, y sea M un espacio métrico "pointed" con al menos 2^n puntos distintos. Entonces, existe un subespacio lineal de* $\mathrm{SNA}(M)$ *isométrico a ℓ_1^n.*

Corolario 37. *Si M es un espacio métrico "pointed" infinito, entonces para todo $n \in \mathbb{N}$,* $\mathrm{SNA}(M)$ *contiene un subespacio n-dimensional isométrico a ℓ_1^n.*

Por tanto, si M es infinito, $\mathrm{SNA}(M)$ no sólo tiene tiene subespacios de dimensión al menos 2: de hecho contiene a todos los ℓ_1^n, $n \in \mathbb{N}$, como subespacios isométricos.

No es difícil ver que $\mathrm{SNA}([0,1])$ contiene una copia isométrica de c_0. Esto lleva a preguntarse qué otros espacios de Banach se pueden formar. La respuepsta, sorprendentemente, es que todos ellos, si se elige el espacio métrico adecuado.

Proposición 38. *El espacio de Banach Y es subespacio isométrico de* $\mathrm{SNA}(B_{Y^*})$.

También es interesante hacerse la pregunta inversa: dado un espacio de Banach Y, ¿cómo de "pequeño" puede ser un espacio métrico M para que Y sea subespacio de $\mathrm{SNA}(M)$? Por el resultado anterior, si Y tiene dual separable, M puede ser separable, pero ¿y si no? El siguiente teorema muestra que esto es, de hecho, una caracterización.

Teorema 39. *Para un espacio de Banach dado Y, equivalen:*

(1) Existe un espacio métrico separable M y un subespacio lineal cerrado $Z \subset \mathrm{Lip}_0(M)$ tal que Z es isométrico a Y y $Z \subset \mathrm{SNA}(M)$.

(2) Existe un espacio de Banach separable X y un subespacio lineal cerrado $Z_1 \subset X^*$ tal que Z_1 es isométrico a Y y $Z_1 \subset \mathrm{NA}(X, \mathbb{R})$.

(3) Y^* es separable.

Por lo tanto, para espacios métricos separables como $M = [0, 1]$, el conjunto $\mathrm{SNA}(M)$ no puede contener subespacios como ℓ_1 con dual no separable. Esto agrega alguna restricción a los espacios métricos separables. De hecho, aparecen otras restricciones para algunos espacios métricos pequeños, como los σ-precompactos, que incluyen por ejemplo a todos los espacios precompactos y todos los espacios $\mathbb{R}^n$ (nótese que los espacios σ-precompactos siempre son separables).

Teorema 40. *Si M es un espacio σ-precompacto, entonces todos los subespacios de Banach de $\mathrm{SNA}(M)$ son separables e isomórficamente polihédricos.*

En cuanto a resultados positivos, hemos mencionado que $\mathrm{SNA}([0, 1])$ contiene a c_0 isométricamente. De hecho, esto se puede extender a una clase extensa de espacios métricos que incluye a todos los normados.

Proposición 41. *Si M es un espacio métrico que contiene a $[0, 1]$ isométricamente, entonces $\mathrm{SNA}(M)$ contiene a c_0 isométricamente.*

En realidad, es posible ver que para todos los espacios métricos M con una cantidad infinita de puntos no aislados, $\mathrm{SNA}(M)$ contiene a c_0 isométricamente. Pero incluso en todos los espacios M sin esta propiedad que estudiamos en profundidad, siempre resultaba posible encontrar c_0

en $\mathrm{SNA}(M)$ isomórficamente. Esto motivó a preguntarnos si este siempre es el caso (véanse [84, Questions 1 and 2]), es decir: si M es infinito, ¿$\mathrm{SNA}(M)$ siempre contiene a c_0 isomórficamente? Recientemente, Avilés, Martínez-Cervantes, Rueda Zoca y Tradacete respondieron afirmativamente esta pregunta mediante una elegante distinción de casos y con la ayuda del teorema de Ramsey.

Teorema 42 ([15, Main Theorem])**.** *Sea M un espacio métrico completo "pointed" infinito. Entonces,* $\mathrm{SNA}(M)$ *contiene a c_0 isomórficamente.*

En cuanto a meter isométricamente a c_0 en $\mathrm{SNA}(M)$, los autores demostraron en [15, Lemma 3.1] que si el espacio métrico involucrado satisface cierta propiedad geométrica (esto se cumple, por ejemplo, para espacios métricos con una cantidad infinita de puntos no aislados y para espacios métricos discretos que no son uniformemente discretos), entonces $\mathrm{SNA}(M)$ contiene a c_0 isométricamente. Para el resto de espacios métricos, dejaron abierta la siguiente pregunta (véase [15, Remark 3.6]): si M es infinito, ¿$\mathrm{SNA}(M)$ contiene a c_0 isométricamente? En [49], proporcionamos una respuesta definitiva a esta pregunta. Para hacerlo, primero encontramos el siguiente resultado, que mejora ligeramente las condiciones de [15, Lemma 3.1].

Lema 43. *Sea Γ un conjunto índice no vacío. Sea M un espacio métrico "pointed" tal que existen dos conjuntos $\{x_\gamma\}_{\gamma\in\Gamma}, \{y_\gamma\}_{\gamma\in\Gamma} \subset M$ con $x_\gamma \neq y_\gamma$, $x_\alpha \neq x_\beta$ para $\gamma, \alpha, \beta \in \Gamma$, $\alpha \neq \beta$. Si $d(x_\alpha, x_\beta) \geqslant d(x_\alpha, y_\alpha) + d(x_\beta, y_\beta)$ para todo $\alpha \neq \beta \in \Gamma$, entonces existe un subespacio lineal de* $\mathrm{SNA}(M)$ *isométrico a $c_0(\Gamma)$.*

Con ayuda de este resultado y algunos otros lemas técnicos, podemos resolver negativamente la pregunta de [15, Remark 3.6]. Proporcionamos dos contraejemplos distintos con comportamientos opuestos. Dado un punto $x \in M$, definimos su *radio de separación* como la cantidad $R(x) :=$

$\inf\{d(x,y)\colon\ y \in M\backslash\{x\}\}$, y decimos que x *alcanza su radio de separación* si ese ínfimo es en realidad un mínimo. Resumimos nuestros resultados.

Teorema 44. *Existen espacios métricos M_1 y M_2 tales que* $\mathrm{SNA}(M_1)$ *y* $\mathrm{SNA}(M_2)$ *no contienen a c_0 isométricamente y tales que M_1 es acotado y ninguno de sus puntos alcanza su radio de separación, y M_2 es "proper" no acotado y todos sus puntos alcanzan su radio de separación.*

Curiosamente, también existen espacios métricos completos "pointed" infinitos y uniformemente discretos M con c_0 contenido isométricamente en $\mathrm{SNA}(M)$ de tal manera que ningún punto alcanza su radio de separación, o que todo punto alcanza su radio de separación. En cuanto a los espacios métricos que no son uniformemente discretos, sorprendentemente, $\mathrm{SNA}(M)$ siempre contiene c_0 isométricamente.

Teorema 45. *Sea M un espacio métrico infinito que no es uniformemente discreto. Entonces, el conjunto* $\mathrm{SNA}(M)$ *contiene una copia isométrica de c_0.*

Finalmente, para el escenario no separable, usando el Lema 43 y un resultado inspirado en [71, Proposition 3], conseguimos el siguiente resultado positivo. Recordemos que dado un espacio métrico M, M' denota el conjunto de puntos de acumulación de M, y $\mathrm{dens}(M)$ denota el carácter de densidad de M.

Teorema 46. *Sea M un espacio métrico "pointed" tal que* $\mathrm{dens}(M') = \Gamma$ *para algún cardinal infinito Γ. Entonces, existe un subespacio lineal de* $\mathrm{SNA}(M)$ *que es isométricamente isomorfo a $c_0(\Gamma)$.*

El documento concluye con un capítulo de Conclusiones (véase la página 277), que incluye algunos comentarios y problemas abiertos. Al final del documento hay una lista extensa de referencias (véase la página 287), seguida de un glosario de conceptos y notaciones.

Resum (Valencià)

En aquesta dissertació, estudiem diverses classes d'aplicacions que poden assolir o no la seua norma o el seu radi numèric naturalment associat. En particular, parlarem sobre operadors i aplicacions bilineals entre espais normats, tensors projectius, operadors nuclears i funcions Lipschitz.

El contingut principal d'aquest document està organitzat en 5 capítols, on cobrim els continguts dels treballs publicats [42, 43, 59, 84] i el treball enviat [49], i s'extrauran algunes notes d'antecedents de l'estudi publicat [40]. En aquesta secció, resumirem en valencià el contingut de cada capítol (consulteu la secció Summary de la pàgina 15 i la secció Resumen de la pàgina 39 per a les respectives traduccions d'aquest resum a l'anglès i al castellà).

Resum del Capítol 1

El Capítol 1 serveix com a introducció. A la Secció 1.1, fem un comentari important sobre com llegir aquest document. A la Secció 1.2, establim notació i conceptes bàsics que es faran servir al llarg del text. A la Secció 1.3, incloem els antecedents històrics necessaris per tal de motivar el treball. Finalment, a la Secció 1.4, expliquem breument l'estructura del document i els continguts dels propers capítols. Comentem alguns fets històrics per motivar millor les altres seccions.

Inspirats pels treballs de James 1957/1963 ([74, 75]), Bishop i Phelps 1961 ([17]), Lindenstrauss 1963 ([93]), Bollobás 1970 ([18]) i molts altres (vegeu [2]) sobre la densitat d'operadors que assoleixen la seua norma, el 2008, Acosta, Aron, García i Maestre van introduir i estudiar la propietat de Bishop-Phelps-Bollobás (vegeu [5]) .

Definició 1 ([5])**.** Un parell d'espais de Banach (X, Y) té la *propietat de Bishop-Phelps-Bollobás* (abreujada *BPBp*) si donat $\varepsilon \in (0,1)$, existeix $\eta(\varepsilon) > 0$ tal que si $T \in \mathcal{L}(X, Y)$ i $x \in S_X$ satisfan $\|T\| = 1$ i $\|T(x)\| > 1 - \eta(\varepsilon)$, existeix $S \in \mathcal{L}(X, Y)$ i $y \in S_X$ tal que $\|S\| = \|S(y)\| = 1$, $\|x - y\| < \varepsilon$ i $\|S - T\| < \varepsilon$.

Notem que si els espais de Banach X i Y satisfan la BPBp, aleshores, en particular, $\mathrm{NA}(X, Y)$ és dens a $\mathcal{L}(X, Y)$, encara que el recíproc no sempre és cert. La BPBp ha estat estudiada per molts autors recentment (vegeu els estudis [3, 40] per a una exposició completa de resultats sobre la BPBp fins al 2022). Diverses variacions interessants de la BPBp també s'han introduït i estudiat recentment a base de fer canvis específics a la definició 1, com ara la $\mathbf{L}_{o,o}$ (la BPBp però per a cada T prèviament fixat, es troba un $\eta(\varepsilon, T)$ depenent també de T, i, a més a més, $S = T$).

Aquest ampli estudi dels operadors que assoleixen les seues normes també s'ha estés a altres tipus d'aplicacions i normes. Per exemple, les aplicacions multilineals, polinomis homogenis, funcions holomorfes, operadors compactes i aplicacions Lipschitz que assoleixen les seues normes s'han estudiat durant molt de temps, i el mateix passa amb els operadors que arriben al seu radi numèric. Naturalment, propietats tipus BPBp també shan introduït i estudiat per a aquests contextos. Ens referim de nou a la secció 1.3 i a l'estudi [40] per a més informació sobre aquestes propietats i més. Aquest escenari és el punt de partida d'aquesta dissertació.

Resum del Capítol 2

Els continguts d'aquest capítol han sigut publicats a

[42] S. Dantas, M. Jung, and Ó. Roldán, Norm-attaining operators which satisfy a Bollobás type theorem, *Banach J. Math. Anal.* **15**(2) (2021), Paper No. 40, 26 pp.

Inspirats per la $\mathbf{L}_{o,o}$ i les seues moltes aplicacions, el Capítol 2 està dedicat a estudiar una classe $\mathcal{A}_{\|\cdot\|}(X,Y) \subset \mathrm{NA}(X,Y)$ d'operadors que compleixen una propietat com la $\mathbf{L}_{o,o}$, és a dir, tals que si gairebé arriben a la seua norma en x, l'assoleixen en un punt proper x_0. La classe anàloga per al radi numèric també s'introdueix i s'estudia. La definició formal daquests conjunts és la següent.

Definició 2. Siguen X, Y dos espais de Banach sobre el cos $\mathbb{K} = \mathbb{R}$ o $\mathbb{C}$.

(i) $\mathcal{A}_{\|\cdot\|}(X,Y)$ representa el conjunt de tots els operadors que assoleixen la seua norma, $T \in \mathcal{L}(X,Y)$ amb $\|T\| = 1$ tals que si $\varepsilon > 0$, existeix $\eta(\varepsilon, T) > 0$ tal que sempre que $x \in S_X$ complisca que $\|T(x)\| > 1 - \eta(\varepsilon, T)$, existeix $x_0 \in S_X$ tal que $\|T(x_0)\| = 1$ i $\|x_0 - x\| < \varepsilon$.

(ii) $\mathcal{A}_{\mathrm{nu}}(X)$ representa el conjunt d'operadors que assoleixen el seu radi numèric, $T \in \mathcal{L}(X,X)$ amb $\nu(T) = 1$ tals que si $\varepsilon > 0$, existeix $\eta(\varepsilon, T) > 0$ tal que sempre que $(x, x^*) \in \Pi(X)$ complisca que $|x^*(T(x))| > 1 - \eta(\varepsilon, T)$, existeix $(x_0, x_0^*) \in \Pi(X)$ tal que $|x_0^*(T(x_0))| = 1$, $\|x_0 - x\| < \varepsilon$, i $\|x_0^* - x^*\| < \varepsilon$.

A la Secció 2.2, es presenta una selecció de resultats i exemples sobre les classes $\mathcal{A}_{\|\cdot\|}$ i $\mathcal{A}_{\mathrm{nu}}$. Per a espais de Banach de dimensió finita, utilitzant la compacitat de la bola unitat i el fet que cada operador assoleix la seua norma i ràdio numèric, obtenim la següent caracterització positiva.

Teorema 3. *Siga X un espai de Banach de dimensió finita.*

(i) $\mathcal{A}_{\|\cdot\|}(X,Y) = \{T \in \mathcal{L}(X,Y) : \|T\| = 1\}$ *per a qualsevol espai de Banach Y,*

(ii) $\mathcal{A}_{\mathrm{nu}}(X) = \{T \in \mathcal{L}(X,X) : \nu(T) = 1\}$.

Per a funcionals, obtenim resultats positius per a una classe extensa d'espais, pero també en trobem de negatius per a altres espais.

Teorema 4. *Siga X un espai de Banach sobre $\mathbb{K}$.*

(i) $\mathrm{NA}(c_0,\mathbb{K}) \cap S_{\ell_1} = \mathcal{A}_{\|\cdot\|}(c_0,\mathbb{K})$.

(ii) *Si X és uniformement convex, aleshores $S_{X^*} = \mathcal{A}_{\|\cdot\|}(X,\mathbb{K})$.*

(iii) *Existeix $x^* \in \mathrm{NA}(\ell_1,\mathbb{K}) \cap S_{\ell_\infty}$ tal que $x^* \notin \mathcal{A}_{\|\cdot\|}(\ell_1,\mathbb{K})$.*

(iv) *Existeix $x^* \in \mathrm{NA}(\ell_\infty,\mathbb{K}) \cap S_{\ell_\infty^*}$ tal que $x^* \notin \mathcal{A}_{\|\cdot\|}(\ell_\infty,\mathbb{K})$.*

Pel que fa a operadors generals sobre un espai de Banach X, notem que tota isometria està en $\mathcal{A}_{\|\cdot\|}(X,X)$, però aquest no és sempre el cas amb $\mathcal{A}_{\mathrm{nu}}(X)$. De fet, fins i tot en el context dels espais de Hilbert com $X = \ell_2$, existeixen (vegeu Exemple 2.2.5) operadors en $\mathcal{A}_{\|\cdot\|}(X,X) \cap \mathcal{A}_{\mathrm{nu}}(X)$, en $\mathcal{A}_{\|\cdot\|}(X,X)\backslash\mathcal{A}_{\mathrm{nu}}(X)$, en $\mathcal{A}_{\mathrm{nu}}(X)\backslash\mathcal{A}_{\mathrm{nu}}(X,X)$, i operadors que no són a $\mathcal{A}_{\|\cdot\|}(X,X)\cup\mathcal{A}_{\mathrm{nu}}(X)$ malgrat estar a $\{T \in \mathrm{NA}(X,X)\cap\mathrm{NRA}(X)\colon\ \nu(T) = \|T\| = 1\}$. Tot això afegeix complexitat a la nostra pregunta.

Una classe important d'operadors per als quals podem obtenir un resultat positiu són els operadors compactes. El resultat següent mostra que sota algunes hipòtesis sobre els espais involucrats, tot operador compacte amb norma 1 (i ràdi numèric 1) està en $\mathcal{A}_{\|\cdot\|}(X,Y)$ (i en $\mathcal{A}_{\mathrm{nu}}(X)$).

Teorema 5. *Siga X un espai reflexiu amb la propietat de Kadec-Klee.*

(i) $S_{\mathcal{K}(X,Y)} \subset \mathcal{A}_{\|\cdot\|}(X,Y)$ *per a tot espai de Banach* Y.

(ii) $\{T \in \mathcal{K}(X,X)\colon\ \nu(T) = \|T\| = 1\} \subset \mathcal{A}_{\mathrm{nu}}(X)$ *si* X *és diferenciable Fréchet.*

En particular, mostrem que sota certes hipòtesis sobre l'espai de Banach X, tot operador compacte $T \in \mathcal{K}(X,X)$ amb $\nu(T) = \|T\| = 1$ assoleix el seu radi numèric. Notem que si X és un espai de Banach de dimensió infinita, la inclusió a (ii) ha de ser estricta, ja que la identitat sempre és a $\mathcal{A}_{\mathrm{nu}}(X)$, però no és compacta. També obtenim la següent conseqüència immediata del resultat anterior.

Corol·lari 6. *Siga* X *un espai de Banach reflexiu amb la propietat de Kadec-Klee i siga* H *un espai de Hilbert.*

(i) *Si* Y *té la propietat de Schur, aleshores* $\mathcal{A}_{\|\cdot\|}(X,Y) = S_{\mathcal{L}(X,Y)}$.

(ii) $\{T \in \mathcal{K}(H,H)\colon\ \nu(T) = \|T\| = 1\} \subset \mathcal{A}_{\mathrm{nu}}(H)$.

Si eliminem algunes de les hipòtesis sobre els espais al Teorema 5, tots dos enunciats deixen de ser certs en general (vegeu els operadors de (2.2.3) i (2.2.5)). A més, el Teorema 5 i el Corol·lari 6 fallen en el context no compacte, com veurem a la Secció 2.3 .

La demostració del resultat següent (inspirat en [1, Exemple 1.9]) ens proporciona una àmplia classe d'operadors compactes $T \in \mathcal{A}_{\mathrm{nu}}(H)$ tals que $1 = \nu(T) < \|T\|$ i, per tant, exemples d'operadors que pertanyen a $\mathcal{A}_{\mathrm{nu}}(H)$ però no a $\mathcal{A}_{\|\cdot\|}(H,H)$ (vegeu la demostració de la Proposició 2.2.9 per a més detalls). En aquest cas obtenim $\mathcal{A}_{\mathrm{nu}}$ en un sentit uniforme, on η només depèn de ε.

Proposició 7. *Siga* H *un espai de Hilbert separable real de dimensió infinita. Aleshores existeix* $T \in \mathcal{L}(H,H)$ *tal que*

(i) *T és un operador compacte.*

(ii) $1 = \nu(T) < \|T\|$ *i T assoleix el seu radi numèric.*

(iii) *Donat $\varepsilon > 0$, existeix $\eta(\varepsilon) > 0$ tal que si $x_0 \in S_H$ compleix que*

$$|\langle T(x_0), x_0\rangle| > 1 - \eta(\varepsilon),$$

existeix $x_1 \in S_H$ tal que $\nu(T) = \langle T(x_1), x_1\rangle = 1$ i $\|x_1 - x_0\| < \varepsilon$.

En particular, $T \in \mathcal{A}_{\mathrm{nu}}(H)$ i $T \notin \mathcal{A}_{\|\cdot\|}(H, H)$.

Els operadors de (2.2.2), (2.2.3) i (2.2.5) mostren que, en general, no hi ha cap relació entre $T \in \mathcal{A}_{\|\cdot\|}(X, Y)$ i $T^* \in \mathcal{A}_{\|\cdot\|}(Y^*, X^*)$. No obstant això, si afegim condicions extra als espais X i Y, obtenim el resultat següent.

Proposició 8. *Siguen X, Y espacis de Banach i $T \in \mathcal{L}(X, Y)$.*

(i) *Si Y és uniformement suau, si $T \in \mathcal{A}_{\|\cdot\|}(X, Y)$, aleshores $T^* \in \mathcal{A}_{\|\cdot\|}(Y^*, X^*)$.*

(ii) *Si X és uniformement convex, si $T^* \in \mathcal{A}_{\|\cdot\|}(Y^*, X^*)$, aleshores $T \in \mathcal{A}_{\|\cdot\|}(X, Y)$.*

(iii) *Si X és reflexiu, aleshores, $T \in \mathcal{A}_{\mathrm{nu}}(X)$ si i sols si $T^* \in \mathcal{A}_{\mathrm{nu}}(X^*)$.*

Notem que no podem eliminar la suavitat i convexitat uniformes a (i) i (ii) (vegeu de nou (2.2.2), (2.2.3), i (2.2.5)). A c_0, obtenim el següent resultat relacionat amb (iii).

Proposició 9. *Siga $T \in \mathcal{A}_{nu}(c_0)$ tal que el rang de $T^* \in \mathcal{L}(\ell_1, \ell_1)$ és a* $\mathrm{span}\{e_1^*, \ldots, e_N^*\}$ *per a algun $N \in \mathbb{N}$. Aleshores, $T^* \in \mathcal{A}_{nu}(\ell_1)$.*

En la Secció 2.3, es donarà una caracterització completa de tots els operadors diagonals que pertanyen a $\mathcal{A}_{\|\cdot\|}(X,X)$ ($X = c_0$ o ℓ_p, $1 \leqslant p \leqslant \infty$), a $\mathcal{A}_{\mathrm{nu}}(X)$ ($X = c_0$ o ℓ_p, $1 \leqslant p < \infty$), a $\mathcal{A}_{\|\cdot\|}(c_0, \ell_p)$ ($1 < p < \infty$) i a $\mathcal{A}_{\|\cdot\|}(\ell_p, c_0)$ ($1 < p < \infty$). Podem resumir aquests resultats com segueix.

Teorema 10. *Siga* (X, Y) *igual a* (c_0, c_0), (ℓ_p, ℓ_p) *(*$1 \leqslant p \leqslant \infty$*) o* (ℓ_p, c_0) *(*$1 \leqslant p < \infty$*). Siga* $T : X \to Y$ *l'operador diagonal de norma* 1 *associat a la successió fitada de complexos* $\{\alpha_n\}_{n=1}^{\infty}$*. Aleshores,* $T \in \mathcal{A}_{\|\cdot\|}(X, Y)$ *si i sols si es donen les dues condicions següents:*

1. *Existeix* $n_0 \in \mathbb{N}$ *tal que* $|\alpha_{n_0}| = 1$.
2. *Si* $J = \{n \in \mathbb{N} : |\alpha_n| = 1\}$*, aleshores* $J = \mathbb{N}$ *o* $\sup_{n\in\mathbb{N}\setminus J} |\alpha_n| < 1$.

Teorema 11. *Donat* $1 \leqslant p < \infty$*, siga* $T : c_0 \to \ell_p$ *l'operador diagonal de norma* 1 *associat a la successió acotada de complexos* $\{\alpha_n\}_{n=1}^{\infty}$*. Aleshores,* $T \in \mathcal{A}_{\|\cdot\|}(c_0, \ell_p)$ *si i sols si existeix* $N \in \mathbb{N}$ *tal que* $\alpha_n = 0$ *per a tot* $n > N$.

Teorema 12. *Siga* $X = c_0$ *o* ℓ_p*,* $1 \leqslant p < \infty$*. Siga* $T : X \to X$ *l'operador diagonal de radi numèric* 1 *associat a la successió fitada de complexos* $\{\alpha_n\}_{n=1}^{\infty}$*. Aleshores,* $T \in \mathcal{A}_{nu}(X)$ *si i sols si es donen les dues condicions següents:*

1. *Existeix* $n_0 \in \mathbb{N}$ *tal que* $|\alpha_{n_0}| = 1$.
2. *Si* $J = \{n \in \mathbb{N} : |\alpha_n| = 1\}$*, aleshores el cardinal de* $\{\alpha_n : n \in J\}$ *és finit i* $\sup_{n\in\mathbb{N}\setminus J} |\alpha_n| < 1$ *quan* $J \neq \mathbb{N}$.

En particular, si $\{\alpha_n\}_{n=1}^{\infty} \subset \mathbb{R}$*,* $T \in \mathcal{A}_{nu}(X)$ *si i sols si* $T \in \mathcal{A}_{\|\cdot\|}(X, X)$.

Com a conseqüència, per a les projeccions canòniques sobre els espais de successions clàssics, $P_N(x) := (x(1), x(2), \ldots, x(N), 0, \ldots)$, obtenim el següent.

Corol·lari 13. *Siga $N \in \mathbb{N}$ donat. Si $X = c_0$ o ℓ_p, $1 \leqslant p \leqslant \infty$, aleshores $P_N \in \mathcal{A}_{\|\cdot\|}(X,X) \cap \mathcal{A}_{\mathrm{nu}}(X)$.*

Finalment, a la Secció 2.4, estudiem la relació entre $\mathcal{A}_{\|\cdot\|}(W,Z)$ i $\mathcal{A}_{\mathrm{nu}}(W \oplus Z)$ per a algunes sumes directes dels espais de Banach W i Z. Donats dos espais de Banach X_1 i X_2, considerem les aplicacions $P_i \in \mathcal{L}(X_1 \oplus X_2, X_i)$ tals que $P_i(x_1,x_2) := x_i$, $i = 1,2$, i $\iota_j \in \mathcal{L}(X_j, X_1 \oplus X_2)$ tals que $\iota_i(x) := xe_i$, on $e_1 = (1,0)$ i $e_2 = (0,1)$. Per als espais de Banach W i Z, si tenim un operador $T \in \mathcal{L}(W,Z)$, aleshores hi ha una manera senzilla de definir $\widetilde{T} \in \mathcal{L}(W \oplus Z)$: considerem $\widetilde{T} := \iota_2 \circ T \circ P_1$, és a dir, $\widetilde{T}(w,z) = (0, T(w))$ per cada $(w,z) \in W \oplus Z$. Per contra, podem definir un procés pseudo-invers de la següent manera: si tenim un operador $S \in \mathcal{L}(W \oplus Z, W \oplus Z)$, aleshores podem considerar $\breve{S} \in \mathcal{L}(W,Z)$ definit com $\breve{S} := P_2 \circ S \circ \iota_1$, és a dir, $\breve{S}(w) = (P_2 \circ S)(w,0)$ per a cada $w \in W$. Obtenim els resultats següents.

Proposició 14. *Siguen W, Z dos espais de Banach, i siga $T \in S_{\mathcal{L}(W,Z)}$. Aleshores,*

(i) *Si $\widetilde{T} \in \mathcal{A}_{\mathrm{nu}}(W \oplus_s Z)$, aleshores $T \in \mathcal{A}_{\|\cdot\|}(W,Z)$, on $s = 1$ o $s = \infty$.*

(ii) *Suposem que W i Z són uniformement suaus. Si $T \in \mathcal{A}_{\|\cdot\|}(W,Z)$, aleshores $\widetilde{T} \in \mathcal{A}_{\mathrm{nu}}(W \oplus_1 Z)$.*

(iii) *Suposem que Z és uniformement convex i W és uniformement suau. Si $T \in \mathcal{A}_{\|\cdot\|}(W,Z)$, aleshores $\widetilde{T} \in \mathcal{A}_{\mathrm{nu}}(W \oplus_\infty Z)$.*

Finalment, notem que (ii) i (iii) ja no es compleixen en general per a espais de Banach arbitraris o per a p-sumes si $1 < p < \infty$, i hi ha $S \in \mathcal{L}(W \oplus_s Z, W \oplus_s Z)$, amb W i Z uniformement suaus i uniformement convexos, tal que $S \in \mathcal{A}_{\mathrm{nu}}(W \oplus_s Z)$ però amb $\breve{S} \notin \mathcal{A}_{\|\cdot\|}(W,Z)$, $s = 1$ o $s = \infty$ (vegeu els comentaris que segueixen a les Proposicions 2.4.1 i 2.4.4).

Resum del Capítol 3

Els continguts d'aquest capítol han sigut publicats a

> [59] D. García, M. Maestre, M. Martín, and Ó. Roldán, On the compact operators case of the Bishop-Phelps-Bollobás property for numerical radius, *Results Math.* **76**(3) (2021), Paper No. 122, 23 pp.

El 2013, Guirao i Kozhushkina van introduir i van estudiar a [69] la següent versió de la BPBp per al radi numèric.

Definició 15 (Combinant [69, Definition 1.2] i [87, Definition 5])**.** Un espai de Banach X té la propietat feble de Bishop-Phelps-Bollobás per al radi numèric (abreujada weak BPBp-nu) si donat $\varepsilon > 0$, existeix $\eta(\varepsilon) > 0$ tal que si $T \in \mathcal{L}(X, X)$ amb $\nu(T) = 1$ i $(x, x^*) \in \Pi(X)$ compleixen que $|x^*(T(x))| > 1 - \eta(\varepsilon)$, existeixen $S \in \mathcal{L}(X, X)$ i $(y, y^*) \in \Pi(X)$ tals que

$$\nu(S) = |y^*(S(y))|, \quad \|x - y\| < \varepsilon, \quad \|x^* - y^*\| < \varepsilon, \quad \text{i} \quad \|T - S\| < \varepsilon.$$

Si, a més, S es pot triar sempre amb $\nu(S) = 1$, diem que X té la propietat de Bishop-Phelps-Bollobás per al radi numèric (abreujada BPBp-nu).

Arran d'aquest article ([69]), molts autors han obtingut múltiples resultats sobre la BPBp-nu (vegeu [3, Section 6] i [40, Section 2.7] per a exposicions dels principals resultats obtinguts sobre aquest tema). El 2018, Dantas, García, Maestre i Martín, van introduir i estudiar la BPBp adaptada als operadors compactes (vegeu [39]). La BPBp-nu i la BPBp per a operadors compactes van motivar a introduir i estudiar en profunditat la BPBp-nu per a operadors compactes (considereu la Definició 15, però amb $T \in \mathcal{K}(X, X)$ i $S \in \mathcal{K}(X, X)$). En explorar les proves existents sobre BPBp-nu i fer-ne xicotetes adaptacions, obtenim una primera llista d'espais que satisfan la BPBp-nu per a operadors compactes. Aquest és l'objectiu de la Secció 3.2

Examples 16. Els espais següents tenen la BPBp-nu per a operadors compactes: espais de dimensió finita ([87, Proposition 2]), $c_0(\Gamma)$ i $\ell_1(\Gamma)$ per a qualsevol conjunt índex Γ ([69, Corollaries 3.3 and 4.2]), i $L_1(\mu)$ per a qualsevol mesura μ ([7, Corollary 2.1] i [87, Theorem 9]).

A continuació, adaptant les nocions d'índex numèric i segon índex numèric al context dels operadors compactes, n_K i n'_K, respectivament, i adaptant els resultats de [87] i [89], mostrem que si un espai de Banach X és uniformement convex i uniformement suau, aleshores té la weak BPBp-nu per a operadors compactes, i si $n_K(X) > 0$ o $n'_K(X) > 0$, aleshores la weak BPBp-nu per a operadors compactes és equivalent a la BPBp-nu per a operadors compactes. En particular, mostrem que per a cada mesura μ i cada $1 < p < \infty$, $L_p(\mu)$ té la BPBp-nu per a operadors compactes.

A [34, Proposition 4.3] es va mostrar que si un espai de Banach X té la BPBp-nu per a operadors compactes, aleshores tot sumand absolut de X de tipus 1 i ∞ també té aquesta propietat, i amb la mateixa funció η. Això permet portar la propietat d'alguns espais a algunes projeccions d'aquests espais. És natural preguntar-se si es pot dir alguna cosa en sentit contrari. A [39, Lemma 2.1] es va presentar una eina que, en particular, ens permet portar la BPBp per a operadors compactes des d'unes projeccions d'un espai al mateix espai. Per a obtenir un resultat anàleg per al radi numèric, cal controlar-ho tot tant a l'espai com al seu dual. El resultat més general obtingut en aquest sentit és el següent lema.

Lema 17. *Siga X un espai de Banach amb $n_K(X) > 0$. Suposem que hi ha una funció $\eta\colon (0,1) \longrightarrow (0,1)$ tal que donats $\delta > 0$, $x_1^*, \ldots, x_n^* \in B_{X^*}$ i $x_1, \ldots, x_\ell \in B_X$, podem trobar operadors de norma 1 $\widetilde{P}\colon X \longrightarrow \widetilde{P}(X)$,*

$i\colon \widetilde{P}(X) \longrightarrow X$ tals que per a $P := i \circ \widetilde{P}\colon X \longrightarrow X$, es compleixen aquestes condicions:

(i) $\|P^*(x_j^*) - x_j^*\| < \delta$, *per a* $j = 1, \ldots, n$.

(ii) $\|P(x_j) - x_j\| < \delta$, *per a* $j = 1, \ldots, \ell$.

(iii) $\widetilde{P} \circ i = \mathrm{Id}_{\widetilde{P}(X)}$.

(iv) $\widetilde{P}(X)$ *té la BPBp-nu per a operadors compactes amb la funció* η.

(v) *O bé* P *és una projecció absoluta i* i *és la inclusió natural, o* $n_K(\widetilde{P}(X)) = n_K(X) = 1$.

Aleshores, X té la BPBp-nu per a operadors compactes.

Al llarg de la Secció 3.3, el Lema 17 s'utilitza per a mostrar que si un espai de Banach X amb $n_K(X) > 0$ pot ser adequadament projectat en alguna xarxa d'espais que tenen la BPBp-nu per a operadors compactes amb una funció η comuna, aleshores de vegades és possible mostrar que X també té aquesta propietat (vegeu la Proposició 3.3.2). Això s'utilitza per obtenir els dos resultats següents.

Corol·lari 18. *Siga X un espai de Banach amb $n_K(X) > 0$. Aleshores les següents afirmacions són equivalents.*

(i) *L'espai $c_0(X)$ té la BPBp-nu per a opeadors compactes.*

(ii) *Hi ha una funció $\eta : (0,1) \longrightarrow (0,1)$ tal que tos els $\ell_\infty^n(X)$, amb $n \in \mathbb{N}$, tenen la BPBp-nu per a operadors compactes amb η.*

A més, si X és de dimensió finita, aquestes propietats es donen quan $c_0(X)$ o $\ell_\infty(X)$ tenen la BPBp-nu.

Corol·lari 19. *Siga X un espai de Banach tal que X^* es isomètricament isomòrfic a ℓ_1. Aleshores X té la BPBp-nu per a operadors compactes.*

A la Secció 3.4, presentem una sèrie d'eines topològiques que permeten cobrir convenientment un espai Hausdorff localment compacte L amb conjunts més xicotets i trobar una partició de la unitat adequada subordinada a aquests conjunts. Això ens fa possible projectar l'espai $C_0(L)$ en algun espai ℓ_∞^p ($p \in \mathbb{N}$) de manera que ens permet utilitzar Lemma 17. Aquesta propietat d'aproximació forta que obtenim a $C_0(L)$ i el seu dual es resumeix en el resultat següent.

Teorema 20. *Siga L un espai localment compacte Hausdorff. Donats $\{f_1, \ldots, f_\ell\} \subset C_0(L)$ amb $\|f_j\| \leqslant 1$ per a $j = 1, \ldots, \ell$, i $\{\mu_1, \ldots, \mu_n\} \subset C_0(L)^*$ amb $\|\mu_j\| \leqslant 1$ per a $j = 1, \ldots, n$, per a cada $\varepsilon > 0$ existeix una projecció de norma 1 $P\colon C_0(L) \longrightarrow C_0(L)$ tal que:*

(1) $\|P^*(\mu_j) - \mu_j\| < \varepsilon$, *per a* $j = 1, \ldots, n$,

(2) $\|P(f_j) - f_j\| < \varepsilon$, *per a* $j = 1, \ldots, \ell$,

(3) $P(C_0(L))$ *es isomètricament isomòrfic a* ℓ_∞^p *per a algun* $p \in \mathbb{N}$.

Finalment, com a conseqüència, obtenim el següent resultat.

Teorema 21. *Si L és un espai localment compacte Hausdorff, aleshores $C_0(L)$ té la BPBp-nu per a operadors compactes.*

En particular, tot espai $C(K)$ (K compacte Hausdorff) i tot espai $L_\infty(\mu)$ (μ qualsevol mesura) té la BPBp-nu per a operadors compactes. Notem que ara per ara continua sent un problema obert si tots els espais $C(K)$ tenen la BPBp-nu, i fins ara només s'han resolt casos particulars en el context real (vegeu [13]), però per a operadors compactes obtenim una resposta definitiva per a aquests espais.

Resum del Capítol 4

Els continguts d'aquest capítol han sigut publicats a

[43] S. Dantas, M. Jung, Ó. Roldán, and A. Rueda Zoca, Norm-attaining tensors and nuclear operators, *Mediterr. J. Math.* **19**(1) (2022), Paper No. 38, 27 pp.

Al Capítol 4, s'introdueixen i estudien les nocions d'assoliment de normes per a tensors projectius de $X\widehat{\otimes}_\pi Y$ i operadors nuclears de $\mathcal{N}(X,Y)$, per a espais de Banach X i Y. Per motivar per què aquestes preguntes poden ser interessants, recordem que dues de les principals preguntes històriques sobre els operadors que assoleixen la seua norma són les següents:

1. És $\mathcal{K}(X,Y) \subset \overline{\mathrm{NA}(X,Y)}$ en general?
2. És $\mathcal{F}(X,Y) \subset \overline{\mathrm{NA}(X,Y)}$ en general?

La primera pregunta va ser resolta negativament per Miguel Martín el 2014 (vegeu [97]). La segona pregunta roman oberta, i molts la consideren actualment com la principal pregunta oberta en la teoria dels operadors que assoleixen les seues normes. Notem que els operadors nuclears es troben entre els operadors de rang finit i els operadors compactes, i els tensors projectius hi estan estretament relacionats i tenen moltes aplicacions en múltiples camps dins de l'anàlisi funcional. Un altre factor important per tal de motivar aquest estudi és el fet que si fóra cert que per a tot espai de Banach de dimensió finita X tot operador nuclear a $\mathcal{N}(X,Y)$ assoleix la seua norma nuclear, aleshores obtindríem una resposta afirmativa a la segona pregunta d'abans. Tanmateix, la suposició va resultar ser falsa, com veurem més endavant.

En aquest capítol, farem servir implícitament totes les identificacions isomètriques $(X\widehat{\otimes}_\pi Y)^* = \mathcal{L}(X, Y^*) = \mathcal{L}(Y, X^*) = \mathcal{B}(X \times Y, \mathbb{K})$. Notem també que si X^* o Y té la propietat d'aproximació, aleshores $X^*\widehat{\otimes}_\pi Y = \mathcal{N}(X, Y)$ (vegeu, per exemple, [107, Corollary 4.8]). Introduïm a continuació les nocions d'assoliment de norma en aquests contextos.

Definició 22. Siguen X, Y dos espais de Banach. Diem que

(i) $z \in X\widehat{\otimes}_\pi Y$ *assoleix la seua norma projectiva* si existeix una successió fitada $(x_n, y_n) \subset X \times Y$ amb $\sum_{n=1}^{\infty} \|x_n\|\|y_n\| < \infty$ tal que $z = \sum_{n=1}^{\infty} x_n \otimes y_n$ i $\|z\|_\pi = \sum_{n=1}^{\infty} \|x_n\|\|y_n\|$. En aquest cas, diem que z és un *tensor que assoleix la seua norma*, o $z \in \mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)$.

(ii) $T \in \mathcal{N}(X, Y)$ *assoleix la seua norma nuclear* si existeix una sucxessió acotada $(x_n^*, y_n) \subset X^* \times Y$ amb $\sum_{n=1}^{\infty} \|x_n^*\|\|y_n\| < \infty$ tal que $T = \sum_{n=1}^{\infty} x_n^* \otimes y_n$ i $\|T\|_{\mathcal{N}} = \sum_{n=1}^{\infty} \|x_n^*\|\|y_n\|$. En este cas, diem que T és un *operador nuclear que assoleix la seua norma*, o $T \in \mathrm{NA}_{\mathcal{N}}(X, Y)$.

A la Secció 4.2, s'obtenen els primers resultats d'assoliment de norma en aquest context. Comencem trobant dues caracteritzacions tècniques que ens permeten afirmar que un tensor o operador nuclear assoleix la seua norma respectiva sempre que hi haja moltes formes bilineals que assoleixen les seues normes en molts punts d'una manera específica (vegeu Teoremes 4.2.1 i 4.2.2). Amb aquests resultats presents i el fet que els espais de dimensió finita, c_0, ℓ_1 i els espais de Hilbert tenen la propietat d'aproximació, obtenim la nostra primera col·lecció de resultats positius.

Proposició 23. *Tot tensor projectiu de $X^*\widehat{\otimes}_\pi Y$ i tot operador nuclear de $\mathcal{N}(X, Y)$ assoleix la seua norma respectiva si X i Y tenen dimensió finita, si $X = Y$ és un espai de Hilbert complex, o si $X = c_0$.*

És interessant comparar l'últim exemple d'ara amb la teoria clàssica dels operadors que assoleixen les normes: si $\mathrm{NA}(X,Y) = \mathcal{L}(X,Y)$ per a cada espai de Banach Y, aleshores en particular X ha de ser reflexiu pel Teorema de James. La proposició anterior ens motiva a preguntar-nos si $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y) = X\widehat{\otimes}_\pi Y$ i $\mathrm{NA}_{\mathcal{N}}(X,Y) = \mathcal{N}(X,Y)$ es compleixen en general per qualsevol espai de Banach X i Y. No obstant això, aquest no és el cas, com mostren els resultats següents.

Lema 24. *Siguen X, Y espais de Banach. Si $B \in \mathcal{B}(X \times Y, \mathbb{K}) = (X\widehat{\otimes}_\pi Y)^*$ assoleix la seua norma com a funcional a un element de l'espai $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)$, aleshores $B \in \mathrm{NA}_{\mathcal{B}}(X \times Y, \mathbb{K})$.*

Proposició 25. *Siguen X, Y espais de Banach. Si $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y) = X\widehat{\otimes}_\pi Y$, aleshores $\overline{\mathrm{NA}_{\mathcal{B}}(X \times Y, \mathbb{K})}^{\|\cdot\|_{\mathcal{B}}} = \mathcal{B}(X \times Y, \mathbb{K})$ (i per tant, també s'obté que $\overline{\mathrm{NA}(X,Y^*)}^{\|\cdot\|} = \mathcal{L}(X,Y^*)$).*

Hi ha molts exemples coneguts d'espais de Banach X i Y que no compleixen que $\overline{\mathrm{NA}(X,Y^*)}^{\|\cdot\|} = \mathcal{L}(X,Y^*)$, per la qual cosa existeixen tensors projectius que no assoleixen la seua norma projectiva. Utilitzant la propietat d'aproximació, també obtenim operadors nuclears que no assoleixen la seua norma nuclear. El següent exemple és de particular interés, ja que mostra que no tots els tensors projectius o operadors nuclears assoleixen les seues normes respectives si s'assumeix que només un dels espais de Banach és de dimensió finita, resolvent negativament un dels factors que fem servir per a motivar aquest estudi.

Example 26. Siga $X = L_1(\mathbb{T})$, on la circumferència unitat $\mathbb{T}$ està equipada amb la mesura de Haar m, i siga Y l'espai de Hilbert de dimensió 2. A [65, Remark 5.7.(2)] es mostra que existeix $T \in \mathcal{B}(X \times Y, \mathbb{K})$ que assoleix la seua norma com a funcional a $X\widehat{\otimes}_\pi Y$ però no com a operador de X a Y^* (i per tant, tampoc com a forma bilineal a $X \times Y$). Pel Lema 24, obtenim que $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y) \neq X\widehat{\otimes}_\pi Y$, i per tant, $\mathrm{NA}_{\mathcal{N}}(Y,X) \neq \mathcal{N}(Y,X)$.

Com que no tots els tensors projectius o operadors nuclears assoleixen la seua norma, és natural preguntar-se ara si tenim resultats de densitat. A la Secció 4.3, proporcionem alguns resultats de densitat positius. Per tal d'aconseguir-los, es fan servir dos enfocaments. Primer, notem que per les dues primeres caracteritzacions, per tal d'obtenir molts tensors projectius i operadors nuclears que assoleixen les seues normes, volem tenir moltes formes bilineals que assoleixen les seues normes en molts punts. La $\mathbf{L}_{o,o}$ assegura l'existència de molts operadors que assoleixen les seues normes en molts punts, i es pot adaptar a aplicacions bilineals de la següent manera.

Definició 27. Diem que $(X \times Y, Z)$ té la $\mathbf{L}_{o,o}$ per a aplicacions bilineals (o simplement, $\mathbf{L}_{o,o,\mathcal{B}}$) si donats $\varepsilon > 0$ i $B \in \mathcal{B}(X \times Y, Z)$ amb $\|B\|_{\mathcal{B}} = 1$, existeix $\eta(\varepsilon, B) > 0$ tal que sempre que $(x, y) \in S_X \times S_Y$ compleix que $\|B(x, y)\| > 1 - \eta(\varepsilon, B)$, existeix $(x_0, y_0) \in S_X \times S_Y$ tal que $\|B(x_0, y_0)\| = 1$, $\|x - x_0\| < \varepsilon$, i $\|y - y_0\| < \varepsilon$.

Obtenim el següent resultat.

Proposició 28. *Siguen X, Y espais de Banach. Si $(X^* \times Y, \mathbb{K})$ té la $\mathbf{L}_{o,o,\mathcal{B}}$, aleshores, $\overline{\mathrm{NA}_{\mathcal{N}}(X, Y)}^{\|\cdot\|_{\mathcal{N}}} = \mathcal{N}(X, Y)$. Si $(X \times Y, \mathbb{K})$ té la $\mathbf{L}_{o,o,\mathcal{B}}$, aleshores, $\overline{\mathrm{NA}_{\pi}(X \widehat{\otimes}_{\pi} Y)}^{\|\cdot\|_{\pi}} = X \widehat{\otimes}_{\pi} Y$.*

En particular, notem que les següents relacions són conegudes, proporcionant resultats positius de densitat.

Examples 29 ([48])**.** Siguen X, Y espais de Banach.

(i) Si $\dim(X), \dim(Y) < \infty$, aleshores $(X \times Y, Z)$ té la $\mathbf{L}_{o,o,\mathcal{B}}$ per a tot de Banach Z.

(ii) Si Y és uniformement convex, aleshores $(X \times Y, \mathbb{K})$ té la $\mathbf{L}_{o,o,\mathcal{B}}$ si i sols si (X, Y^*) té la $\mathbf{L}_{o,o}$.

(iii) Si $1 < p, q < \infty$, aleshores $(\ell_p \times \ell_q, \mathbb{K})$ té la $\mathbf{L}_{o,o,\mathcal{B}}$ si i sols si $p > q'$.

No obstant això, notem que la $\mathbf{L}_{o,o,\mathcal{B}}$ per a formes bilineals és una propietat molt restrictiva, ja que requereix que tots dos espais siguen reflexius per començar, i també hi ha parells d'espais reflexius sense la propietat, com acabem de veure. Per tant, necessitem un enfocament diferent per tal d'obtenir més resultats positius. Per fer servir allò que sabem sobre el context de dimensió finita, seria convenient tenir bons subespais dels nostres espais. Les normes projectives no respecten els subespais en general, però sí es comporten bé amb els subespais 1-complementats, per això ens interessa tenir una propietat que assegure l'existència de molts subespais 1-complementats adequats dels nostres espais. Per tant, considerem la propietat π mètrica.

Definició 30. Siga X un espai de Banach. Diem que X té la *propietat π mètrica* si donats $\varepsilon > 0$ i $\{x_1, \dots, x_n\} \subset S_X$ una col·lecció finita a la esfera unitat, podem encontrar un subespai 1-complementat de dimensió finita $M \subset X$ tal que per a cada $i \in \{1, \dots, n\}$ existeix $x_i' \in M$ amb $\|x_i - x_i'\| < \varepsilon$.

El concepte anterior realment és equivalent a la propietat d'aproximació π mètrica (una propietat d'aproximació on els operadors d'aproximació són tots projeccions amb norma 1), i això ens permet trobar molts més exemples d'espais per als quals tenim densitat (vegeu [23], [76] i [94] per a més informació sobre la propietat π). Es compleixen les propietats següents.

Teorema 31. *Siga Y un espai uniformement convex o un espai amb la propietat π mètrica. Si X (respectivament, X^*) té la propietat π mètrica, aleshores es té que $X \widehat{\otimes}_\pi Y = \overline{\mathrm{NA}_\pi(X \widehat{\otimes}_\pi Y)}^{\|\cdot\|_\pi}$ (respectivament, $\mathcal{N}(X, Y) = \overline{\mathrm{NA}_{\mathcal{N}}(X, Y)}^{\|\cdot\|_{\mathcal{N}}}$).*

Example 32. Els següents espais tenen la propietat π mètrica.

(i) Espais de Banach amb una descomposició de dimensió finita amb constant 1 (en conseqüència, tot espai de Banach amb una base de Schauder es pot renormar per a tindre la propietat π mètrica),

(ii) $L_p(\mu)$ ($1 \leqslant p < \infty$, μ qualsevol mesura) i duals isomètrics de L_1,

(iii) $X \oplus_a Y$, si X, Y tenen la propietat π mètrica i $|\cdot|_a$ és una norma absoluta,

(iv) $X = \left[\bigoplus_{n\in\mathbb{N}} X_n\right]_{c_0}$ o $\left[\bigoplus_{n\in\mathbb{N}} X_n\right]_{\ell_p}$, per a $1 \leqslant p < \infty$, si X_n té la propietat π mètrica per a tot n,

(v) $X\widehat{\otimes}_\pi Y$ i $X\widehat{\otimes}_\varepsilon Y$ quan X, Y tenen la propietat π mètrica.

Això mostra que a molts espais, la densitat es compleix. Remetem al article recent [41, Section 4] per a més resultats de densitat positius relacionats amb la RNP, espais duals i la propietat π mètrica (per exemple, si X^* i Y^* tenen la RNP i un d'ells té la propietat d'aproximació, aleshores $\overline{\mathrm{NA}_\pi(X^*\widehat{\otimes}_\pi Y^*)}^{\|\cdot\|_\pi} = X^*\widehat{\otimes}_\pi Y^*$, i si Y és qualsevol espai dual, aleshores $\overline{\mathrm{NA}_\pi(c_0\widehat{\otimes}_\pi Y)}^{\|\cdot\|_\pi} = c_0\widehat{\otimes}_\pi Y$). En aquest punt, és natural preguntar-se ara si sempre tenim densitat de tensors projectius o operadors nuclears que assoleixen les normes. Tanmateix, malgrat la nostra àmplia col·lecció de resultats positius, a la Secció 4.4, obtenim el següent resultat negatiu per als tensors.

Teorema 33. *Siga $\mathcal{R}$ l'espai de Read. Existeixen subespais X de c_0 i Y de $\mathcal{R}$ tals que el conjunt de tensors de $X\widehat{\otimes}_\pi Y^*$ que assoleixen les seues normes projectives no és dens en $X\widehat{\otimes}_\pi Y^*$.*

Cal assenyalar que la pregunta anàloga per als operadors nuclears roman oberta.

Finalment, notem que encara que no se sap si qualsevol operador de rang finit pot aproximar-se per operadors que assoleixen la seua norma, l'afirmació anàloga per als tensors no es compleix en general.

Proposició 34. *Hi ha tensors de rang finit que no es poden aproximar per tensors que assoleixen la seua norma projectiva.*

Resum del Capítol 5

El Capítol 5 té dues mitats ben diferenciades.

La primera mitat d'aquest capítol ha sigut publicada a

[84] V. Kadets and Ó. Roldán, Closed linear spaces consisting of strongly norm attaining Lipschitz mappings, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **116** (2022), Paper No. 162, 12 pp.

i la segona mitat ha aparegut a l'article enviat

[49] S. Dantas, R. Medina, A. Quilis, and Ó. Roldán, On isometric embeddings into the set of strongly norm-attaining Lipschitz functions. Preprint.

El 2001, Godefroy va preguntar si per a cada espai de Banach de dimensió infinita X, el conjunt de funcionals que assoleixen les seues normes $\mathrm{NA}(X, \mathbb{K})$ sempre conté espais lineals de dimensió 2 (vegeu [64, Problem III]). Aquesta pregunta va ser contestada negativament per Rmoutil el 2017: si $\mathcal{R}$ és el renormament de c_0 de Read ([103]), aleshores $\mathrm{NA}(\mathcal{R}, \mathbb{K})$ no conté espais lineals de dimensió 2. Al Capítol 5, estudiem la qüestió de l'espacialitat anàloga per al conjunt de funcions Lipschitz (reals) que assoleixen la seua norma fortament.

Siga M un espai mètric "pointed", és a dir, un espai mètric amb un punt distingit 0. $\mathrm{Lip}_0(M)$ és l'espai de Banach de les funcions Lipschitz

$f : M \to \mathbb{R}$ tals que $f(0) = 0$ dotat de la norma Lipschitz

$$\|f\| := \sup\left\{\frac{|f(y) - f(x)|}{d(x, y)} : \ x, y \in M, \ \ , x \neq y\right\}.$$

Es diu que una funció Lipschitz $f \in \mathrm{Lip}_0(M)$ assoleix la seua norma fortament si existeixen $x, y \in M$, $x \neq i$, tals que $\|f\| = \frac{|f(y)-f(x)|}{d(x,y)}$. El conjunt de funcions Lipschitz que assoleixen la seua norma fortament a M es denota $\mathrm{SNA}(M)$.

L'assoliment fort de norma de funcions Lipschitz, així com altres assoliments de norma més febles, s'han estudiat àmpliament durant els darrers anys, des dels primers treballs sobre el tema ([66, 83]). Se sap que l'assoliment fort de norma és en realitat bastant estricte (per exemple, segons [83, Lemma 2.2], si una funció assoleix fortament la seua norma en un parell (x, y), ha d'assolir-la llarg de tot el segment $[x, y]$ i ser afí sempre que estiga definida). Per aquesta raó, en molts espais mètrics M, $\mathrm{SNA}(M)$ passa a no ser dens a $\mathrm{Lip}_0(M)$, encara que també s'han obtingut resultats positius per a algun altres espais mètrics.

Clarament, si M té cardinal $n \in \mathbb{N}$, aleshores $\mathrm{SNA}(M) = \mathrm{Lip}_0(M)$, i és un espai de Banach. Al Capítol 5 abordem la pregunta següent: si M és infinit, $\mathrm{SNA}(M)$ sempre conté espais lineals de dimensió més gran que 1? Per com és d'estricte aquest assoliment de norma, i tenint en compte el treball de Rmoutil per a funcionals, es pot pensar que la resposta a aquesta pregunta pot ser negativa. No obstant això, veurem que açò s'allunya de ser cert. Per fer-ho, ens basem en diverses tècniques, com ara el teorema d'extensió de McShane (que ens permet estendre funcions Lipschitz d'un espai mètric M_1 a un espai mètric més gran M_2 conservant la seua norma), espais Lipschitz-free, i algunes altres eines. En aquest capítol, assumirem implícitament que tots els espais mètrics són complets (a causa del teorema d'extensió de McShane) i "pointed", i que tots els espais vectorials són reals.

En general, no és cert que si un espai de Banach X està a $\mathrm{SNA}(M)$ per a algun espai mètric M, aleshores podem estendre'l amb McShane i trobar el mateix espai X a $\mathrm{SNA}(M_2)$ per a cada espai mètric més gran M_2. Malgrat això, amb la norma $\|\cdot\|_1$, obtenim el següent resultat.

Lema 35. *Siga M un espai mètric "pointed" tal que per a algun subespai K de M, $\mathrm{SNA}(K)$ conté un subespai lineal isomètric a ℓ_1^n per a algun $n \in \mathbb{N}$. Aleshores, $\mathrm{SNA}(M)$ també conté un subespai isomètric a ℓ_1^n.*

Utilitzant aquest resultat i alguna altra tècnica, ara podem donar una resposta definitiva a la nostra pregunta.

Teorema 36. *Siga $n > 1$ un nombre natural, i siga M un espai mètric "pointed" amb almenys 2^n punts diferentss. Aleshores, existeix un subespai lineal de $\mathrm{SNA}(M)$ isomètric a ℓ_1^n.*

Corol·lari 37. *Si M és un espai mètric "pointed" infinit, aleshores per a tot $n \in \mathbb{N}$, $\mathrm{SNA}(M)$ conté un subespai n-dimensional isomètric a ℓ_1^n.*

Per tant, si M és infinit, $\mathrm{SNA}(M)$ no sols té subespais de dimensió almenys 2: de fet conté tots els ℓ_1^n, $n \in \mathbb{N}$, com a subespais isomètrics.

No és difícil veure que $\mathrm{SNA}([0,1])$ conté una còpia de c_0. Això porta a preguntar quins altres espais de Banach es poden formar. La resposta, sorprenentment, és que tots si es tria l'espai mètric adequat.

Proposició 38. *Si Y és qualsevol espai de Banach, aleshores Y és un subespai isomètric de $\mathrm{SNA}(B_{Y^*})$.*

També és interessant fer-se la pregunta inversa: donat un espai de Banach Y, com de "xicotet" pot ser un espai mètric M perquè Y siga subespai de $\mathrm{SNA}(M)$? Pel resultat anterior, si Y té dual separable, M pot ser separable, però i si no? El teorema següent mostra que això és, de fet, una caracterització.

Teorema 39. *Per a un espai de Banach donat Y, són afirmacions equivalents:*

(1) Existeix un espai mètrico separable M i un subespai lineal tancat $Z \subset \mathrm{Lip}_0(M)$ tal que Z és isomètric a Y i $Z \subset \mathrm{SNA}(M)$.

(2) Existeix un espai de Banach separable X i un subespai lineal tancat $Z_1 \subset X^*$ tal que Z_1 és isomètric a Y i $Z_1 \subset \mathrm{NA}(X, \mathbb{R})$.

(3) Y^* és separable.

Per tant, per a espais mètrics separables com $M = [0, 1]$, el conjunt $\mathrm{SNA}(M)$ no pot contenir subespais com ℓ_1 amb dual no separable. Això afegeix alguna restricció als espais mètrics separables. De fet, apareixen altres restriccions per a alguns espais mètrics xicotets, com ara els espais σ-precompactes, que inclouen per exemple tots els espais precompactes i tots els espais $\mathbb{R}^n$ (noteu que els espais σ-precompactes sempre són separables).

Teorema 40. *Si M és un espai σ-precompacte, aleshores tots els subespais de Banach de* $\mathrm{SNA}(M)$ *són separables i isomórficament polihèdrics.*

Pel que fa a resultats positius, hem esmentat que $\mathrm{SNA}([0, 1])$ conté c_0 isomètricament. De fet, això es pot estendre a una classe extensa d'espais mètrics que inclou tots els normats.

Proposició 41. *Si M és un espai mètric que conté* $[0, 1]$ *isomètricament, aleshores* $\mathrm{SNA}(M)$ *conté c_0 isomètricament.*

En realitat, és possible veure que per a tots els espais mètrics M amb una quantitat infinita de punts no aïllats, $\mathrm{SNA}(M)$ conté c_0 isomètricament. Però fins i tot en tots els espais M sense aquesta propietat que estudiem en profunditat, sempre resultava possible trobar c_0 a $\mathrm{SNA}(M)$ isomòrficament. Això va motivar a preguntar-nos si aquest sempre és el cas (vegeu [84, Questions 1 and 2]), és a dir: si M és infinit, $\mathrm{SNA}(M)$ sempre conté c_0 isomòrficament? Recentment, Avilés, Martínez-Cervantes, Rueda Zoca i Tradacete van respondre afirmativament aquesta pregunta mitjançant una elegant distinció de casos i ajudats del teorema de Ramsey.

Teorema 42 ([15, Main Theorem])**.** *Siga M un espai mètric complet "pointed" infinit. Aleshores,* $\mathrm{SNA}(M)$ *conté c_0 isomórficament.*

Quant a ficar isomètricament c_0 a $\mathrm{SNA}(M)$, els autors van demostrar a [15, Lemma 3.1] que si l'espai mètric involucrat satisfà certa propietat geomètrica (això es compleix, per exemple, per espais mètrics amb una quantitat infinita de punts no aïllats i per a espais mètrics discrets que no són uniformement discrets), aleshores $\mathrm{SNA}(M)$ conté c_0 isomètricament. Per a la resta d'espais mètrics, van deixar oberta la pregunta següent (vegeu [15, Remark 3.6]): si M és infinit, $\mathrm{SNA}(M)$ conté c_0 isomètricament? A [49], proporcionem una resposta definitiva a aquesta pregunta. Per fer-ho, primer trobem el següent resultat, que millora lleugerament les condicions de l'esmentat [15, Lemma 3.1].

Lema 43. *Siga Γ un conjunt índex no buit. Siga M un espai mètric "pointed" tal que existeixen dos conjunts $\{x_\gamma\}_{\gamma\in\Gamma}, \{y_\gamma\}_{\gamma\in\Gamma} \subset M$ amb $x_\gamma \neq y_\gamma$, $x_\alpha \neq x_\beta$ per a $\gamma, \alpha, \beta \in \Gamma$, $\alpha \neq \beta$. Si $d(x_\alpha, x_\beta) \geqslant d(x_\alpha, y_\alpha) + d(x_\beta, y_\beta)$ per a todo $\alpha \neq \beta \in \Gamma$, aleshores existeix un subespai lineal de* $\mathrm{SNA}(M)$ *isomètric a $c_0(\Gamma)$.*

Ajudant-nos d'aquest resultat i d'altres lemes tècnics, podem resoldre negativament la pregunta de [15, Remark 3.6]. Proporcionem dos contraexemples diferents amb comportaments oposats. Donat un punt $x \in M$, definim el seu *ràdi de separació* com la quantitat $R(x) := \inf\{d(x, y) \colon y \in M\backslash\{x\}\}$, i diem que *$x$ assoleix el seu radi de separació* si aquest ínfim és en realitat un mínim. Resumim els nostres resultats.

Teorema 44. *Hi ha espais mètrics M_1 i M_2 tals que* $\mathrm{SNA}(M_1)$ *i* $\mathrm{SNA}(M_2)$ *no contenen c_0 isomètricament i tals que M_1 és fitat i cap dels seus punts assoleix el seu radi de separació, i M_2 és "proper" no acotat i tots els seus punts assoleixen el seu radi de separació.*

Curiosament, també existeixen espais mètrics complets "pointed" infinits i uniformement discrets M amb c_0 contingut isomètricament a $\mathrm{SNA}(M)$ de tal manera que cap punt assoleix el seu radi de separació, o que tot punt assoleix el radi de separació. Pel que fa als espais mètrics que no són uniformement discrets, sorprenentment, $\mathrm{SNA}(M)$ sempre conté c_0 isomètricament.

Teorema 45. *Siga M un espai mètric infinit que no és uniformement discret. Aleshores, el conjunt* $\mathrm{SNA}(M)$ *conté una còpia isomètrica de c_0.*

Finalment, per a l'escenari no separable, usant el Lema 43 i un resultat inspirat en [71, Proposition 3], aconseguim el següent resultat positiu. Recordem que donat un espai mètric M, M' denota el conjunt de punts d'acumulació de M i $\mathrm{dens}(M)$ denota el caràcter de densitat de M.

Teorema 46. *Siga M un espai mètric "pointed" tal que* $\mathrm{dens}(M') = \Gamma$ *per a algun cardinal infinit Γ. Aleshores, existeix un subespai lineal de* $\mathrm{SNA}(M)$ *que es isomètricament isomòrfic a $c_0(\Gamma)$.*

El document conclou amb un capítol de Conclusions (vegeu la pàgina 277), que inclou alguns comentaris y problemes oberts. Al final del document hi ha una llista extensa de referències (vegeu la pàgina 287), seguida d'un glossari de conceptes i notacions.

Chapter 1

Introduction

1.1 About the text

This document is a PhD dissetation on Functional Analysis. The results in this document will include detailed proofs and references.

One may choose to read this document in its printed version if preferred. Nevertheless, anyone that chooses to read this document via a PDF viewer should be aware of some characteristics that these tools offer:

1. The PDF version of the text will contain hyperlinks to particular sections, results, references, and so on.

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1.2 Notation and preliminaries

The notations and concepts that we will use in this document can be found in books such as [53, 54]. In this section we will briefly introduce or recall some of the notations and basic concepts that will be used throughout the document.

The symbols $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{N}$ represent, respectively, the sets of *complex numbers*, *real numbers*, *integers*, and *naturals* (that is, positive integers, not including the number 0). Unless specified otherwise, all vector spaces in this document are defined over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

If (M, d), or just M, is a metric space, $c \in M$, and $R > 0$, we denote as $B(c, R)$ and $S(c, R)$ the *closed ball* and the *sphere* of center c and radius R, respectively, and an R*-net* (or an R*-separated set*) is a set $A \subset M$ such that for every $x, y \in A$ with $x \neq y$, we have $d(x, y) \geqslant R$. Recall that if $K > 0$, a mapping $f : M \to \mathbb{R}$ is K*-Lipschitz* if $|f(y) - f(x)| \leqslant Kd(x, y)$ for all $x, y \in M$ with $x \neq y$, and f is *Lipschitz* if it is K-Lipschitz for some $K > 0$.

The usual notations $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ will be used for *norms* and *inner products* in normed spaces and Hilbert spaces, respectively. Let $(X, \|\cdot\|)$, or just X, be a normed space. $B_X := B(0, 1)$ and $S_X := S(0, 1)$ respectively represent the *closed unit ball* and the *unit sphere* of X. X^* represents the *topological dual* of X, and X^{**} denotes the *bidual*. The elements of X^* are called *functionals*. The dual (evaluation) action of a functional $x^* \in X^*$ over a point $x \in X$ is denoted as $x^*(x)$. The canonical embedding of X into its bidual is the mapping $J_X : X \to X^{**}$ such that $J_X(x)(x^*) := x^*(x)$ for all $x \in X$ and all $x^* \in X^*$, which is an isometric isomorphism between X and $J_X(X)$.

Let X and Y be two normed spaces. $L(X, Y)$ is the set of *linear mappings* from X to Y, and $\mathcal{L}(X, Y)$ is the set of *linear and bounded mappings*

(equivalently, linear and continuous) from X to Y. The elements of $\mathcal{L}(X,Y)$ are called *operators*. Note that X^* is just $\mathcal{L}(X,\mathbb{K})$. $\mathcal{K}(X,Y)$ and $\mathcal{F}(X,Y)$ respectively represent the set of *compact* and of *finite-rank* operators from X to Y. Unless stated otherwise, we will say that two Banach spaces coincide if there exists an isometric isomorphism between them. Recall that $\mathcal{L}(X,Y)$ is a normed space (in fact, it is a Banach space whenever Y is a Banach space), when endowed with the *operator norm*, given by
$$\|T\| := \sup\{\|T(x)\|\colon\ \|x\| = 1\}.$$

An operator $T \in \mathcal{L}(X,Y)$ is said to *attain its norm*, or to be *norm-attaining*, if there exists $x_0 \in S_X$ such that $\|T(x_0)\| = \|T\|$. The *set of norm-attaining operators from X to Y* is denoted as $\mathrm{NA}(X,Y)$. If $T \in \mathcal{L}(X,Y)$, the operator $T^* \in \mathcal{L}(Y^*,X^*)$ defined by $T^*(y^*)(x) := y^*(T(x))$ for all $y^* \in Y^*$ and $x \in X$, is called the *adjoint operator* of T. It is a well-known fact that an operator is compact if and only if its adjoint is also compact.

A set of points $A \subset S_X$ is *norming* for a set of functionals $B \subset X^*$ if for all $b \in B$, $\|b\| = \sup\{\|b(a)\|\colon\ a \in A\}$. If $B = X^*$, we just say that A is norming. Similarly, a set of functionals $B \subset S_{X^*}$ is *norming* for a set of points $A \subset X$ if for all $a \in A$ we have $\|a\| = \sup\{\|b(a)\|\colon\ b \in B\}$. If $A = X$, we just say that B is norming.

Let X, Y, Z be Banach spaces. The notation $\mathcal{B}(X \times Y, Z)$ represents the set of *bilinear mappings* from $X \times Y$ to Z. This is a Banach space when endowed with the norm

$$\|T\|_{\mathcal{B}} := \sup\{\|T(x,y)\|\colon\ (x,y) \in S_X \times S_Y\}.$$

If $Z = \mathbb{K}$, then these mappings are called *bilinear forms*. It is clear that $\mathcal{B}(X \times Y, \mathbb{K})$, $\mathcal{L}(X, Y^*)$, and $\mathcal{L}(Y, X^*)$ are all isometrically isomorphic with the natural identifications $T(x, y) \equiv T(x)(y) \equiv T(y)(x)$. A bilinear mapping $T \in \mathcal{B}(X \times Y, Z)$ *attains its norm* if there exists some pair $(x_0, y_0) \in S_X \times S_Y$ such that $\|T\|_{\mathcal{B}} = \|T(x_0, y_0)\|$. The set of norm-attaining bilinear mappings from $X \times Y$ to Z is denoted $\mathrm{NA}_{\mathcal{B}}(X \times Y, Z)$. Note also that if a mapping $T \in \mathcal{B}(X \times Y, \mathbb{K})$ attains its norm, then its associated operators $T \in \mathcal{L}(X, Y^*)$ and $T \in \mathcal{L}(X, Y^*)$ also attain their respective norms, although the converse is not true in general.

Let X be a Banach space and let $T \in \mathcal{L}(X, X)$. The *numerical range* of T is $V(T) := \{x^*(T(x)) \colon (x, x^*) \in \Pi(X)\}$, where $\Pi(X) := \{(x, x^*) \in S_X \times S_{X^*} \colon x^*(x) = 1\}$ is the *set of states* of X (note that the numerical range of $T \in \mathcal{L}(X, X)$ is an extension of the concept of numerical range in Hilbert spaces, $W(T) := \{\langle T(x), x\rangle \colon (x, x) \in \Pi(H)\}$, for $T \in \mathcal{L}(H, H)$, H Hilbert space). The *numerical radius* of T is

$$\nu(T) := \sup\{|\lambda| : \lambda \in V(T)\}$$

(see the books [19, 20] for a solid background on this topic). T is said to *attain its numerical radius* if there exists some $(x_0, x_0^*) \in \Pi(X)$ such that $x^*(T(x)) = \nu(T)$. The set of numerical radius attaining operators on X is denoted as $\mathrm{NRA}(X)$. It is easy to see that ν is a seminorm and that $0 \leqslant \nu(T) \leqslant \|T\|$ for every $T \in \mathcal{L}(X, X)$. The *numerical index* of X is a way of measuring how similar ν and $\|\cdot\|$ are, and it is defined as

$$\begin{aligned} n(X) &:= \max\{k \geqslant 0 \colon k\|T\| \leqslant \nu(T) \text{ for all } T \in \mathcal{L}(X, X)\} \\ &= \inf\{\nu(T) \colon T \in S_{\mathcal{L}(X,X)}\}. \end{aligned}$$

If $n(X) = 1$, then ν and $\|\cdot\|$ coincide for every operator, if $n(X) > 0$, then ν is a norm equivalent to $\|\cdot\|$, and if $n(X) = 0$, then ν is not an equivalent norm to $\|\cdot\|$. It is worth noting that $n(X)$ can attain every possible value in $[0,1]$ in the real case, and that if X is complex, then $n(X) \geqslant \frac{1}{e}$. For spaces with numerical index 0, we can define a related concept that is also useful: the *second numerical index* of X, defined as

$$n'(X) = \inf\{\nu(S)\colon\ T \in \mathcal{L}(X,X),\ \|T + \mathcal{Z}(X)\| = 1\},$$

where $\mathcal{Z}(X) = \{S \in \mathcal{L}(X,X)\colon\ \nu(S) = 0\}$ is the Lie group of the skew-hermitian operators (see [89, p. 1004] for the details), and $\|T + \mathcal{Z}(X)\|$ is just the quotient norm of $\mathcal{L}(X,X)/\mathcal{Z}(X)$. These concepts will be adapted to compact operators in Section 3.1, and we refer to [26], [81], [82], [89, Subsection 1.1], and references therein for more information and background.

In Chapter 4, we will make extensive use of tensors. Let us recall some basics (we refer to [107] for more background and information). Let V, W be two vector spaces. Their *tensor product* $V \otimes W$ is the linear subspace of $L(\mathcal{B}(X \times Y, \mathbb{K}), \mathbb{K})$ (algebraic dual) spanned by the evaluation mappings

$$(v \otimes w)(A) = A(v,w), \quad A \in \mathcal{B}(X \times Y, \mathbb{K}),\ v \in V,\ w \in W.$$

Tensors allow us to *linearize* bilinear mappings thanks to the following universal property: if $\mu : V_1 \times V_2 \to V_1 \otimes V_2$ is the canonical mapping such that $\mu(v_1, v_2) = v_1 \otimes v_2$, then for any bilinear mapping $f \in \mathcal{B}(V_1 \times V_2, W)$, there is a unique linear mapping $\overline{f} : V_1 \otimes V_2 \to W$ with $\overline{f}(v_1 \otimes v_2) = f(v_1, v_2)$. The notions of *projective tensor product*, *nuclear operator* and *injective tensor product*, as well as relations between them, will be introduced in Section 4.1.1.

Finally, we will briefly recall the notations for the classical Banach spaces. c_0 is the space of sequences that converge to 0 endowed with the supremum norm. If $1 \leqslant p \leqslant \infty$, ℓ_p is the space of sequences whose p-norm is finite, endowed with $\|\cdot\|_p$; moreover, if $n \in \mathbb{N}$, ℓ_p^n is the finite-dimensional space $(\mathbb{K}^n, \|\cdot\|_p)$. If I is an arbitrary index set, $c_0(I, \mathbb{K})$ is the Banach space of functions $x : I \to \mathbb{K}$ for which the set $\{i \in I\colon |x(i)| \geqslant \varepsilon\}$ is finite for every $\varepsilon > 0$, endowed with the supremum norm $\|x\|_\infty := \sup\{|x(i)|\colon i \in I\}$. If $\mathbb{K}$ is clear or irrelevant in the context, we can omit it and just write $c_0(I)$. Note that the elements of $c_0(I, \mathbb{K})$ are nets with at most a countable amount of non-zero elements. Note also that $c_0(\mathbb{N}, \mathbb{K})$ is just the space c_0 over $\mathbb{K}$. If two index sets I_1 and I_2 are bijective, $c_0(I_1, \mathbb{K})$ is isometrically isomorphic to $c_0(I_1, \mathbb{K})$, and so, if their cardinality is Γ, we will denote $c_0(\Gamma, \mathbb{K})$ to the space $c_0(I, \mathbb{K})$ for any set I of cardinality Γ. For this reason, for simplicity, we will usually just write $c_0(\Gamma, \mathbb{K})$ (or just $c_0(\Gamma)$) both when Γ represents a set and when it represents its cardinal. Similarly, one can define its dual space, $\ell_1(\Gamma, \mathbb{K})$, as the set of functions $x : \Gamma \to \mathbb{K}$ with countable support and such that the norm $\|x\|_1 := \sum_{\gamma\in\Gamma} |x(\gamma)|$ is finite. Like before, if $\mathbb{K}$ is clear or irrelevant in the context it can be omitted, and $\ell_1(\mathbb{N}, \mathbb{K})$ is just the space ℓ_1 over $\mathbb{K}$. Note that if $\Gamma = \{1, \ldots, n\}$ for some $n \in \mathbb{N}$, the spaces $c_0(\Gamma, \mathbb{K})$ and $\ell_1(\Gamma, \mathbb{K})$ are just the spaces ℓ_∞^n and ℓ_1^n over $\mathbb{K}$. Recall that $c_0^* = \ell_1$, $\ell_1^* = \ell_\infty$, and if $1 < p < \infty$ and $1 < q < \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, then $\ell_p^* = \ell_q$ in an isometrically isomorphic sense, and the same is true for their finite-dimensional versions.

Let Y be a Banach space. Let K be a compact Hausdorff topological space, and let L be a locally compact Hausdorff topological space. We denote by $C(K, Y)$ the Banach space of continuous functions $f : K \to Y$ endowed with the supremum norm $\|f\|_\infty := \sup\{\|f(x)\| : x \in K\}$, and we denote by $C_0(L, Y)$ the set of continuous functions $f : L \to Y$ which vanish at infinity (that is, for all $\varepsilon > 0$, there is a compact set $K \subset L$ such that $\|f(x)\| < \varepsilon$ for all $x \in L \backslash K$), which is again a Banach space when endowed with the supremum norm $\|f\|_\infty := \sup\{\|f(x)\| : x \in L\}$. If $Y = \mathbb{K}$, we can omit it and just write $C(K)$ and $C_0(L)$, respectively.

Note that $C_0(\mathbb{N})$ is just c_0. Let $1 \leqslant p \leqslant \infty$, and (Ω, Σ, μ) a measure space. We denote $L_p(\mu, Y)$ (if $Y = \mathbb{K}$, it can be omitted from the notation) the set of strongly measurable functions $f : \Omega \to Y$ identified when they differ in a set of measure 0, such that $\|f\|^p$ is integrable if $p < \infty$ or f is essentially bounded if $p = \infty$, where

$$\begin{cases} \|f\|_p := \left(\int_\Omega \|f(x)\|^p \,\mathrm{d}\mu\right)^{1/p}, & \text{if } p < \infty, \\ \|f\|_\infty := \sup\{\|f(x)\| \colon\ x \in \Omega\}. \end{cases}$$

We end this section recalling the following definition. A Banach space X has the *Radon-Nikodým property* (abbreviated RNP) if the Radon-Nikodým theorem is valid in X, that is: if Σ is a σ-algebra of subsets of Ω and $\mu : \Sigma \to X$ is a vector measure of bounded variation that is absolutely continuous with respect to a finite, positive measure λ, then there exists a λ-Bochner integrable function $f : \Omega \to X$ such that $\mu(E) = \int_E f \,\mathrm{d}\mu$, for all $E \in \Sigma$. For instance, reflexive spaces, ℓ_1, and subspaces of spaces with the RNP have the RNP, but $L_1([0,1])$ and $C(K)$ for an infinite compact set K do not have the RNP. The RNP, in its many equivalent reformulations, has historically proven to be a powerful tool in order to get density results for norm-attaining operators (see [21] for instance).

1.3 Historical background

In 1957/1964, James proved that a Banach space is reflexive if and only if all its functionals attain their norm (see [74, 75]). In 1961, Bishop and Phelps showed that, actually, if X is a Banach space, the norm-attaining functionals on X always form a dense set in the dual, or in other words, any functional $x^* \in X^*$ can be approximated by a nearby functional

$y^* \in X^*$ that attains its norm (see [17]). They wondered if this density also held in general for operators between Banach spaces X and Y, that is, if $\mathrm{NA}(X,Y)$ is always dense in $\mathcal{L}(X,Y)$. Lindenstrauss showed in 1963 that this is not the case in general by providing some counterexamples, but he also introduced a series of techniques and properties that allow to get positive results in some cases (see [93]). According to Lindenstrauss' paper, a Banach space X is said to satisfy Property A if for every Banach space Y, $\mathrm{NA}(X,Y)$ is dense in $\mathcal{L}(X,Y)$, and a Banach space Y is said to satisfy Property B if for every Banach space X, $\mathrm{NA}(X,Y)$ is dense in $\mathcal{L}(X,Y)$. With this notation, he showed for instance that every reflexive space has Property A, and that every space satisfying a certain geometrical property called Property β (which is satisfied, for instance, by finite-dimensional polyhedral spaces and by any Banach space X with $c_0 \subset X \subset \ell_\infty$) has Property B. Ever since this work was published, the question of for what Banach spaces X and Y we have that $\mathrm{NA}(X,Y)$ is dense in $\mathcal{L}(X,Y)$ has intrigued many mathematicians for the last 6 decades, and hundreds of works have been made about this topic. Just to name some, Bourgain, Huff, Johnson, Schachermayer, Uhl, Wolfe, and Zizler continued the study about the set of all linear operators which attain their norms ([21, 72, 77, 110, 113, 115]). We refer to [2] for a survey with the most important results on norm-attaining operators up to 2006.

In 1970, Bollobás improved the statement of the Bishop-Phelps theorem from [17] by showing that, in fact, for a Banach space X, if $x^* \in S_{X^*}$ almost attains its norm at $x \in S_X$, then they can be approximated by some $y^* \in S_{X^*}$ and $y \in S_X$ such that y^* attains its norm at y (see [18]; see also [25, Corollary 2.4.b] for the sharpest version of this result). Note that this implies in particular that the set of norm-attaining functionals is dense in the dual, so this is indeed a strengthening of the result by Bishop and Phelps. Inspired by this idea of a double approximation of operators and points, in 2008, Acosta, Aron, García, and Maestre introduced as follows and studied the Bishop-Phelps-Bollobás property.

Definition 1.3.1 ([5, Definition 1.1])**.** A pair of Banach spaces (X, Y) has the *Bishop-Phelps-Bollobás property* (abbreviated BPBp) if given $\varepsilon \in (0, 1)$, there exists $\eta(\varepsilon) > 0$ such that whenever $T \in \mathcal{L}(X, Y)$ and $x \in S_X$ satisfy $\|T\| = 1$ and $\|T(x)\| > 1 - \eta(\varepsilon)$, there are $S \in \mathcal{L}(X, Y)$ and $y \in S_X$ such that $\|S\| = \|S(y)\| = 1$, $\|x - y\| < \varepsilon$, and $\|S - T\| < \varepsilon$.

Note that if a pair of Banach spaces X and Y satisfy the BPBp, then $\mathrm{NA}(X, Y)$ is dense in $\mathcal{L}(X, Y)$, although the converse is not true in general. (X, Y) is known to have the BPBp in many cases, such as when both spaces are finite-dimensional, when Y has Property β of Lindenstrauss, or when X is uniformly convex. The BPBp has been studied by many authors in the past several years (see the paper [10] and the surveys [3, 40] and the references therein for a complete exposition of results about the BPBp up to 2022). Several interesting variations of the BPBp have also been introduced and studied lately by doing specific changes to Definition 1.3.1, such as the following ones:

- The BPBop: the BPBp but $S = T$.
- The $\mathbf{L}_o$: the BPBp but for each previously fixed T, you find an $\eta(\varepsilon, T)$ depending also on T.
- The $\mathbf{L}_{o,o}$: the $\mathbf{L}_o$ but also $S = T$.

See [38, 45, 47, 86, 108, 112] for more information, and [44, 46, 47] for "point" counterparts of those properties. We also refer to [40, Sections 4 and 5] for a detailed summary of the known results about all these properties and the relations between them. Those variants turned out to be strongly connected to the geometry of the unit ball of the involved Banach spaces.

This wide study of norm-attaining operators has also been extended to other kinds of mappings and norms. For instance, norm-attaining multilinear mappings, homogeneous polynomials, holomorphic functions, compact operators, and Lipschitz mappings have all been studied for a long time, and BPBp properties have also been introduced and studied for these contexts. In fact, nowadays we have a large literature about various classes of functions which attain their norms and satisfy a Bollobás type result (see, for instance, [1, 5, 33, 47] and the references therein, and we refer once more to the surveys [3, 40] for a complete exposition of results on the Bishop-Phelps-Bollobás property in several contexts).

In his 1972 PhD dissertation, Sims raised a question that is, in nature, related to the one that Lindenstrauss tackled in 1963: the norm-denseness of the set of numerical radius attaining operators on a Banach space X (see [111]). Many authors have contributed to this question ever since (see for instance Acosta's PhD dissertation, where this question is studied systematically, [1], and see also [69] and the references therein for a summary of the main known results on the topic). In 2013, Guirao and Kozhushkina ([69]) introduced and studied a version of the BPBp for numerical radius, defined as follows.

Definition 1.3.2 (Combining [69, Definition 1.2] and [87, Definition 5]). A Banach space X has the *weak Bishop-Phelps-Bollobás property for the numerical radius* (weak BPBp-nu, for short) if given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that, whenever $T \in \mathcal{L}(X, X)$ with $\nu(T) = 1$ and $(x, x^*) \in \Pi(X)$ satisfy $|x^*(T(x))| > 1 - \eta(\varepsilon)$, there exist $S \in \mathcal{L}(X, X)$ and $(y, y^*) \in \Pi(X)$ such that

$$\nu(S) = |y^*(S(y))|, \quad \|x - y\| < \varepsilon, \quad \|x^* - y^*\| < \varepsilon, \quad \text{and} \quad \|T - S\| < \varepsilon.$$

If, moreover, S can be chosen so that $\nu(S) = 1$, we say that X has the *Bishop-Phelps-Bollobás property for the numerical radius* (abbreviated BPBp-nu, although some authors use the notation BPBp-ν as well).

Many Banach spaces are known to have the BPBp-nu (see [3, Section 6] and [40, Section 2.7] for a complete exposition of the known results on the topic). A list of some important ones: $c_0(\Gamma)$ and $\ell_1(\Gamma)$ spaces (see [69]), finite-dimensional Banach spaces (see [87]), $L_1(\mathbb{R})$ (see [55]) and, in fact, $L_p(\mu)$ spaces for $1 \leqslant p < \infty$ and for any measure μ (see [87, 89]), and real $C(K)$ spaces whenever the Hausdorff compact space K has local compensation (for instance when K is metrizable, see [13]). In Chapter 2, we introduce and study some classes of operators for which a property like the $\mathbf{L}_{o,o}$ holds, as well as their analogous for numerical radius (that is, we will study norm-attaining operators that whenever they almost attain their norm at a point, they attain it at a nearby point, and the same for numerical radius).

As mentioned earlier, the theory of norm-attaining operators has also been extended and studied for some classes of operators. It is particularly relevant for us the study of norm-attaining compact operators (see for instance [77, 97, 98] and the references therein for a solid background on the topic). Inspired by the work [5], and extending some ideas from [77], Dantas, García, Maestre, and Martín introduced and studied a version of the BPBp for compact operators (it is like Definition 1.3.1, but with both T and S being compact, see [39]). The numerical radius attaining compact operators have also been studied (see for instance [22]). In Chapter 3 we introduce and study a version of the BPBp-nu for compact operators.

Two of the main open questions in norm-attaining theory were whether every finite-rank operator can be approximated by norm-attaining operators (remains as an open question nowadays) and whether every compact operator can be approximated by norm-attaining operators (answered in the negative by Miguel Martín in [97]). Nuclear operators are in between finite-rank operators and compact operators, and projective tensors are closely related (we refer to the book [107] for a solid background on tensor products and nuclear operators). In Chapter 4, we introduce and study norm-attainment concepts for nuclear operators and projective tensors.

Norm-attainment notions have also been studied for Lipschitz mappings. Strongly norm-attaining Lipschitz mappings were first introduced and studied in [66, 83], and ever since, they have been studied by many authors (see for instance [24, 29] and the references therein, and see also [32, Section 1] for a very clean exposition of various kinds of norm-attainment for Lipschitz mappings and the relations between them). The possibility to embed c_0 and ℓ_∞ isometrically in $\mathrm{Lip}_0(M)$ was solved in [36, 37] (see also [71]).

In 2001, Godefroy asked if for every infinite-dimensional Banach space X, the set of norm-attaining functionals over X always contained a 2-dimensional linear subspace (see [64, Problem III]), and Rmoutil showed in 2017 that this is not always the case (see [104]). In Chapter 5, we study the analogous question for the set of strongly norm-attaining Lipschitz mappings over an infinite metric space M.

1.4 Structure of the text

As hinted before and exposed in the Summary (see page 15), the rest of the document will be structured in chapters as follows. In Chapter 2, the class $\mathcal{A}_{\|\cdot\|}(X,Y)$ of operators $T \in \mathcal{L}(X,Y)$ such that whenever they almost attain their norm at a point x they do attain it at a nearby point y, and the analogous corresponding class $\mathcal{A}_{\mathrm{nu}}(X)$ for the numerical radius, are introduced and studied. In Section 2.1, the main concepts will be motivated and defined. In Section 2.2, a first collection of results and examples about these sets will be provided. For instance, several properties are obtained for finite-dimensional Banach spaces, functionals, compact operators, and adjoint operators, and a vast amount of examples show how sharp the conditions of these results are. We will see several examples of operators that belong to both of our sets, to just one of them, or to neither of them. In Section 2.3, a complete characterization will be given of all the diagonal operators that belong to $\mathcal{A}_{\|\cdot\|}(X,X)$ ($X=c_0$ or ℓ_p, $1\leqslant p\leqslant\infty$), to $\mathcal{A}_{\mathrm{nu}}(X)$ ($X=c_0$ or ℓ_p, $1\leqslant p<\infty$), to $\mathcal{A}_{\|\cdot\|}(c_0,\ell_p)$ ($1<p<\infty$) and to $\mathcal{A}_{\|\cdot\|}(\ell_p,c_0)$ ($1<p<\infty$). In particular, it is shown that every canonical projection P_N belongs to both $\mathcal{A}_{\|\cdot\|}(X,X)$ and $\mathcal{A}_{\mathrm{nu}}(X)$ when $X=c_0$ or ℓ_p ($1\leqslant p\leqslant\infty$). Finally, in Section 2.4, relations between the sets $\mathcal{A}_{\|\cdot\|}(W,Z)$ and $\mathcal{A}_{\mathrm{nu}}(W\oplus Z)$ will be studied for some particular types of direct sums and Banach spaces, and several results, remarks, and examples will be exhibited. The contents of this chapter have been taken from the published paper [42].

In Chapter 3, the Bishop-Phelps-Bollobás property for the numerical radius (BPBp-nu) will be adapted to the setting of compact operators and studied. In Section 3.1, the topic will be introduced and motivated. In Section 3.2, from the known spaces that have the BPBp-nu, a wide list of Banach spaces having the BPBp-nu for compact operators will be immediately obtained, including finite-dimensional spaces, $c_0(\Gamma)$ and $\ell_1(\Gamma)$ spaces, and every $L_p(\mu)$ space ($1\leqslant p<\infty$, μ any measure). In Section 3.3, some technical tools will be provided to show that if certain spaces have the BPBp-nu for compact operators, then so do some other spaces. These tools will be extensively used for the rest of the chapter. It will be shown, for instance, that every isometric predual of ℓ_1 has the property. Finally, in Section 3.4, some sort of approximation property is obtained in all $C_0(L)$ spaces, and this is used to show that every $C_0(L)$

space has the BPBp-nu for compact operators whenever L is a locally compact Hausdorff space. The contents of this chapter have been taken from the published paper [59].

In Chapter 4, a notion of norm-attainment is introduced and studied in the setting of projective tensor products and nuclear operators. In Section 4.1, the main notions are introduced and motivated. In Section 4.2, a first collection of results and examples will be provided. It will be shown in particular that there exist spaces where every projective tensor and every nuclear operator attains their respective norms, but that there are also spaces where this does not hold. In Section 4.3, the density of the norm-attaining projective tensors and nuclear operators will be studied. Several results and examples will be provided for which this density holds. Finally, in Section 4.4 we will see that there also exist Banach spaces where such a density does not hold for the case of projective tensors. The contents of this chapter have been taken from the following published paper [43].

Finally, in Chapter 5, the spaceability of the set of strongly norm-attaining Lipschitz functionals $\mathrm{SNA}(M)$ over metric spaces M will be studied. This chapter will have two parts (the first one from the published paper [84], and the second one from the submitted paper [49]). The first part will be devoted to show that if M is infinite, then $\mathrm{SNA}(M)$ contains linear spaces of dimension greater than 1. In Section 5.1, the topic will be introduced and motivated. In Section 5.2, it is shown that if M is infinite, then $\mathrm{SNA}(M)$ actually has all the ℓ_1^n $(n \in \mathbb{N})$ as subspaces isometrically. In Section 5.3, several related questions are tackled in order to study the possible sizes of such subspaces. It is shown for instance that every Banach space Y is a subspace of $\mathrm{SNA}(M)$ for an appropriate metric space M. It is also shown that if a Banach space Y is a subspace of $\mathrm{SNA}(M)$, then the separability of Y^* is equivalent to the separability of M. On top of that, a positive result is obtained for metric spaces containing $[0,1]$ isometrically, and a set of restrictions are obtained for σ-compact metric spaces. In [15], the authors show that if M is infinite, then $\mathrm{SNA}(M)$ contains an isomorphic copy of c_0, answering [84, Questions 1 and 2], and they left as an open question whether or not this embedding could always be isometric (see [15, Remark 3.6]). In the second part of Chapter 5, we will provide a definitive negative answer to their question by finding 2 counterexamples with very different behaviours as metric spaces: one where no points have a closest point, and one where every point has a closest point. We also show that if M is not uniformly discrete, then the answer to their question is actually positive. Finally, using some technical lemmas, we obtain a positive result in the non-separable case.

Finally, the Conclusions chapter (page 277) contains a list of final remarks, including some open questions on the topics treated in the chapters. The document concludes with an extensive list of references and a glossary (see page 287).

Chapter 2

Classes of operators that satisfy local Bollobás properties

2.1 Introduction and Motivation

In this chapter we will study a set of bounded linear operators which satisfy a very specific version of a Bollobás type theorem. This was motivated by the natural question of what pairs of Banach spaces (X, Y) over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ satisfy a strong version of Bollobás theorem where any operator almost attaining its norm at a point will necessarily attain it at a nearby point. More specifically, for X and Y, given $\varepsilon > 0$, is it true that there exists $\eta(\varepsilon) > 0$ such that whenever $\|T(x)\| > 1 - \eta$ ($T \in S_{\mathcal{L}(X,Y)}$, $x \in S_X$), there exists $x_0 \in S_X$ such that $\|x - x_0\| < \varepsilon$ and T attains its norm at x_0? Pairs of Banach spaces (X, Y) for which that claim holds are said to satisfy the Bishop-Phelps-Bollobás operator property (abbreviated BPBop), which is a stronger form of the Bishop-

Phelps-Bollobás property (BPBp). It turns out that the BPBop is very restrictive, to the point where the answer for this question is negative whenever the dimensions of the involved Banach spaces are bigger than or equal to 2 (see [45, Theorem 2.1]) but, on the other hand, this property characterizes uniformly convex Banach spaces when we consider the problem for the case of functionals, that is, when $Y = \mathbb{K}$ (see [86, Theorem 2.1]). Since there is no hope for a uniform version for the operator case of this problem (in the sense that η depends only on the given $\varepsilon > 0$) and the functional case is completely characterized, it seems to be reasonable to consider the same problem but now taking η depending not just on ε but also on the previously fixed operator T. This new property, named the $\mathbf{L}_{o,o}$, was studied for instance in [38, 47, 48, 108, 112], and many positive results were obtained, unlike in the uniform case. In this chapter, instead of studying spaces that satisfy the $\mathbf{L}_{o,o}$ property globally (for every operator), we will study classes of operators that satisfy such a property, as well as its analogous version for the numerical radius. Let us give the precise definitions.

Definition 2.1.1. Let X, Y be Banach spaces.

(i) $\mathcal{A}_{\|\cdot\|}(X,Y)$ stands for the set of all norm-attaining operators $T \in \mathcal{L}(X,Y)$ with $\|T\| = 1$ such that if $\varepsilon > 0$, then there is $\eta(\varepsilon, T) > 0$ such that whenever $x \in S_X$ satisfies $\|T(x)\| > 1 - \eta(\varepsilon, T)$, there is $x_0 \in S_X$ such that $\|T(x_0)\| = 1$ and $\|x_0 - x\| < \varepsilon$.

(ii) $\mathcal{A}_{\text{nu}}(X)$ stands for the set of all numerical radius attaining operators $T \in \mathcal{L}(X,X)$ with $\nu(T) = 1$ such that if $\varepsilon > 0$, then there is $\eta(\varepsilon, T) > 0$ such that whenever $(x, x^*) \in \Pi(X)$ satisfies $|x^*(T(x))| > 1 - \eta(\varepsilon, T)$, there is $(x_0, x_0^*) \in \Pi(X)$ such that $|x_0^*(T(x_0))| = 1$, $\|x_0 - x\| < \varepsilon$, and $\|x_0^* - x^*\| < \varepsilon$.

Note that if (X, Y) satisfy the $\mathbf{L}_{o,o}$ property, then $S_{\mathcal{L}(X,Y)} = \mathcal{A}_{\|\cdot\|}(X, Y)$, and since every operator must attain its norm, X must be reflexive by James' theorem. Studying the set $\mathcal{A}_{\|\cdot\|}$ gives us more freedom in the sense that we do not have to restrict ourselves to any condition on the involved spaces, just on the definition of a specific operator. Note also that, as we will see in Chapter 4, the $\mathbf{L}_{o,o}$ property has been recently used in [43] as a tool to prove that, for some Banach spaces, every nuclear operator can be approximated (in the nuclear norm) by nuclear operators which attain their nuclear norms (see Proposition 4.3.3). This makes us think that studying the sets $\mathcal{A}_{\|\cdot\|}$ and $\mathcal{A}_{\text{nu}}$ might be helpful to get similar results in the context of tensor products by using an analogous definition of the set $\mathcal{A}_{\|\cdot\|}$ for bilinear mappings. Thanks to the natural isometric identification between the bilinear mappings on $X \times Y$ and the operators from X into Y^*, our study on the sets $\mathcal{A}_{\|\cdot\|}$ and $\mathcal{A}_{\text{nu}}$ for operators might derive in new progresses on both nuclear operators and projective tensors which attain their nuclear and projective norms, respectively.

Let us briefly describe the outline of this chapter. In Section 2.2, a first collection of results and examples about these sets will be provided. For instance, several properties are obtained for finite-dimensional Banach spaces, functionals, compact operators, and adjoint operators, and a vast amount of examples show how sharp the conditions of the results are. We will see several examples of operators that belong to both of our sets, to just one of them, or to neither of them. In Section 2.3, a complete characterization will be given of all the diagonal operators that belong to $\mathcal{A}_{\|\cdot\|}(X, X)$ ($X = c_0$ or ℓ_p, $1 \leqslant p \leqslant \infty$), to $\mathcal{A}_{\text{nu}}(X)$ ($X = c_0$ or ℓ_p, $1 \leqslant p < \infty$), to $\mathcal{A}_{\|\cdot\|}(c_0, \ell_p)$ ($1 < p < \infty$) and to $\mathcal{A}_{\|\cdot\|}(\ell_p, c_0)$ ($1 < p < \infty$). In particular, it is shown that every canonical projection P_N belongs to both $\mathcal{A}_{\|\cdot\|}(X, X)$ and $\mathcal{A}_{\text{nu}}(X)$ when $X = c_0$ or ℓ_p ($1 \leqslant p \leqslant \infty$). Finally, in Section 2.4, relations between the sets $\mathcal{A}_{\|\cdot\|}(W, Z)$ and $\mathcal{A}_{\text{nu}}(W \oplus Z)$

will be studied for some particular types of direct sums, and several results, remarks, and examples will be exhibited.

2.2 First results

In this section, we present a first collection of results on the topic. We will start discussing the operators with a finite-dimensional domain and the functionals. Recall that in finite-dimensional Banach spaces, every operator T attains its norm (and its numerical radius whenever it can be defined) by compactness of the closed unit ball. The following result shows that, when X is finite-dimensional, we can entirely describe the sets $\mathcal{A}_{\|\cdot\|}(X,Y)$ and $\mathcal{A}_{\text{nu}}(X)$. Moreover, the ideas can be extended to show that $\mathcal{A}_{\|\cdot\|}(c_0,\mathbb{K}) = S_{\ell_1} \cap \text{NA}(c_0,\mathbb{K})$.

Theorem 2.2.1. *Let X be a finite-dimensional Banach space. Then*

(i) $\mathcal{A}_{\|\cdot\|}(X,Y) = \{T \in \mathcal{L}(X,Y) : \|T\| = 1\}$ *for any Banach space Y,*

(ii) $\mathcal{A}_{\text{nu}}(X) = \{T \in \mathcal{L}(X,X) : \nu(T) = 1\}$,

(iii) *Every norm one functional on c_0 which attains the norm belongs to $\mathcal{A}_{\|\cdot\|}(c_0,\mathbb{K})$.*

Proof. Items (i) and (ii) are proved by using the compactness of the closed unit ball of the finite-dimensional space X as in [5, Proposition 2.4] or [38, Theorem 2.4]. We include the details for the sake of completeness.

To prove (i), note that we just have to show that $S_{\mathcal{L}(X,X)} \subset \mathcal{A}_{\|\cdot\|}(X,X)$. Indeed, if this is not the case, then there are $\varepsilon_0 > 0$ and a norm-attaining operator $T \in S_{\mathcal{L}(X,X)}$ such that for all $n \in \mathbb{N}$, there is $x_n \in S_X$ with

$$1 \geqslant |T(x_n)| \geqslant 1 - \frac{1}{n},$$

and whenever $x \in S_X$ satisfies $\|x - x_n\| < \varepsilon_0$, we have $|T(x)| < 1$. By compactness, there is a subsequence of $\{x_n\}_{n=1}^{\infty}$, which we denote again by $\{x_n\}_{n=1}^{\infty}$, and there is $x_0 \in B_X$ such that $x_n \longrightarrow x_0$ as $n \to \infty$. Since $\|x_n\| = 1$ for every $n \in \mathbb{N}$, we have that $\|x_0\| = 1$, so $x_0 \in S_X$. Now, since $|T(x_n)| \overset{n\to\infty}{\longrightarrow} 1$, we get that $|T(x_0)| = 1$, which is a contradiction.

The proof of (ii) is very similar. Notice that we just have to show that $\{T \in \mathcal{L}(X, X) : \nu(T) = 1\} \subset \mathcal{A}_{\text{nu}}(X)$. If this is not the case, then there are $\varepsilon_0 > 0$ and a numerical radius attaining operator $T \in \mathcal{L}(X, X)$ with $\nu(T) = 1$ such that for all $n \in \mathbb{N}$, there is $(x_n, x_n^*) \in \Pi(X)$ with

$$1 \geqslant |x_n^*(T(x_n))| \geqslant 1 - \frac{1}{n},$$

and whenever $(x, x^*) \in \Pi(X)$ satisfies $\|x - x_n\| < \varepsilon_0$ and $\|x^* - x_n^*\| < \varepsilon_0$, we have $|x^*(T(x))| < 1$. By compactness, there are subsequences of $\{x_n\}_{n=1}^{\infty}$ and $\{x_n^*\}_{n=1}^{\infty}$ which we denote again by $\{x_n\}_{n=1}^{\infty}$ and $\{x_n^*\}_{n=1}^{\infty}$, and there are $x_0 \in B_X$ and $x_0^* \in B_{X^*}$ such that $x_n \longrightarrow x_0$ and $x_n^* \longrightarrow x_0^*$ as $n \to \infty$. Since $x_n^*(x_n) = 1$ for every $n \in \mathbb{N}$, we have that $x_0^*(x_0) = 1$. This shows that $(x_0, x_0^*) \in \Pi(X)$. Now, since $|x_n^*(T(x_n))| \overset{n\to\infty}{\longrightarrow} 1$, we get that $|x_0^*(T(x_0))| = 1$, which is a contradiction.

To prove (iii), suppose $x^* \in S_{c_0^*}$ attains its norm at some point in B_{c_0}. Then, there exists $n_0 \in \mathbb{N}$ so that $x^*(n) = 0$ for every $n > n_0$. Let $\Psi : (\mathbb{K}^{n_0}, \|\cdot\|_\infty) \to c_0$ be the canonical embedding into c_0 that sends $(k_1, \cdots, k_{n_0}) \mapsto (k_1, \cdots, k_{n_0}, 0, 0, \cdots)$. It is easy to see that $\|\Psi\| = 1$. Moreover, $\|x^* \circ \Psi\| = 1$, so (i) implies that $x^* \circ \Psi \in \mathcal{A}_{\|\cdot\|}(\mathbb{K}^{n_0}, \mathbb{K})$. Given $\varepsilon > 0$, define

$$\delta(\varepsilon, x^*) := \min\left\{\frac{\varepsilon}{2}, \eta\left(\frac{\varepsilon}{2}, x^* \circ \Psi\right)\right\}$$

and suppose that $|x^*(x_0)| > 1 - \delta(\varepsilon, x^*)$ for some point $x_0 \in S_{c_0}$. Let $z_0 \in \mathbb{K}^{n_0}$ be the point such that $z_0(n) = x_0(n)$ for $1 \leqslant n \leqslant n_0$. Then,

$$\left|(x^* \circ \Psi)\left(\frac{z_0}{\|z_0\|_\infty}\right)\right| > 1 - \delta(\varepsilon, x^*) > 1 - \eta\left(\frac{\varepsilon}{2}, x^* \circ \Psi\right)$$

so, there is $u_0 \in S_{\mathbb{K}^{n_0}}$ such that $|(x^* \circ \Psi)(u_0)| = 1$ and $\|u_0 - \frac{z_0}{\|z_0\|_\infty}\|_\infty < \frac{\varepsilon}{2}$. Finally, let $v_0 \in c_0$ be such that $v_0(n) = u_0(n)$ for $1 \leqslant n \leqslant n_0$ and $v_0(n) = x_0(n)$ for $n > n_0$. It follows that x^* attains its norm at $v_0 \in S_{c_0}$ and

$$\begin{aligned}\|v_0 - x_0\| = \|u_0 - z_0\|_\infty &\leqslant \left\|u_0 - \frac{z_0}{\|z_0\|_\infty}\right\|_\infty + \left\|\frac{z_0}{\|z_0\|_\infty} - z_0\right\|_\infty \\ &< \frac{\varepsilon}{2} + (1 - \|z_0\|) \leqslant \varepsilon. \end{aligned}$$ ■

Concerning functionals, we have just seen in item (iii) that, for the space c_0, $\mathcal{A}_{\|\cdot\|}(c_0, \mathbb{K}) = S_{\ell_1} \cap \mathrm{NA}(c_0, \mathbb{K})$. It is natural to wonder what happens in other classical sequence spaces. Recall that given a Banach space X, for every $0 < \varepsilon \leqslant 2$, the *modulus of convexity* of $\|\cdot\|$ is given by

$$\delta_X(\varepsilon) := \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \ x, y \in B_X, \ \|x - y\| \geqslant \varepsilon\right\},$$

and X is called *uniformly convex* if $\delta_X(\varepsilon) > 0$ for all $0 < \varepsilon \leqslant 2$ (for instance, if $1 < p < \infty$, then ℓ_p is uniformly convex). In [86, Theorem 2.1] it is shown that a Banach space X is uniformly convex if and only if $(X, \mathbb{K})$ has the BPBop, which in particular implies that if X is uniformly convex, then every norm-one functional is in the set $\mathcal{A}_{\|\cdot\|}(X, \mathbb{K})$ with an uniform η not depending on the operators (note that if X is uniformly convex, then every element in X^* attains its norm by James' theorem). One may wonder if in our more relaxed setting, where we do not ask for such uniformity, we can remove the hypothesis of X being uniformly convex and still get a positive result, but this is not the case general, as we will just see.

Proposition 2.2.2. *Let X be a Banach space.*

(i) *If X is uniformly convex, then $S_{X^*} = \mathcal{A}_{\|\cdot\|}(X, \mathbb{K})$.*

(ii) *There is $x^* \in \mathrm{NA}(\ell_1, \mathbb{K}) \cap S_{\ell_\infty}$ such that $x^* \notin \mathcal{A}_{\|\cdot\|}(\ell_1, \mathbb{K})$.*

(iii) *There is $x^* \in \mathrm{NA}(\ell_\infty, \mathbb{K}) \cap S_{\ell_\infty^*}$ such that $x^* \notin \mathcal{A}_{\|\cdot\|}(\ell_\infty, \mathbb{K})$.*

Proof. A stronger claim than (i) is contained in [86, Theorem 2.1].

Let us prove (ii) now. Consider the norm one functional

$$z^* := \left(1, \frac{1}{2}, \frac{2}{3}, \dots, \frac{n-1}{n}, \dots\right) \in \ell_\infty.$$

Notice that z^* is a norm-attaining functional and it is not difficult to see that the rotations of the unit vector $e_1 \in S_{\ell_1}$ are the only norming points of z^*, that is, if $|z^*(z)| = 1$ with $z \in S_{\ell_1}$, then z is of the form $z = e^{i\theta} e_1$ for some $\theta \in [0, 2\pi)$. Given $\varepsilon > 0$, suppose that there is such a $\eta(\varepsilon, z^*) > 0$. We take $k \in \mathbb{N}$, $k \geqslant 2$, to be such that $\frac{1}{k} < \eta(\varepsilon, z^*)$ and then $|z^*(e_k)| > 1 - \eta(\varepsilon, z^*)$. This means that there is $z \in S_{\ell_1}$ such that $|z^*(z)| = 1$ and $\|z - e_k\|_1 < \varepsilon$. This implies that $z = e^{i\theta} e_1$ and $\|e^{i\theta} e_1 - e_k\|_1 = 2$, which is a contradiction.

For item (iii), consider the functional $x^* := \left(\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \dots\right)$ on ℓ_∞, which is as an element in S_{ℓ_1} (hence it is embedded in $S_{\ell_\infty^*}$). If there is $z = (z(n))_{n=1}^\infty \in S_{\ell_\infty}$ such that $|x^*(z)| = \|x^*\| = 1$, then

$$1 = |x^*(z)| = \left| \sum_{n=1}^\infty \frac{1}{2^n} z(n) \right| \leqslant \sum_{n=1}^\infty \frac{1}{2^n} |z(n)| \leqslant 1.$$

From this, we get that $z(n) = e^{i\theta}$ for all $n \in \mathbb{N}$. Now, assuming that such a $\eta(\varepsilon, x^*) > 0$ exists, we take $k \in \mathbb{N}$ with $2^k \eta(\varepsilon, x^*) > 1$ and consider the element $e_1 + \ldots + e_k \in S_{\ell_\infty}$. Then, $|x^*(e_1 + \ldots + e_k)| > 1 - \eta(\varepsilon, x^*)$. So, there is $x \in S_{\ell_\infty}$ such that $|x^*(x)| = 1$ and $\|x - (e_1 + \ldots + e_k)\|_\infty < \varepsilon$, which leads to a contradiction since $\|x - (e_1 + \ldots + e_k)\|_\infty \geqslant 1$. ■

Recall that, given a Banach space X, the *modulus of smoothness* of $\|\cdot\|$ is

$$\rho_X(\tau) = \sup \left\{ \frac{\|x + \tau h\| + \|x - \tau h\| - 2}{2} : \ \|x\| = \|h\| = 1 \right\},$$

and we say that X is *uniformly smooth* if $\lim_{\tau \searrow 0} \frac{\rho_X(\tau)}{\tau} = 0$. A Banach space is uniformly convex if and only if its dual is uniformly smooth. Note that it is immediate that an operator which has norm one but does not attain the norm cannot be in $\mathcal{A}_{\|\cdot\|}(X, Y)$, by definition. Analogously, the same argument for operators that do not attain their numerical radius applies for the set $\mathcal{A}_{\mathrm{nu}}(X)$. Nevertheless, the following example shows that there exists an operator T with $\|T\| = \nu(T) = 1$ which attains both its norm and numerical radius but belongs neither to $\mathcal{A}_{\|\cdot\|}(X, X)$ nor to $\mathcal{A}_{\mathrm{nu}}(X)$, for some uniformly convex and uniformly smooth Banach space X.

Example 2.2.3. Let $p > 0$ and $q > 0$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. We consider the spaces ℓ_p and ℓ_q as $\ell_p(\ell_p^2)$ and $\ell_q(\ell_q^2)$, respectively, where $\ell_p^2 = (\mathbb{K}^2, \|\cdot\|_p)$. For each $n \in \mathbb{N}$, we define $T_n \in \mathcal{L}(\ell_p^2, \ell_p^2)$ by

$$T_n(x, y) := \left(\left(1 - \frac{1}{2n}\right) x, y \right) \quad \left((x, y) \in \ell_p^2\right).$$

Now, define $T \in \mathcal{L}(\ell_p, \ell_p)$ as

$$T(z) := (T_n(x(n), y(n)))_{n=1}^\infty = \left(\left(1 - \frac{1}{2n}\right) x(n), y(n) \right)_{n=1}^\infty$$

for every $z = ((x(n), y(n)))_{n=1}^{\infty} \in \ell_p$. Following the same steps as in [38, Theorem 2.21.(ii)], we see that T attains its norm but $T \notin \mathcal{A}_{\|\cdot\|}(\ell_p, \ell_p)$. Let us also see that $T \notin \mathcal{A}_{\text{nu}}(X)$. Let e_i^2 be the canonical unit vectors of ℓ_p^2 and ℓ_q^2 for $i = 1, 2$, that is, $e_1^2 = (1, 0)$ and $e_2^2 = (0, 1)$. Consider

$$e_{i,n} := ((0,0), \ldots, (0,0), \underbrace{e_i^2}_{n\text{-th}}, (0,0), \ldots) \in S_{\ell_p}$$

and

$$e_{i,n}^* := ((0,0), \ldots, (0,0), \underbrace{e_i^2}_{n\text{-th}}, (0,0), \ldots) \in S_{\ell_q}$$

for $i = 1, 2$. Since $|e_{2,n}^*(T(e_{2,n}))| = 1$, T attains its numerical radius and $\nu(T) = \|T\| = 1$. Suppose that $T \in \mathcal{A}_{\text{nu}}(\ell_p)$ and consider $\frac{1}{2n} < \eta(\varepsilon, T)$ for a given $\varepsilon \in (0, 1)$. Since $\nu(T) = \|e_{1,n}\|_p = \|e_{1,n}^*\|_q = e_{1,n}^*(e_{1,n}) = 1$ and $|e_{1,n}^*(T(e_{1,n}))| > 1 - \eta(\varepsilon, T)$, there is $(w, w^*) \in \Pi(\ell_p)$ such that $|w^*(T(w))| = 1$, $\|w - e_{1,n}\|_p < \varepsilon$, and $\|w^* - e_{1,n}^*\|_q < \varepsilon$. Since $\|T\| = 1$ and $|w^*(T(w))| = 1$, it follows that $\|T(w)\|_p = 1$. If we denote $w = ((u(n), v(n)))_{n=1}^{\infty} \in S_{\ell_p}$, then it is not difficult to see that $u(j) = 0$ for all $j \in \mathbb{N}$. This implies that $\|w - e_{1,n}\|_p = \|((0, v(n)))_{n=1}^{\infty} - e_{1,n}\|_p \geqslant 1$, which is a contradiction.

Remark 2.2.4. Due to the relation between the norm of an operator and its numerical radius, it is natural to wonder whether the fact that an operator is in $\mathcal{A}_{\|\cdot\|}(X, X)$ for some Banach space X implies that it also belongs to $\mathcal{A}_{\text{nu}}(X)$ (or viceversa). However, this is by no means the case, even in the context of Hilbert spaces. The following example shows that, in fact, every scenario is possible regarding our sets.

Example 2.2.5. Consider these operators on the Hilbert space ℓ_2:

$$T_1(x) := x, \quad \text{for all } x \in \ell_2,$$
$$T_2(x) := \left(x(1), \frac{1}{2}x(2), \frac{2}{3}x(3), \frac{3}{4}x(4), \frac{4}{5}x(5), \ldots\right), \quad \text{for all } x \in \ell_2,$$
$$T_3(x) := (2x(2), -2x(1), x(3), 0, 0, \ldots), \quad \text{for all } x \text{ in the real } \ell_2,$$
$$T_4(x) := (0, x(1), x(2), x(3), x(4), \ldots), \quad \text{for all } x \text{ in the complex } \ell_2.$$

Then, as we will observe in Theorems 2.3.3 and 2.3.5, $T_1 \in \mathcal{A}_{\|\cdot\|}(\ell_2, \ell_2) \cap \mathcal{A}_{\text{nu}}(\ell_2)$, but $T_2 \notin \mathcal{A}_{\|\cdot\|}(\ell_2, \ell_2) \cup \mathcal{A}_{\text{nu}}(\ell_2)$, even though it satisfies that $\|T_2\| = \nu(T_2) = 1$ and $T_2 \in \text{NA}(\ell_2, \ell_2) \cap \text{NRA}(\ell_2)$. It can also be seen that $T_3 \in \mathcal{A}_{\text{nu}}(\ell_2) \backslash \mathcal{A}_{\|\cdot\|}(\ell_2, \ell_2)$ (see the proof of Proposition 2.2.9). Finally, note that every isometry on X belongs to $\mathcal{A}_{\|\cdot\|}(X, X)$, and so, $T_4 \in \mathcal{A}_{\|\cdot\|}(\ell_2, \ell_2)$, but it cannot belong to $\mathcal{A}_{\text{nu}}(\ell_2)$, as it is known that the numerical range $W(T_4)$ of T_4 is the open unit disk $\mathbb{D}$ in the complex plane (see, for example, [70, Example 2]), which implies that $\nu(T_4) = 1$, but $|\langle x, T_4(x)\rangle| < 1$ for every $x \in S_{\ell_2}$.

We will obtain next a positive result for the class of compact operators. A Banach space X is said to satisfy the *Kadec-Klee property* when the weak and norm topologies coincide on the unit sphere S_X. It is well known, for instance, that every locally uniformly rotund (LUR) space satisfies the Kadec-Klee property, although the converse is not true, e.g., ℓ_1^2 (recall that X is called *locally uniformly rotund (LUR)* if for all $x, x_n \in X$ satisfying that $\lim_{n\to\infty} 2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2 = 0$, we have $\lim_{n\to\infty} \|x_n - x\| = 0$). Recall also that, by the Šmulian lemma, the norm of X is Fréchet differentiable at x if and only if $(x_n^*) \subset S_{X^*}$ is convergent whenever $\lim_n x_n^*(x) = 1$. In the next result, under some assumptions on the involved Banach spaces, we show that some subsets of the space of all compact operators belong to the classes $\mathcal{A}_{\|\cdot\|}$ and $\mathcal{A}_{\text{nu}}$.

Theorem 2.2.6. *Let X be a reflexive space which satisfies the Kadec-Klee property. Then,*

(i) $S_{\mathcal{K}(X,Y)} \subset \mathcal{A}_{\|\cdot\|}(X,Y)$ *for every Banach space Y.*

(ii) $\{T \in \mathcal{K}(X,X) : \nu(T) = \|T\| = 1\} \subset \mathcal{A}_{\mathrm{nu}}(X)$ *whenever X is Fréchet differentiable.*

Proof. It is shown in [108, Theorem 2.12] that, under these conditions, the pair (X,Y) has the $\mathbf{L_{o,o}}$ property for compact operators, which implies in particular our item (i) (note that compact operators from X to Y always attain their norms if X is reflexive).

Let us prove (ii) by contradiction. Suppose that it is not true. Then, there are $\varepsilon_0 \in (0,1)$ and a compact operator $T \in \mathcal{K}(X,X)$ with $\nu(T) = \|T\| = 1$ such that for every $n \in \mathbb{N}$, there is $(x_n, x_n^*) \in \Pi(X)$ such that

$$1 \geqslant |x_n^*(T(x_n))| \geqslant 1 - \frac{1}{n} \tag{2.2.1}$$

and whenever $(x, x^*) \in \Pi(X)$ satisfies $\|x - x_n\| < \varepsilon_0$ and $\|x^* - x_n^*\| < \varepsilon_0$, we have $|x^*(T(x))| < 1$. By reflexivity of X, there is a subsequence of $\{x_n\}_{n=1}^\infty$, which we denote again by $\{x_n\}_{n=1}^\infty$, and $x_0 \in B_X$ such that $x_n \xrightarrow{w} x_0$. Thus, $T(x_n) \xrightarrow{n\to\infty} T(x_0)$ in norm. From this and $1 = \nu(T) = \|T\| \geqslant \|T(x_n)\| \geqslant |x_n^*(T(x_n))| \xrightarrow{n\to\infty} 1$, we get that $\|T(x_0)\| = 1$. This shows that $x_0 \in S_X$. Since w and norm topologies coincide in S_X, we have that $x_n \xrightarrow{n\to\infty} x_0$ in norm. Notice now that for each $n \in \mathbb{N}$, we have

$$1 \geqslant |x_n^*(T(x_0))| \geqslant |x_n^*(T(x_n))| - \|x_0 - x_n\|.$$

Since $\{x_n\}_{n=1}^\infty$ converges to x_0 in norm, by using the inequality (2.2.1), we get that $|x_n^*(T(x_0))|$ converges to 1. Thus, there exists a subsequence of $\{x_n^*\}_{n=1}^\infty$, which we denote again by $\{x_n^*\}_{n=1}^\infty$, and some $\theta \in [0, 2\pi)$ such that $x_n^*(T(x_0))$ converges to $e^{i\theta}$. Let $S \in \mathcal{K}(X, X)$ be the operator defined by $S := e^{-i\theta}T$. One clearly has that $S(x_0) \in S_X$ and $x_n^*(S(x_0))$ converges to 1. By Šmulian lemma, there is $x_0^* \in B_{X^*}$ such that $x_n^* \overset{n\to\infty}{\longrightarrow} x_0^*$ in norm. Since $x_n^*(x_n) = 1$ for every $n \in \mathbb{N}$, we get that $x_0^*(x_0) = 1$. So, $x_0^* \in S_{X^*}$ and then $(x_0, x_0^*) \in \Pi(X)$. Finally, in view of (2.2.1) and $|x_n^*(T(x_n))| \overset{n\to\infty}{\longrightarrow} |x_0^*(T(x_0))|$, we get that $|x_0^*(T(x_0))| = 1$. This is a contradiction. ■

In fact, the above argument shows that, under the assumptions on (ii), every compact operator T which has norm and numerical radius 1 attains its numerical radius. Notice also that the identity operator always belongs to $\mathcal{A}_{\text{nu}}(X)$ whereas it is not compact unless X is finite-dimensional, so in the infinite-dimensional setting, the inclusion in Theorem 2.2.6.(ii) must be strict. Recall that a Banach space has the *Schur property* if the weak and norm topologies coincide (this happens for example in ℓ_1). Since every operator from a reflexive space into a space which satisfies the Schur property is known to be compact and Hilbert spaces satisfy all the hypothesis of Theorem 2.2.6, we have the following consequence.

Corollary 2.2.7. *Let X be a reflexive Banach space with the Kadec-Klee property and let H be a Hilbert space.*

(i) *If Y has the Schur property, then $\mathcal{A}_{\|\cdot\|}(X, Y) = S_{\mathcal{L}(X,Y)}$.*

(ii) *If $T \in \mathcal{K}(H, H)$ is with $\nu(T) = \|T\| = 1$, then $T \in \mathcal{A}_{\text{nu}}(H)$.*

Next, we will show with some examples that if some of the conditions from these results are dropped, the claims stop being true in general. We present now a numerical radius attaining compact operator $S \notin \mathcal{A}_{\text{nu}}(X)$ with $\nu(S) = \|S\| = 1$ defined on a Banach space X which is not reflexive, its norm is nowhere Fréchet differentiable, and satisfies the Schur property (and, in particular, the Kadec-Klee property).

Example 2.2.8. Consider c_0 as a real space. Define the operator $T \in \mathcal{L}(c_0, c_0)$ by

$$(T(x))(1) = \sum_{j=1}^{\infty} \frac{1}{2^j} x(j) \text{ and } (T(x))(k) = 0 \quad (k \geqslant 2) \tag{2.2.2}$$

for every $x = (x(j))_{j=1}^{\infty} \in c_0$. It is proved in [1, Proposition 2.8] that $\|T\| = \nu(T) = 1$ but T attains neither its norm nor numerical radius. In particular, T belongs neither to $\mathcal{A}_{\|\cdot\|}(c_0, c_0)$ nor to $\mathcal{A}_{\mathrm{nu}}(c_0)$. We claim that $S := T^*$ is a compact numerical radius attaining operator with $\nu(S) = \|S\| = 1$ but does not belong to $\mathcal{A}_{\mathrm{nu}}(\ell_1)$. Indeed, first notice that $S \in \mathcal{L}(\ell_1, \ell_1)$ is given by

$$S(y) = \sum_{j=1}^{\infty} \frac{y(1)}{2^j} e_j \quad (y = (y(j))_{j=1}^{\infty} \in \ell_1). \tag{2.2.3}$$

Moreover, $\nu(S) = \nu(T) = 1$. Note that, if $z = (1, 1, 1, \dots) \in S_{\ell_\infty}$, then $z(e_1) = 1$, and $z(S(e_1)) = \sum_{j=1}^{\infty} \frac{1}{2^j} = 1$, which implies that S attains the numerical radius (and the norm). Before proving that $S \notin \mathcal{A}_{\mathrm{nu}}(\ell_1)$, let us first observe that $S \in \mathcal{A}_{\|\cdot\|}(\ell_1, \ell_1)$. Indeed, given $\varepsilon > 0$, take $x \in S_{\ell_1}$ such that $\|S(x)\|_1 > 1 - \frac{\varepsilon}{2}$, that is, $\sum_{j=1}^{\infty} \frac{|x(1)|}{2^j} > 1 - \frac{\varepsilon}{2}$. Thus, $|x(1)| > 1 - \frac{\varepsilon}{2}$ and $\sum_{j=2}^{\infty} |x(j)| \leqslant \frac{\varepsilon}{2}$. Consider $y = \left(\frac{x(1)}{|x(1)|}, 0, 0, \dots \right) \in S_{\ell_1}$, then

$$\begin{aligned} \|S(y)\|_1 = 1 \quad \text{and} \quad \|x - y\|_1 &= |x(1) - y(1)| + \sum_{j=2}^{\infty} |x(j)| \\ &\leqslant (1 - |x(1)|) + \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

This shows that $S \in \mathcal{A}_{\|\cdot\|}(\ell_1, \ell_1)$.

Next, we claim that S cannot be in $\mathcal{A}_{\text{nu}}(\ell_1)$. Indeed, observe that if $(y,z) \in \Pi(\ell_1)$ satisfy $|z(S(y))| = 1$, then

$$\sum_{j=1}^{\infty} |y(j)| = 1, \ \sum_{j=1}^{\infty} y(j)z(j) = 1, \ \left|\sum_{j=1}^{\infty} \frac{1}{2^j} y(1)z(j)\right| = 1, \text{ and } \sup_{j\in\mathbb{N}} |z(j)| = 1.$$

From the third equality, we have

$$1 = \left|\sum_{j=1}^{\infty} \frac{1}{2^j} y(1)z(j)\right| \leqslant |y(1)| \sum_{j=1}^{\infty} \frac{1}{2^j} = |y(1)| \leqslant 1.$$

This implies that the only possible candidates are $y = (1,0,0,0,\ldots)$ and $z = (1,1,1,1,\ldots)$ or $y = (-1,0,0,0,\ldots)$ and $z = (-1,-1,-1,-1,\ldots)$. We will proceed by contradiction. Suppose that for a given $\varepsilon \in (0,1)$, there is $\eta(\varepsilon, S) > 0$. Let $n_0 \in \mathbb{N}$ be such that $\sum_{j=1}^{n_0} \frac{1}{2^j} > 1 - \eta(\varepsilon, S)$. Set $y_0 = (1,0,0,\ldots) \in S_{\ell_1}$ and $z_0 = (1,1,\ldots,1,\underbrace{1}_{n_0\text{-th}},0,0,\ldots) \in S_{\ell_\infty}$. Then, $(y_0,z_0) \in \Pi(\ell_1)$ and $|z_0(S(y_0))| = \sum_{j=1}^{n_0} \frac{1}{2^j} > 1 - \eta(\varepsilon, S)$. So, there is $(y,z) \in \Pi(\ell_1)$ such that $|z(S(y))| = 1$, $\|y - y_0\|_1 < \varepsilon$, and $\|z - z_0\|_\infty < \varepsilon$. But this is not possible since $\|z - z_0\|_\infty \geqslant |z(n_0+1) - z_0(n_0+1)| \geqslant 1$.

Let us recall that in Corollary 2.2.7 we proved that if a compact operator T defined on a Hilbert space is such that $\nu(T) = \|T\| = 1$, then T must belong to the set $\mathcal{A}_{\text{nu}}(H)$. However, the following result (inspired by [1, Example 1.9]) provides us a wide class of compact operators $T \in \mathcal{A}_{\text{nu}}(H)$ such that $1 = \nu(T) < \|T\|$ and so, in particular, examples of operators which belong to $\mathcal{A}_{\text{nu}}(H)$ but not to $\mathcal{A}_{\|\cdot\|}(H)$.

Proposition 2.2.9. *Let H be a separable infinite-dimensional real Hilbert space. Then, there is $T \in \mathcal{L}(H,H)$ such that*

(i) *T is a compact operator.*

(ii) *$1 = \nu(T) < \|T\|$ and T attains its numerical radius.*

(iii) *given $\varepsilon > 0$, there is $\eta(\varepsilon) > 0$ such that whenever $x_0 \in S_H$ satisfies*

$$|\langle T(x_0), x_0\rangle| > 1 - \eta(\varepsilon),$$

there is $x_1 \in S_H$ with $\nu(T) = \langle T(x_1), x_1\rangle = 1$ and $\|x_1 - x_0\| < \varepsilon$.

In particular, $T \in \mathcal{A}_{\mathrm{nu}}(H)$ and $T \notin \mathcal{A}_{\|\cdot\|}(H, H)$.

Proof. Let $0 < \alpha \leqslant 1$ and $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence such that $|\alpha_1| > 1$, $-1 < \alpha_n < 1$ for $n \geqslant 2$, and $\alpha_n \longrightarrow 0$ as $n \to \infty$. Let $\{J_1, J_2, J_3\}$ be a partition of $\mathbb{N}$ such that $|J_1| = |J_2| = \aleph_0$, $|J_3| = \ell < \infty$. Write the subsets J_1, J_2 as $J_1 = \{n_k : k \geqslant 1\}$, $J_2 = \{m_k : k \geqslant 1\}$ where $n_1 \leqslant n_2 \leqslant \ldots$, $m_1 \leqslant m_2 \leqslant \ldots$ and each n_k corresponds to m_k via a one-to-one correspondence between J_1 and J_2. Define $T \in \mathcal{L}(H, H)$ by

$$T(e_{n_k}) = -\alpha_k e_{m_k}\ ,\ T(e_{m_k}) = \alpha_k e_{n_k}\ ,\ T(e_n) = \alpha e_n \quad (k \in \mathbb{N}, n \in J_3),$$

where $\{e_n : n \geqslant 1\}$ is an orthonormal basis of H. Note first that for every $x \in H$, we have

$$\begin{aligned} T(x) &= \sum_{n=1}^{\infty} \langle x, e_n\rangle T(e_n) \\ &= \sum_{k=1}^{\infty} (-\alpha_k \langle x, e_{n_k}\rangle e_{m_k} + \alpha_k \langle x, e_{m_k}\rangle e_{n_k}) + \sum_{n \in J_3} \alpha \langle x, e_n\rangle e_n. \end{aligned}$$

The item (i) is clear, as T is a limit of finite-rank operators. Indeed, since J_3 is finite, we may take j sufficiently large so that $j \notin J_3$ and $j \in J_1 \cup J_2$. Assume $j \in J_1$. If $j = n_k$ for some $k \geqslant 1$, then $\|Te_j\| = |\alpha_k|$. Since $\alpha_n \to 0$ as $n \to \infty$, we have $\|Te_j\| \to 0$ as $j \to \infty$. For a given $\varepsilon > 0$, choose j_0 such that $\|Te_j\| < \varepsilon$ for $j \geqslant j_0$. For each $n \geqslant 1$, set $T_n := \sum_{j=1}^{n} \langle \cdot, e_j\rangle T(e_j)$. Let us observe that for every $x \in S_H$ and

$n \geqslant m \geqslant j_0$ (j_0 is large enough so that $j_0 \notin J_3$), we have that

$$\begin{aligned}\|(T_n - T_m)(x)\| &= \left\| \sum_{j=m}^{n} \langle x, e_j \rangle T(e_j) \right\| = \left(\sum_{j=m}^{n} \|\langle x, e_j \rangle T(e_j)\|^2 \right)^{1/2} \\ &< \varepsilon \left(\sum_{j=m}^{n} |\langle x, e_j \rangle|^2 \right)^{1/2} \leqslant \varepsilon\end{aligned}$$

since $\{Te_j\}_{j=m}^{n}$ are orthogonal. This shows that (T_n) is a Cauchy sequence in $\mathcal{L}(H, H)$ which converges to T. Since each T_n is finite-rank, it follows that T is compact.

Let us calculate the norm and numerical radius of T. Note for each $x \in S_H$, we have

$$\begin{aligned}\langle T(x), x \rangle = \sum_{k=1}^{\infty} (\alpha_k \langle e_{n_k}, x \rangle \langle x, e_{m_k} \rangle - \alpha_k \langle e_{m_k}, x \rangle \langle x, e_{n_k} \rangle) \\ + \sum_{n \in J_3} \alpha \langle e_n, x \rangle \langle x, e_n \rangle.\end{aligned}$$

The first two terms are canceled out because H is real, and then

$$\langle T(x), x \rangle = \alpha \sum_{n \in J_3} |\langle x, e_n \rangle|^2 \tag{2.2.4}$$

for $x \in S_H$, which implies that $\nu(T) \leqslant \alpha$. Since $|\langle Te_n, e_n \rangle| = \alpha$ for every $n \in J_3$, we have that T attains its numerical radius and $\nu(T) = \alpha$.

On the other hand, let us notice that, for every $x \in H$, we have

$$\begin{aligned}\|T(x)\|^2 &= \sum_{j=1}^{\infty} |\langle T(x), e_j \rangle|^2 \\ &= \sum_{k=1}^{\infty} \left(|\alpha_k \langle x, e_{m_k} \rangle|^2 + |\alpha_k \langle x, e_{n_k} \rangle|^2 \right) + \sum_{n \in J_3} |\alpha \langle x, e_n \rangle|^2.\end{aligned}$$

It follows that $\|T\| \leqslant \max\{\|\{\alpha_n\}\|_\infty, |\alpha|\}$. However, we also have

$$\begin{aligned}\|T\| &\geqslant \sup\{\|T(e_n)\| : n \geqslant 1\} = \sup\{|\alpha_k|, |\alpha| : k \geqslant 1\} \\ &= \max\{\|\{\alpha_n\}\|_\infty, |\alpha|\};\end{aligned}$$

hence $\|T\| = \max\{\|\{\alpha_n\}\|_\infty, |\alpha|\}$. In particular, since $|\alpha_1| > 1$, we have $\|T\| > 1 \geqslant \alpha = \nu(T)$. This proves item (ii).

Now we prove that $T \in \mathcal{A}_{\text{nu}}(H)$ when $\alpha = 1$. Given $\varepsilon \in (0,1)$, let $x_0 \in S_H$ be such that $|\langle T(x_0), x_0\rangle| > 1 - \frac{\varepsilon^2}{4}$. By equation (2.2.4), we have that

$$\sum_{n\in J_3} |\langle x_0, e_n\rangle|^2 = |\langle T(x_0), x_0\rangle| > 1-\frac{\varepsilon^2}{4}, \text{ and then } \sum_{k\in J_1\cup J_2} |\langle x_0, e_k\rangle|^2 < \frac{\varepsilon^2}{4}.$$

Let π_3 be the projection of H onto the closed subspace $H_3 = \overline{\text{span}}\{e_n : n \in J_3\}$ (where $\text{span}\{A\}$ denotes the vector space spanned by the elements of A, and $\overline{\text{span}}(A)$ is its closure). Then we have $\pi_3(x_0) = \sum_{n\in J_3}\langle x_0, e_n\rangle e_n$ and

$$\langle T(\pi_3(x_0)), \pi_3(x_0)\rangle = \sum_{n\in J_3} |\langle \pi_3(x_0), e_n\rangle|^2 = \sum_{n\in J_3} |\langle x_0, e_n\rangle|^2.$$

It follows that T attains its numerical radius at $\|\pi_3(x_0)\|^{-1}\pi_3(x_0) \in S_H$.

Moreover, the following chain of inequalities clearly holds.

$$\begin{aligned}\left\| \frac{\pi_3(x_0)}{\|\pi_3(x_0)\|} - x_0 \right\| &\leqslant \left\| \frac{\pi_3(x_0)}{\|\pi_3(x_0)\|} - \pi_3(x_0) \right\| + \|\pi_3(x_0) - x_0\| \\ &\leqslant |1 - \|\pi_3(x_0)\|| + \left(\sum_{k \in J_1 \cup J_2} |\langle x_0, e_k \rangle|^2 \right)^{1/2} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

■

Remark 2.2.10. Let us notice that item (iii) of Proposition 2.2.9 says that T belongs to the set $\mathcal{A}_{\mathrm{nu}}$ in a uniform sense, that is, the η does not depend on the operator T defined there. We do not know how often this happens in general.

Observe that it is not true in general that $T^* \in \mathcal{A}_{\|\cdot\|}$ if $T \in \mathcal{A}_{\|\cdot\|}$ or viceversa (see the operators from (2.2.3) and (2.2.5)). However, if we put some extra assumptions on the spaces X and Y, then we can obtain the following.

Proposition 2.2.11. *Let X, Y be Banach spaces and $T \in \mathcal{L}(X,Y)$.*

(i) *Suppose that Y is uniformly smooth. If $T \in \mathcal{A}_{\|\cdot\|}(X,Y)$, then $T^* \in \mathcal{A}_{\|\cdot\|}(Y^*, X^*)$.*

(ii) *Suppose that X is uniformly convex. If $T^* \in \mathcal{A}_{\|\cdot\|}(Y^*, X^*)$, then $T \in \mathcal{A}_{\|\cdot\|}(X,Y)$.*

(iii) *Suppose that X is reflexive. Then, $T \in \mathcal{A}_{\mathrm{nu}}(X)$ if and only if $T^* \in \mathcal{A}_{\mathrm{nu}}(X^*)$.*

Proof. Note that (ii) is just a consequence of (i) since, in this case, X is, in particular, reflexive. Let us prove (i). Let Y be a uniformly smooth

Banach space. Let $T \in \mathcal{A}_{\|\cdot\|}(X,Y)$. Then, $\|T^*\| = \|T\| = 1$ and T^* is also norm-attaining. In order to prove that $T^* \in \mathcal{A}_{\|\cdot\|}(Y^*,X^*)$, let $\varepsilon \in (0,1)$ be given and consider $\eta(\varepsilon,T) > 0$. Set

$$\eta(\varepsilon,T^*) := \min\left\{\eta\left(\frac{\delta_{Y^*}(\varepsilon)}{2},T\right), \frac{\delta_{Y^*}(\varepsilon)}{2}\right\} > 0,$$

where the mapping $\varepsilon \mapsto \delta_{Y^*}(\varepsilon)$ is the modulus of convexity of Y^*. Pick $y_1^* \in S_{Y^*}$ to satisfy $\|T^*(y_1^*)\| > 1 - \eta(\varepsilon,T^*)$. There is $x_1 \in S_X$ such that $\mathrm{Re}(y_1^*(T(x_1))) = \mathrm{Re}(x_1(T^*(y_1^*))) = \|T^*(y_1^*)\| > 1 - \eta(\varepsilon,T^*)$. This implies that $\|T(x_1)\| > 1 - \eta(\varepsilon,T^*)$. Since $T \in \mathcal{A}_{\|\cdot\|}(X,Y)$, there is $x_2 \in S_X$ such that $\|T(x_2)\| = 1$ and $\|x_2 - x_1\| < \frac{\delta_{Y^*}(\varepsilon)}{2}$. Take $y_2^* \in S_{Y^*}$ to be such that $\mathrm{Re}(y_2^*(T(x_2))) = \|T(x_2)\| = 1$ and notice that $\mathrm{Re}(y_1^*(T(x_2))) > 1 - \delta_{Y^*}(\varepsilon)$. Then, $\|y_1^* + y_2^*\| > 2 - 2\delta_{Y^*}(\varepsilon)$. This shows that $\|y_2^* - y_1^*\| < \varepsilon$. As T^* attains its norm at y_2^* which is close to y_1^*, and therefore, $T^* \in \mathcal{A}_{\|\cdot\|}(Y^*,X^*)$.

Now we prove (iii). Since X is reflexive, we just have to prove one direction. Assume $T \in \mathcal{A}_{\mathrm{nu}}(X)$. Note that $T^* \in \mathcal{L}(X^*,X^*)$ clearly also attains its numerical radius. Now let $\varepsilon > 0$ be given and set $\eta(\varepsilon,T^*) := \eta(\varepsilon,T) > 0$. Let $(x_1^*,x_1^{**}) \in \Pi(X^*)$ be such that $|x_1^{**}(T^*(x_1^*))| > 1 - \eta(\varepsilon,T^*)$. Since X is reflexive, there is $x_1 \in S_X$ such that $x_1 = x_1^{**}$. Then

$$|x_1^*(T(x_1))| = |x_1(T^*(x_1^*))| = |x_1^{**}(T^*(x_1^*))| > 1 - \eta(\varepsilon,T^*) = 1 - \eta(\varepsilon,T).$$

Then there is $(x_2,x_2^*) \in \Pi(X)$ such that $|x_2^*(T(x_2))| = 1$, $\|x_2 - x_1\| < \varepsilon$ and $\|x_2^* - x_1^*\| < \varepsilon$. So, $T^* \in \mathcal{A}_{\mathrm{nu}}(X^*)$ as desired. ■

Given $T \in \mathcal{L}(c_0,c_0)$ and $N \in \mathbb{N}$, it is not difficult to see that $T^*(\ell_1) \subset \mathrm{span}\{e_1^*,\ldots,e_N^*\}$ if and only if $T = T \circ P_N$, where P_N is the natural N-th projection on c_0. A property related to Proposition 2.2.11.(iii) above can be proved for the non-reflexive space c_0 under this condition.

Proposition 2.2.12. *Let $T \in \mathcal{A}_{nu}(c_0)$ be an operator such that the range of $T^* \in \mathcal{L}(\ell_1, \ell_1)$ is in* $\operatorname{span}\{e_1^*, \dots, e_N^*\}$ *for some $N \in \mathbb{N}$. Then, $T^* \in \mathcal{A}_{nu}(\ell_1)$.*

Proof. Let $\varepsilon > 0$. Set $\eta(\varepsilon, T^*) := \min\{\frac{\varepsilon}{3}, \eta\left(\frac{\varepsilon}{3}, T\right)\} > 0$. Let $(x_1^*, x_1^{**}) \in \Pi(\ell_1)$ be such that $|x_1^{**}(T^*(x_1^*))| > 1 - \eta(\varepsilon, T^*)$. Let $n_0 > N$ be big enough so that $\sum_{n=1}^{n_0} |x_1^*(n)| > 1 - \eta(\varepsilon, T^*)$. Define $(x_2^*, x_2^{**}) \in \ell_1 \times \ell_\infty$ as follows:

(a) $x_2^*(n) = (\sum_{n=1}^{n_0} |x_1^*(n)|)^{-1} x_1^*(n)$ for $1 \leqslant n \leqslant n_0$ and $x_2^*(n) = 0$ for $n > n_0$,

(b) $x_2^{**}(n) = x_1^{**}(n)$ for $1 \leqslant n \leqslant n_0$ and $x_2^{**}(n) = 0$ for $n > n_0$.

As $x_1^*(n)x_1^{**}(n) = |x_1^*(n)|$ for every $n \in \mathbb{N}$, we get that $(x_2^*, x_2^{**}) \in \Pi(\ell_1)$. Note that $\|x_2^* - x_1^*\| < 2\eta(\varepsilon, T^*) < \frac{2\varepsilon}{3}$. Now,

$$\begin{aligned} |x_2^*(T(x_2^{**}))| = |x_2^{**}(T^*(x_2^*))| &= \left| \sum_{n=1}^{N} x_2^{**}(n)(T^*(x_2^*))(n) \right| \\ &= \left(\sum_{n=1}^{n_0} |x_1^*(n)| \right)^{-1} \left| \sum_{n=1}^{N} x_1^{**}(n)(T^*(x_1^*))(n) \right| > 1 - \eta(\varepsilon, T^*). \end{aligned}$$

Hence, there exists $(x_3, x_3^*) \in \Pi(c_0)$ such that $|x_3^*(Tx_3)| = 1$, $\|x_3 - x_2^{**}\| < \frac{\varepsilon}{3}$, and $\|x_3^* - x_2^*\| < \frac{\varepsilon}{3}$. Notice that $|x_3(n)| < \frac{\varepsilon}{3}$ for every $n > n_0$; hence $x_3^*(n) = 0$ for every $n > n_0$. Define $x_3^{**} \in B_{\ell_\infty}$ by $x_3^{**}(n) = x_3(n)$ for $1 \leqslant n \leqslant n_0$ and $x_3^{**}(n) = x_1^{**}(n)$ for $n > n_0$. Then, $(x_3^*, x_3^{**}) \in \Pi(\ell_1)$, $\|x_3^* - x_1^*\| < \varepsilon$, and $\|x_3^{**} - x_1^{**}\| < \frac{\varepsilon}{3}$. Finally,

$$|x_3^{**}(T^*(x_3^*))| = \left| \sum_{n=1}^{N} x_3^{**}(n)(T^*(x_3^*))(n) \right| = \left| \sum_{n=1}^{N} x_3(n)(T^*(x_3^*))(n) \right| = 1.$$

■

In Proposition 2.2.11, if we drop off some of the hypothesis, then it is possible to construct operators which do not satisfy the conclusion of that result. Recall that, in Example 2.2.8, we constructed an operator T on c_0 such that $T^* \in \mathcal{A}_{\|\cdot\|}(\ell_1, \ell_1)$ but $T \notin \mathcal{A}_{\|\cdot\|}(c_0, c_0)$ (see (2.2.2)). Next, we present an operator S such that $S \in \mathcal{A}_{\|\cdot\|}(X, X)$ but $S^* \notin \mathcal{A}_{\|\cdot\|}(X^*, X^*)$.

Example 2.2.13. The operator T defined in (2.2.2) is such that $T^{**} \notin \mathcal{A}_{\|\cdot\|}(\ell_\infty, \ell_\infty)$ although $T^* \in \mathcal{A}_{\|\cdot\|}(\ell_1, \ell_1)$. Indeed, $T^{**} \in \mathcal{L}(\ell_\infty, \ell_\infty)$ is given by

$$(T^{**}(z))(1) = \sum_{j=1}^{\infty} \frac{1}{2^j} z(j) \quad \text{and} \quad (T^{**}(z))(k) = 0 \ \forall\ k \geqslant 2 \tag{2.2.5}$$

for $z \in \ell_\infty$. Then, for the vector $u_0 = (1, 1, 1, \ldots) \in S_{\ell_\infty}$, we have $\|T^{**}(u_0)\| = 1 = \|T^{**}\|$. Let $z_0 \in S_{\ell_\infty}$ be such that $\|T^{**}(z_0)\|_\infty = 1$. This implies that $|z_0(j)| = 1$ for all $j \in \mathbb{N}$. For a given $\varepsilon \in (0, 1)$, suppose that there is $\eta(\varepsilon, T^{**}) > 0$. Let $n_0 \in \mathbb{N}$ be such that $2^n \eta(\varepsilon, T^{**}) > 1$ for every $n \geqslant n_0$. Consider the vector $z \in S_{\ell_\infty}$ defined as $z_1(n) = 1$ for $1 \leqslant n \leqslant n_0$ and $z_1(n) = 0$, otherwise. Then, $\|T^{**}(z_1)\| = \sum_{j=1}^{n_0} \frac{1}{2^j} > 1 - \eta(\varepsilon, T^{**})$. However, the vector z_1 cannot be close to norming points of T^{**} by definition. This shows that $T^{**} \notin \mathcal{A}_{\|\cdot\|}(\ell_\infty, \ell_\infty)$.

2.3 Diagonal operators

In Example 2.2.5, we saw two examples of diagonal operators $T_1, T_2 \in \mathrm{NA}(\ell_2, \ell_2) \cap \mathrm{NRA}(\ell_2)$ and with $\|T_1\| = \|T_2\| = \nu(T_1) = \nu(T_2) = 1$, but such that $T_1 \in \mathcal{A}_{\|\cdot\|}(\ell_2, \ell_2) \cap \mathcal{A}_{\mathrm{nu}}(\ell_2)$ and $T_2 \notin \mathcal{A}_{\|\cdot\|}(\ell_2, \ell_2) \cup \mathcal{A}_{\mathrm{nu}}(\ell_2)$. The purpose of this section is to characterize the diagonal operators which belong to the sets $\mathcal{A}_{\|\cdot\|}$ and $\mathcal{A}_{\mathrm{nu}}$ for the classical Banach sequence spaces. We give a complete characterization for these operators which belong to $\mathcal{A}_{\|\cdot\|}(X, X)$ whenever $X = c_0$ or ℓ_p with $1 \leqslant p \leqslant \infty$ and for $\mathcal{A}_{\mathrm{nu}}(X)$

whenever $X = c_0$ or ℓ_p with $1 \leqslant p < \infty$, and we also study what diagonal operators belong to $\mathcal{A}_{\|\cdot\|}(X, Y)$ when $X = c_0$ and $Y = \ell_p$ $(1 < p < \infty)$ and viceversa. Before we get into the details of the proofs, let us show some intuitions on how the statements can be found.

Example 2.3.1. Consider the real space c_0. Consider the following diagonal operators $T : c_0 \to c_0$ defined as $T(x) = (\alpha_1 x(1), \alpha_2 x(2), \ldots)$ for all $x \in c_0$ with $\sup\{|\alpha_n| \colon n \in \mathbb{N}\} = 1$, all of which satisfying $\|T\| = \nu(T) = 1$:

1. If $\alpha_n := \frac{n}{n+1}$, then T cannot be in $\mathcal{A}_{\|\cdot\|}(c_0, c_0) \cup \mathcal{A}_{\text{nu}}(c_0)$, since T does not attain its norm or numerical radius. So at least one of the α_n needs to have absolute value 1 to be in our sets.
2. If $\alpha_1 = 1$ and $\alpha_n = 1 - \frac{1}{n}$ for $n > 1$, then T is also not in $\mathcal{A}_{\|\cdot\|}(c_0, c_0) \cup \mathcal{A}_{\text{nu}}(c_0)$, since the only points $x \in S_{c_0}$ where it attains its norm are of the form $s \cdot e_1$ with $|s| = 1$, but the sequence $\{|T(e_{n+1})|\}_{n=1}^{\infty}$ is strictly increasing and converges to 1, none of the points e_{n+1} being close to e_1, and a similar thing happens with the numerical radius. So to be in our sets, not only there must be some α_n with $|\alpha_n| = 1$, but also, those that are not 1 have to be far from 1 (that is, 1 cannot be an accumulation point of $\{|\alpha_n|\}_{n=1}^{\infty}$).
3. Finally, if $\alpha_1 = 1$ and $\alpha_n = \frac{1}{n}$ for $n > 1$, then T is in $\mathcal{A}_{\|\cdot\|}(c_0, c_0) \cap \mathcal{A}_{\text{nu}}(c_0)$, since the only points where the norm is almost attained are close to some point $x \in S_{c_0}$ with $|x(1)| = 1$ (similar for $\mathcal{A}_{\text{nu}}$).

The intuitions presented above hint us towards necessary and sufficient conditions for a diagonal operator to be in $\mathcal{A}_{\|\cdot\|}(c_0, c_0) \cup \mathcal{A}_{\text{nu}}(c_0)$ in the real case, and they can be adapted to other spaces and to the complex case with some suitable changes. We will get into the results of the section now.

The following lemma describes the norm-attaining diagonal operators defined on c_0 or ℓ_p spaces. Although it might be well-known in the literature, we present a short proof of it for the sake of completeness and we will use it to prove Theorem 2.3.3.

Lemma 2.3.2. *Let $X = c_0$ or ℓ_p with $1 \leqslant p \leqslant \infty$. Let $T \in \mathcal{L}(X, X)$ be an operator defined as*

$$T(x) = (\alpha_n x(n))_{n=1}^{\infty} \quad (x = (x(n))_{n=1}^{\infty} \in X),$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a bounded sequence of complex numbers. Given $x \in S_X$, T attains its norm at x if and only if the following is satisfied:

(i) *Case $X = c_0$: there exists $n_0 \in \mathbb{N}$ such that $|\alpha_{n_0}| = \|T\|$ and $|x(n_0)| = 1$.*

(ii) *Case $X = \ell_\infty$: either the same condition as in c_0 holds or there exists a subsequence of the natural numbers, $\{n_k\}_{k=1}^{\infty}$, such that $|\alpha_{n_k}|$ converges to $\|T\|$ and $|x(n_k)|$ converges to 1 as $k \to \infty$.*

(iii) *Case $X = \ell_p$ with $1 \leqslant p < \infty$: setting $J = \{n \in \mathbb{N} \colon |\alpha_n| = 1\}$, J is non-empty and $x(n) = 0$ for all $n \in \mathbb{N} \backslash J$.*

Proof. We claim first that $\|T\| = \nu(T) = \beta := \sup_{n \in \mathbb{N}} |\alpha_n|$. Indeed, clearly $\nu(T) \leqslant \|T\| \leqslant \beta$. Moreover, if there exists some $N \in \mathbb{N}$ such that $|\alpha_N| = S$, then it is clear that

$$|e_N^*(T(e_N))| = \|T(e_N)\| = \beta.$$

On the other hand, if no such N exists, then there is a subsequence $\{\alpha_{n_k}\}_{k=1}^{\infty}$ of $\{\alpha_n\}_{n=1}^{\infty}$ such that $\{|\alpha_{n_k}|\}_{k=1}^{\infty}$ converges to β. It suffices now to notice that

$$\lim_{k \to \infty} |e_{n_k}^*(T(e_{n_k}))| = \lim_{k \to \infty} \|T(e_{n_k})\| = \beta$$

to prove the other inequality and hence the claim.

Now, to prove the lemma, note that the proofs for $X = c_0$ and $X = \ell_\infty$ are an immediate consequence of the fact that

$$\|T\| = \|T(x)\| = \sup_{n\in\mathbb{N}} |\alpha_n x(n)| \leqslant \sup_{n\in\mathbb{N}} |\alpha_n| \leqslant \|T\|,$$

and the proof for $X = \ell_p$ with $1 \leqslant p < \infty$ is a consequence of

$$\|T(x)\|^p = \sum_{n=1}^{\infty} |\alpha_n|^p |x(n)|^p = \sum_{n\in J} |x(n)|^p + \sum_{n\in\mathbb{N}\setminus J} |\alpha_n|^p |x(n)|^p \leqslant \sum_{n=1}^{\infty} |x_n|^p.$$

■

Theorem 2.3.3. *Let $X = c_0$ or ℓ_p, $1 \leqslant p \leqslant \infty$. Let $T \in \mathcal{L}(X, X)$ be a norm one operator defined as*

$$T(x) = (\alpha_n x(n))_{n=1}^{\infty} \quad (x = (x(n))_{n=1}^{\infty} \in X),$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a bounded sequence of complex numbers. Then, the following assertions are equivalent:

(i) $T \in \mathcal{A}_{\|\cdot\|}(X, X)$,

(ii) *Both of these conditions are satisfied:*

(a) There exists some $n_0 \in \mathbb{N}$ such that $|\alpha_{n_0}| = 1$.

(b) If we denote $J = \{n \in \mathbb{N} \colon |\alpha_n| = 1\}$, then either $J = \mathbb{N}$ or $\sup_{n\in\mathbb{N}\setminus J} |\alpha_n| < 1$.

Proof. We prove the result for $X = c_0$ first.

(i) $\Longrightarrow$ (ii): By Lemma 2.3.2, it suffices to show that $\sup_{n\in\mathbb{N}\setminus J} |\alpha_n| < 1$ when $J \neq \mathbb{N}$. Assume to the contrary that $\sup_{n\in\mathbb{N}\setminus J} |\alpha_n| = 1$. Pick a sequence $\{n_k\}_k \subset \mathbb{N}\backslash J$ such that $|\alpha_{n_k}| \geqslant 1 - \frac{1}{k}$ for each $k \in \mathbb{N}$. Given $\varepsilon \in (0,1)$, choose $N \in \mathbb{N}$ so that $N^{-1} < \eta(\varepsilon, T)$, then $\|T(e_{n_N})\| > 1 - \eta(\varepsilon, T)$. Thus there exists $x_0 \in S_{c_0}$ such that T attains its norm at x_0 and $\|x_0 - e_{n_N}\| < \varepsilon$. Now, Lemma 2.3.2 implies that there exists $k \in J$ such that $|x_0(k)| = 1 = |\alpha_k|$. This contradicts $\|x_0 - e_{n_N}\| < \varepsilon$.

(ii) $\Longrightarrow$ (i): If $J = \mathbb{N}$, then T attains its norm at every point in S_{c_0}. Suppose that $J \neq \mathbb{N}$ and $\sup_{n \in \mathbb{N}\backslash J} |\alpha_n| < 1$. Assume to the contrary that $T \notin \mathcal{A}_{\|\cdot\|}(c_0, c_0)$, then there is some $\varepsilon_0 \in (0,1)$ such that for each $n \in \mathbb{N}$, there is some $x_n \in S_{c_0}$ such that $1 \geqslant \|T(x_n)\| \geqslant 1 - \frac{1}{n}$, and whenever $x \in S_{c_0}$ satisfies that $\|x - x_n\| < \varepsilon_0$, we have that $\|T(x)\| < 1$. Let $n_0 \in \mathbb{N}$ be such that

$$\sup_{n \in \mathbb{N}\backslash J} |\alpha_n| < 1 - \frac{1}{n_0} \quad \text{and} \quad \frac{1}{n_0} < \varepsilon_0.$$

Since $\|T(x_{n_0})\| \geqslant 1 - \frac{1}{n_0}$, we can choose $k \in J$ such that $|x_{n_0}(k)| \geqslant 1 - \frac{1}{n_0}$. Let $y_{n_0} \in S_{c_0}$ be the point such that

(1) $y_n(j) := x_n(j)$ for all $j \in \mathbb{N}\backslash\{k\}$,

(2) $y_n(k) := \dfrac{x_n(k)}{|x_n(k)|}$.

It is clear that $\|T(y_{n_0})\| = \|y_{n_0}\| = 1$ and $\|y_{n_0} - x_{n_0}\| \leqslant \frac{1}{n_0} < \varepsilon_0$. This contradiction completes the proof of this case.

The proof for $X = \ell_\infty$ is very similar. Nevertheless, we include the details for the sake of completeness.

(i) $\Longrightarrow$ (ii): Suppose first that there is not any $n_0 \in \mathbb{N}$ with $|\alpha_{n_0}| = 1$. Then, by Lemma 2.3.2, for each $x \in S_{\ell_\infty}$ with $\|T(x)\| = 1$, there exists a subsequence of the natural numbers $\{n_k\}_{k=1}^{\infty}$ such that $|\alpha_{n_k}|$ and $|x(n_k)|$ both converge to 1 as $k \to \infty$. Given $\varepsilon \in (0,1)$, let $k_0 \in \mathbb{N}$ be such that

$\frac{1}{k_0} < \eta(\varepsilon, T)$. Then, for all $k > k_0$,

$$\|T(e_{n_k})\| = |\alpha_{n_k}| > 1 - \eta(\varepsilon, T).$$

However, for each $x \in S_{\ell_\infty}$ such that $\|T(x)\| = 1$, it is clear that $\|x - e_{n_k}\| = 1 > \varepsilon$ for all $k > k_0$, which is a contradiction. Therefore, item (1) from the statement must hold.

It suffices now to show that $\sup_{n \in \mathbb{N} \setminus J} |\alpha_n| < 1$ when $J \neq \mathbb{N}$. Assume to the contrary that $\sup_{n \in \mathbb{N} \setminus J} |\alpha_n| = 1$. Pick a sequence $\{n_k\}_k \subset \mathbb{N} \setminus J$ such that $|\alpha_{n_k}| \geqslant 1 - \frac{1}{k}$ for each $k \in \mathbb{N}$. Given $\varepsilon \in (0, 1)$, choose $N \in \mathbb{N}$ so that $N^{-1} < \eta(\varepsilon, T)$, then $\|T(e_{n_N})\| > 1 - \eta(\varepsilon, T)$. Thus there exists $x_0 \in S_{\ell_\infty}$ such that T attains its norm at x_0 and $\|x_0 - e_{n_N}\| < \varepsilon$. However, like before, that is a contradiction.

(ii) $\implies$ (i): If $J = \mathbb{N}$, then T attains its norm at every point in S_{ℓ_∞}. Suppose that $J \neq \mathbb{N}$ and $\sup_{n \in \mathbb{N} \setminus J} |\alpha_n| < 1$. Assume to the contrary that $T \notin \mathcal{A}_{\|\cdot\|}(\ell_\infty, \ell_\infty)$, then there is some $\varepsilon_0 \in (0, 1)$ such that for each $n \in \mathbb{N}$, there is some $x_n \in S_{\ell_\infty}$ such that $1 \geqslant \|T(x_n)\| \geqslant 1 - \frac{1}{n}$, and whenever $x \in S_{\ell_\infty}$ satisfies that $\|x - x_n\| < \varepsilon_0$, we have that $\|T(x)\| < 1$. Let $n_0 \in \mathbb{N}$ be such that

$$\sup_{n \in \mathbb{N} \setminus J} |\alpha_n| < 1 - \frac{1}{n_0} \quad \text{and} \quad \frac{1}{n_0} < \varepsilon_0.$$

Since $\|T(x_{n_0})\| \geqslant 1 - \frac{1}{n_0}$, we can choose $k \in J$ such that $|x_{n_0}(k)| \geqslant 1 - \frac{1}{n_0}$. Let $y_{n_0} \in S_{\ell_\infty}$ be the point such that

- $y_n(j) := x_n(j)$ for all $j \in \mathbb{N} \setminus \{k\}$,
- $y_n(k) := \dfrac{x_n(k)}{\|x_n(k)\|}$.

It is clear that $\|T(y_{n_0})\| = \|y_{n_0}\| = 1$ and $\|y_{n_0} - x_{n_0}\| \leqslant \frac{1}{n_0} < \varepsilon_0$. This contradiction completes the proof of this case.

Finally, let us prove now the result for $X = \ell_p$ with $1 \leqslant p < \infty$.

(i) $\Longrightarrow$ (ii): It suffices to check that $\sup_{n\in\mathbb{N}\setminus J} |\alpha_n| < 1$ when $J \neq \mathbb{N}$. Assume to the contrary that $\sup_{n\in\mathbb{N}\setminus J} |\alpha_n| = 1$. Given $\varepsilon \in (0,1)$, pick $n_0 \in \mathbb{N}\setminus J$ so that $|\alpha_{n_0}| > 1 - \eta(\varepsilon, T)$. Thus, $\|Te_{n_0}\| > 1 - \eta(\varepsilon, T)$. By Lemma 2.3.2, if T attains its norm at $x \in S_{\ell_p}$, then $|x(n_0)| = 0$ which implies that $\|x - e_{n_0}\| \geqslant 1 > \varepsilon$.

(ii) $\Longrightarrow$ (i): If $J = \mathbb{N}$, then we are done. Suppose that $J \neq \mathbb{N}$ and $\beta := \sup_{n\in\mathbb{N}\setminus J} |\alpha_n| < 1$. Assuming that T does not belong to $\mathcal{A}_{\|\cdot\|}(\ell_p, \ell_p)$, there exists $\varepsilon_0 \in (0,1)$ such that for each $n \in \mathbb{N}$, there is some $x_n \in S_{\ell_p}$ such that $1 \geqslant \|T(x_n)\| \geqslant 1 - \frac{1}{n}$, and whenever $x \in S_{\ell_p}$ satisfies that $\|x - x_n\| < \varepsilon_0$, we have that $\|T(x)\| < 1$. Note that

$$\left(1 - \frac{1}{n}\right)^p \leqslant \sum_{k\in J} |x_n(k)|^p + \beta \sum_{k\in\mathbb{N}\setminus J} |x_n(k)|^p < \sum_{k=1}^{\infty} |x_n(k)|^p = 1.$$

This implies that $\sum_{k\in J} |x_n(k)|^p$ converges to 1 and $\sum_{k\in\mathbb{N}\setminus J} |x_n(k)|^p$ converges to 0 as $n \to \infty$. Set $A_n := (\sum_{k\in J} |x_n(k)|^p)^{\frac{1}{p}}$ and choose $n_0 \in \mathbb{N}$ such that $1 - A_{n_0}^p < \frac{\varepsilon_0^p}{2}$. Define $y_{n_0} \in S_{\ell_p}$ by

$$y_{n_0}(k) = \frac{x_{n_0}(k)}{A_{n_0}} \quad \text{for every } \; k \in J \quad \text{and} \quad y_{n_0}(k) = 0 \quad \text{for every } \; k \in \mathbb{N}\setminus J.$$

By Lemma 2.3.2 that $\|Ty_{n_0}\| = 1$. However,

$$\|y_{n_0} - x_{n_0}\|^p \leqslant (1 - A_{n_0})^p + \sum_{j\in\mathbb{N}\setminus J} |x_{n_0}(k)|^p \leqslant 2(1 - A_{n_0}^p) < \varepsilon_0^p.$$

■

Next we are going to prove the counterpart of Lemma 2.3.2 and Theorem 2.3.3 for numerical radius. As in the $\mathcal{A}_{\|\cdot\|}$ case, it gives a complete characterization for the set $\mathcal{A}_{\text{nu}}$ for diagonal operators on c_0 and ℓ_p spaces

$(1 \leqslant p < \infty)$. Let us notice that Lemma 2.3.4 below establishes necessary conditions for a numerical radius one diagonal operator on c_0 and ℓ_p to attain its numerical radius. We will use it to prove Theorem 2.3.5, and again, we present a short proof of it for the sake of completeness.

Lemma 2.3.4. *Let $X = c_0$ or ℓ_p, $1 \leqslant p < \infty$. Let $T \in \mathcal{L}(X, X)$ be a numerical radius one operator defined as*

$$T(x) = (\alpha_n x(n))_{n=1}^{\infty} \quad (x = (x(n))_{n=1}^{\infty} \in X),$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a bounded sequence of complex numbers. If T attains its numerical radius at $(x, x^*) \in \Pi(X)$, then we have the following:

(i) *There exists $n_0 \in \mathbb{N}$ such that $|\alpha_{n_0}| = 1$.*

(ii) *For $X = c_0$, $\operatorname{Re}(x^*(n)x(n)) = |x^*(n)x(n)| = |x^*(n)|$ for every $n \in \mathbb{N}$.*
For $X = \ell_1$, $\operatorname{Re}(x^*(n)x(n)) = |x^*(n)x(n)| = |x(n)|$ for every $n \in \mathbb{N}$.
For $X = \ell_p$, $1 < p < \infty$, $\operatorname{Re}(x^*(n)x(n)) = |x^*(n)x(n)| = |x(n)|^p = |x^*(n)|^q$ for every $n \in \mathbb{N}$, where $\frac{1}{p} + \frac{1}{q} = 1$.

(iii) *There exists $\theta \in [0, 2\pi)$ such that $\alpha_n = e^{i\theta}$ on the set $\{n \in \mathbb{N} : |x(n)| \cdot |x^*(n)| \neq 0\}$.*

Proof. Note first that, like on Lemma 2.3.2, we have $\nu(T) = \|T\| = \sup_{n\in\mathbb{N}} |\alpha_n|$. From that, item (i) is clear, as the operator T attains its norm as well. Let us prove now item (ii). For the space $X = c_0$, first of all, as $(x, x^*) \in \Pi(c_0)$, then for all $n \in \mathbb{N}$, we have

$$\operatorname{Re}(x^*(n)x(n)) \leqslant |x^*(n)x(n)| \leqslant |x^*(n)|,$$

but we also know that

$$1 = \sum_{n=1}^{\infty} x^*(n)x(n) = \sum_{n=1}^{\infty} |x^*(n)|.$$

This proves what we wanted for $X = c_0$.

For ℓ_1 we will use the same exact argument, changing the roles of $x(n)$ and $x^*(n)$: as $(x, x^*) \in \Pi(\ell_1)$, then for all $n \in \mathbb{N}$, we have

$$\mathrm{Re}(x^*(n)x(n)) \leqslant |x^*(n)x(n)| \leqslant |x(n)|,$$

but we also know that

$$1 = \sum_{n=1}^{\infty} x^*(n)x(n) = \sum_{n=1}^{\infty} |x(n)|.$$

This proves what we wanted for $X = \ell_1$.

Finally, for $X = \ell_p$ $(1 < p < \infty)$, we argue similarly, taking into account the equality case of Hölder's inequality. Indeed, note that if $(x, x^*) \in \Pi(\ell_p)$, then

$$1 = \sum_{n=1}^{\infty} x^*(n)x(n) \leqslant \sum_{n=1}^{\infty} |x^*(n)x(n)| \leqslant \|x^*\|_q \cdot \|x\|_p = 1,$$

and hence, Hölder's inequality becomes an equality in this case. Note that then, for each $n \in \mathbb{N}$, $\mathrm{Re}(x^*(n)x(n)) = |x^*(n)x(n)|$, and by Hölder's equality case, the following identities must hold for each $n \in \mathbb{N}$:

$$\frac{|x^*(n)|^q}{\|x^*\|_q^q} = \frac{|x(n)|^p}{\|x\|_p^p}, \quad \text{and} \quad \mathrm{Re}(x^*(n)x(n)) = \frac{|x^*(n)|^q}{q} + \frac{|x(n)|^p}{p}.$$

This proves item (ii) for $X = \ell_p$.

Finally, to see (iii), observe that

$$1 = |x^*(T(x))| = \left|\sum_{n=1}^{\infty} \alpha_n x^*(n)x(n)\right| \leqslant \sum_{n=1}^{\infty} |\alpha_n x^*(n)x(n)| \leqslant 1.$$

Therefore, using (ii), there exists $\theta \in [0, 2\pi)$ such that $\alpha_n = e^{i\theta}$ on $\{n \in \mathbb{N} : |x(n)| \cdot |x^*(n)| \neq 0\}$. ■

Theorem 2.3.5. *Let $X = c_0$ or ℓ_p, $1 \leqslant p < \infty$. Let $T \in \mathcal{L}(X, X)$ be a numerical radius one operator defined as*

$$T(x) = (\alpha_n x(n))_{n=1}^{\infty} \quad (x = (x(n))_{n=1}^{\infty} \in X),$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a bounded sequence of complex numbers. Then, the following assertions are equivalent:

(i) $T \in \mathcal{A}_{\mathrm{nu}}(X)$.

(ii) *The following both conditions hold:*

(a) There exists some $n_0 \in \mathbb{N}$ such that $|\alpha_{n_0}| = 1$.

(b) If $J = \{n \in \mathbb{N} \colon |\alpha_n| = 1\}$, then the cardinality of the set $\{\alpha_n : n \in J\}$ is finite and $\sup_{n \in \mathbb{N} \setminus J} |\alpha_n| < 1$ when $J \neq \mathbb{N}$.

Before proving Theorem 2.3.5, let us notice that when $\{\alpha_n\}_{n=1}^{\infty}$ is a bounded sequence of *real* numbers, we have that the set $\{\alpha_n : n \in J\} \subseteq \{1, -1\}$, that is, it is automatically finite. Combining Theorem 2.3.3 and Theorem 2.3.5, we get the following immediate consequence.

Corollary 2.3.6. *Let $X = c_0$ or ℓ_p, $1 \leqslant p < \infty$. Let $T \in \mathcal{L}(X, X)$ be a numerical radius one operator defined as*

$$T(x) = (\alpha_n x(n))_{n=1}^{\infty} \quad (x = (x(n))_{n=1}^{\infty} \in X),$$

where $\{\alpha_n\}_{n=1}^{\infty}$ *is a bounded sequence of real numbers. Then, the following assertions are equivalent:*

(i) $T \in \mathcal{A}_{\|\cdot\|}(X, X)$.

(ii) $T \in \mathcal{A}_{\text{nu}}(X)$.

(iii) *Both of the following conditions are satisfied:*

(a) There exists some $n_0 \in \mathbb{N}$ *such that* $|\alpha_{n_0}| = 1$.

(b) If $J = \{n \in \mathbb{N} \colon |\alpha_n| = 1\}$, *then* $J = \mathbb{N}$ *or* $\sup_{n\in\mathbb{N}\setminus J} |\alpha_n| < 1$ *when* $J \neq \mathbb{N}$.

Proof of Theorem 2.3.5. Let us prove first the result for $X = c_0$.

(i) $\Longrightarrow$ (ii): By Lemma 2.3.4, the set J is non-empty. Assume that the set $\{\alpha_n : n \in J\}$ is an infinite set. Write $\{\alpha_n : n \in J\} = \{e^{i\theta_1}, \ldots, e^{i\theta_n}, \ldots\}$. Then, there exists a subsequence $\{n_k\}_{k=1}^{\infty} \subset J$ such that $e^{i\theta_{n_k}}$ converges to some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Given $\varepsilon \in (0, 1/2)$, let $k_0 \in \mathbb{N}$ be such that $|e^{i\theta_{n_k}} - \lambda| < \eta(\varepsilon, T)$ for every $k \geqslant k_0$. Then, for $k \neq k' \geqslant k_0$, we obtain that $\left|\frac{e^{i\theta_{n_k}} + e^{i\theta_{n_{k'}}}}{2} - \lambda\right| < \eta(\varepsilon, T)$. Pick $n \neq n'$ in J so that $\alpha_n = e^{i\theta_{n_k}}$ and $\alpha_{n'} = e^{i\theta_{n_{k'}}}$. Then, $((e_n + e_{n'}), \frac{1}{2}(e_n^* + e_{n'}^*)) \in \Pi(c_0)$ and $\left|\frac{1}{2}(e_n^* + e_{n'}^*)\,(T\,(e_n + e_{n'}))\right| = \left|\frac{e^{i\theta_{n_k}} + e^{i\theta_{n_{k'}}}}{2}\right| > 1 - \eta(\varepsilon, T)$. However, if T attains its numerical radius at $(x, x^*) \in \Pi(c_0)$, then, by Lemma 2.3.4, there exists $\theta \in [0, 2\pi)$ such that $\alpha_m = e^{i\theta}$ on $A := \{m \in \mathbb{N} : |x^*(m)| \neq 0\}$. If $n, n' \notin A$, then for $k \in A$, $\|x - (e_n + e_{n'})\| \geqslant |e_k^*(x - (e_n + e_{n'}))| = |x(k)| = 1 > \varepsilon$. Otherwise, without loss of generality, we may assume that $n \in A$. As $\alpha_n \neq \alpha_{n'}$, we have that $n' \notin A$, i.e., $|x^*(n')| = 0$. It follows that $\left\|x^* - \frac{1}{2}(e_n^* + e_{n'}^*)\right\| \geqslant |(x^* - \frac{1}{2}(e_n^* + e_{n'}^*))(e_{n'})| = \frac{1}{2} > \varepsilon$. This proves that $\{\alpha_m : m \in J\}$ must be a finite set. Finally, arguing similarly to Theorem 2.3.3, we can deduce that $\sup_{m\in\mathbb{N}\setminus J} |\alpha_m| < 1$ when $J \neq \mathbb{N}$. Indeed, assume that $\sup_{n\in\mathbb{N}\setminus J} = 1$ and suppose that $T \in \mathcal{A}_{\text{nu}}(c_0)$.

Then, for $\varepsilon \in (0,1)$, let $\eta(\varepsilon, T)$ be given. For each $k \in \mathbb{N}$, there exists tome $n_k \in \mathbb{N}\backslash J$ such that $|\alpha_{n_k}| > 1 - \frac{1}{k}$. Choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \eta(\varepsilon, T)$. Then $|e^*_{n_N}(T(e_{n_N}))| \geqslant 1 - \frac{1}{N} > 1 - \eta(\varepsilon, T)$. However, for each $(x, x^*) \in \Pi(c_0)$ such that $|x^*(T(x))| = 1$, it is clear (see the proof of Theorem 2.3.3) that $\|x - e_{n_N}\| > \varepsilon$, so we have a contradiction.

(ii) $\Longrightarrow$ (i): Let us say that $\{\alpha_n : n \in J\} = \{e^{i\theta_1}, \ldots, e^{i\theta_m}\}$ for some $m \in \mathbb{N}$. Assume to the contrary that T does not belong to $\mathcal{A}_{\mathrm{nu}}(c_0)$. Then, there exists some $\varepsilon_0 \in (0,1)$ such that for each $n \in \mathbb{N}$, there is $(x_n, x_n^*) \in \Pi(c_0)$ such that $1 \geqslant |x_n^*(T(x_n))| \geqslant 1 - \frac{1}{n}$, and whenever $(x, x^*) \in \Pi(c_0)$ is such that $\|x - x_n\| < \varepsilon_0$ and $\|x^* - x_n^*\| < \varepsilon_0$, we have that $|x^*(T(x))| < 1$. If $J \neq \mathbb{N}$, then, by Lemma 2.3.4,

$$\begin{aligned}
1 &= \sum_{k=1}^{\infty} x_n^*(k) x_n(k) \\
&> \sum_{k \in J_1} x_n^*(k) x_n(k) + \ldots + \sum_{k \in J_m} x_n^*(k) x_n(k) + \beta \sum_{k \in \mathbb{N}\backslash J} x_n^*(k) x_n(k) \\
&\geqslant \left| e^{i\theta_1} \sum_{k \in J_1} x_n^*(k) x_n(k) + \ldots + e^{i\theta_m} \sum_{k \in J_m} x_n^*(k) x_n(k) \right| + \\
&\quad + \left| \sum_{k \in \mathbb{N}\backslash J} \alpha_k x_n^*(k) x_n(k) \right| \geqslant 1 - \frac{1}{n},
\end{aligned}$$

for every $n \in \mathbb{N}$, where $J_k = \{n \in \mathbb{N} : \alpha_n = e^{i\theta_k}\}$ and $\beta := \sup_{n \in \mathbb{N}\backslash J} |\alpha_n| < 1$. By passing to a subsequence if necessary, we may assume that $\sum_{k \in J_l} x_n^*(k) x_n(k)$ converges as $n \to \infty$ for each $1 \leqslant l \leqslant m$. As $e^{i\theta_l} \neq e^{i\theta_{l'}}$ for all $1 \leqslant l \neq l' \leqslant m$, we can choose $1 \leqslant s \leqslant m$ so that

$$\sum_{k \in J_l} x_n^*(k) x_n(k) \to 0 \quad \text{for all} \quad l \neq s, \quad \text{and} \quad \sum_{k \in J_s} x_n^*(k) x_n(k) \to 1$$

as $n \to \infty$. Also, notice that $\sum_{k\in\mathbb{N}\setminus J} x_n^*(k)x_n(k) \to 0$ as $n \to \infty$. Pick $n_0 \in \mathbb{N}$ large enough so that

$$\sum_{k\in J_l} |x_{n_0}^*(k)| < \frac{\varepsilon}{3m} \ \text{ for all } \ l \neq s,\ 1 - \sum_{k\in J_s} |x_{n_0}^*(k)| < \frac{\varepsilon}{3},$$

and

$$\sum_{k\in\mathbb{N}\setminus J} x_{n_0}^*(k)x_{n_0}(k) < \frac{\varepsilon}{3}.$$

Let $y_{n_0} = x_{n_0} \in S_{c_0}$ and define $y_{n_0}^* \in S_{\ell_1}$ as

$$y_{n_0}^*(k) = \frac{x_{n_0}^*(k)}{\gamma} \ \text{ for every } \ k \in J_s \text{ and } \ y_{n_0}^*(k) = 0 \ \text{ for every } \ k \in \mathbb{N}\setminus J_s,$$

where $\gamma = \sum_{k\in J_s} |x_{n_0}^*(k)|$. Then, $(y_{n_0}, y_{n_0}^*) \in \Pi(c_0)$,

$$y_{n_0}^*(T(y_{n_0}^*))| = \left|\sum_{k\in J_s} \frac{e^{i\theta_s}x_{n_0}^*(k)x_{n_0}(k)}{\gamma}\right| = 1,$$

and

$$\|y_{n_0}^* - x_{n_0}^*\| \leqslant (m-1)\frac{\varepsilon}{3m} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon.$$

This is a contradiction. For the case when $J = \mathbb{N}$, we have

$$\begin{aligned} 1 = \sum_{k=1}^{\infty} x_n^*(k)x_n(k) &= \sum_{k\in J_1} x_n^*(k)x_n(k) + \ldots + \sum_{k\in J_m} x_n^*(k)x_n(k) \\ &\geqslant \left|e^{i\theta_1}\sum_{k\in J_1} x_n^*(k)x_n(k) + \ldots + e^{i\theta_m}\sum_{k\in J_m} x_n^*(k)x_n(k)\right| \geqslant 1 - \frac{1}{n}, \end{aligned}$$

for every $n \in \mathbb{N}$. Arguing as above, we may choose $1 \leqslant s \leqslant m$ and $n_0 \in \mathbb{N}$ such that

$$\sum_{k\in J_l} |x_{n_0}^*(k)| < \frac{\varepsilon}{2m} \ \text{ for all } \ l \neq s, \ \text{ and } \ 1 - \sum_{k\in J_s} |x_{n_0}^*(k)| < \frac{\varepsilon}{2}.$$

By defining $(y_{n_0}, y^*_{n_0}) \in \Pi(c_0)$ as above, we get a contradiction.

The proof of the case $X = \ell_1$ is almost identical to the proof for $X = c_0$ by a duality argument, so we will omit it.

Let us prove the result for $X = \ell_p$ with $1 < p < \infty$. Let q be such that $\frac{1}{p} + \frac{1}{q} = 1$.

(i) $\Longrightarrow$ (ii): Note that Lemma 2.3.4 implies (1). Assume that the set $\{\alpha_n : n \in J\}$ is an infinite set, say $\{\alpha_n : n \in J\} = \{e^{i\theta_1}, \ldots, e^{i\theta_n}, \ldots\}$. Then, there exists a subsequence $\{n_k\}_{k=1}^{\infty} \subset J$ such that $e^{i\theta_{n_k}}$ converges to some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Given $\varepsilon \in (0, (\frac{1}{2})^{\frac{1}{q}})$, let $k_0 \in \mathbb{N}$ be such that $|e^{i\theta_{n_k}} - \lambda| < \eta(\varepsilon, T)$ for every $k \geqslant k_0$. Then, for $k \neq k' \geqslant k_0$, we obtain that $\left|\frac{e^{i\theta_{n_k}} + e^{i\theta_{n_{k'}}}}{2} - \lambda\right| < \eta(\varepsilon, T)$. Pick $n \neq n'$ in J so that $\alpha_n = e^{i\theta_{n_k}}$ and $\alpha_{n'} = e^{i\theta_{n_{k'}}}$. Thus, $((\frac{1}{2})^{\frac{1}{p}}(e_n + e_{n'}), (\frac{1}{2})^{\frac{1}{q}}(e^*_n + e^*_{n'})) \in \Pi(\ell_p)$ and

$$\left|\left(\frac{1}{2}\right)^{\frac{1}{q}} (e^*_n + e^*_{n'}) \left(T\left(\left(\frac{1}{2}\right)^{\frac{1}{p}} (e_n + e_{n'})\right)\right)\right| = \left|\frac{e^{i\theta_{n_k}} + e^{i\theta_{n_{k'}}}}{2}\right| > 1 - \eta(\varepsilon, T).$$

However, if T attains its numerical radius at $(x, x^*) \in \Pi(\ell_p)$, then, by Lemma 2.3.4, there exists $\theta \in [0, 2\pi)$ such that $\alpha_m = e^{i\theta}$ on $A := \{m \in \mathbb{N} : |x^*(m)| \neq 0\}$. If $n, n' \notin A$, i.e., $|x^*(n)| = |x^*(n')| = 0$, then Lemma 2.3.4 implies that $|x(n)| = |x(n')| = 0$. Thus, $\left\|x - \left(\frac{1}{2}\right)^{\frac{1}{p}} (e_n + e_{n'})\right\|^p \geqslant \frac{1}{2} + \frac{1}{2} = 1 > \varepsilon$. Otherwise, without loss of generality, we may assume that $n \in A$. As $\alpha_n \neq \alpha_{n'}$, we have that $n' \notin A$, i.e., $|x^*(n')| = 0$. It follows that $\left\|x^* - \left(\frac{1}{2}\right)^{\frac{1}{q}} (e^*_n + e^*_{n'})\right\|^q \geqslant \frac{1}{2} > \varepsilon^q$. This proves that $\{\alpha_m : m \in J\}$ must be a finite set. Again, like in Theorem 2.3.3, we can deduce that $\sup_{m \in \mathbb{N}\setminus J} |\alpha_m| < 1$ when $J \neq \mathbb{N}$. Indeed, assume once more that $\sup_{n \in \mathbb{N}\setminus J} = 1$ and suppose that $T \in \mathcal{A}_{\text{nu}}(c_0)$. Then, for $\varepsilon \in (0, 1)$, let $\eta(\varepsilon, T)$ be given. For each $k \in \mathbb{N}$, there exists tome $n_k \in \mathbb{N}\setminus J$ such that $|\alpha_{n_k}| > 1 - \frac{1}{k}$. Choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \eta(\varepsilon, T)$. Then

$|e^*_{n_N}(T(e_{n_N}))| \geqslant 1 - \frac{1}{N} > 1 - \eta(\varepsilon, T)$. However, for each $(x, x^*) \in \Pi(c_0)$ such that $|x^*(T(x))| = 1$, it is clear (see the proof of Theorem 2.3.3) that $\|x - e_{n_N}\| > \varepsilon$, so we have a contradiction.

The implication (ii) $\Longrightarrow$ (i) is very similar to the one for $X = c_0$. We will include the details for the sake of completeness. Let us say again that $\{\alpha_n : n \in J\} = \{e^{i\theta_1}, \ldots, e^{i\theta_m}\}$ for some $m \in \mathbb{N}$. Assume to the contrary that T does not belong to $\mathcal{A}_{\mathrm{nu}}(\ell_p)$. Then, there exists some $\varepsilon_0 \in (0,1)$ such that for each $n \in \mathbb{N}$, there is $(x_n, x_n^*) \in \Pi(\ell_p)$ such that $1 \geqslant |x_n^*(T(x_n))| \geqslant 1 - \frac{1}{n}$, and whenever $(x, x^*) \in \Pi(\ell_p)$ is such that $\|x - x_n\| < \varepsilon_0$ and $\|x^* - x_n^*\| < \varepsilon_0$, we have that $|x^*(T(x))| < 1$. If $J \neq \mathbb{N}$, then, by Lemma 2.3.4, the following chain of inequalities holds

$$\begin{aligned}
1 &= \sum_{k=1}^{\infty} x_n^*(k) x_n(k) \\
&> \sum_{k \in J_1} x_n^*(k) x_n(k) + \ldots + \sum_{k \in J_m} x_n^*(k) x_n(k) + \beta \sum_{k \in \mathbb{N} \setminus J} x_n^*(k) x_n(k) \\
&\geqslant \left| e^{i\theta_1} \sum_{k \in J_1} x_n^*(k) x_n(k) + \ldots + e^{i\theta_m} \sum_{k \in J_m} x_n^*(k) x_n(k) \right| + \\
&\qquad + \left| \sum_{k \in \mathbb{N} \setminus J} \alpha_k x_n^*(k) x_n(k) \right| \geqslant 1 - \frac{1}{n},
\end{aligned}$$

for every $n \in \mathbb{N}$, where $J_k = \{n \in \mathbb{N} : \alpha_n = e^{i\theta_k}\}$ and $\beta := \sup_{n \in \mathbb{N} \setminus J} |\alpha_n| < 1$. By passing to a subsequence if necessary, we may assume that $\sum_{k \in J_l} x_n^*(k) x_n(k)$ converges as $n \to \infty$ for each $1 \leqslant l \leqslant m$. As $e^{i\theta_l} \neq e^{i\theta_{l'}}$ for all $1 \leqslant l \neq l' \leqslant m$, we can choose $1 \leqslant s \leqslant m$ so that

$$\sum_{k \in J_l} x_n^*(k) x_n(k) \to 0 \quad \text{for all} \quad l \neq s, \quad \text{and} \quad \sum_{k \in J_s} x_n^*(k) x_n(k) \to 1$$

as $n \to \infty$. Also, notice that $\sum_{k \in \mathbb{N} \setminus J} x_n^*(k) x_n(k) \to 0$ as $n \to \infty$. Pick $n_0 \in \mathbb{N}$ large enough so that the following inequalities hold:

$$\sum_{k\in J_l} |x_{n_0}^*(k)|^q < \frac{\varepsilon^q}{3m}, \quad \text{for all } l \neq s,$$

$$\left(1 - \left(\sum_{k\in J_s} |x_{n_0}^*(k)|^q\right)^{1/q}\right)^q < \frac{\varepsilon^q}{3},$$

$$\sum_{k\in J_l} |x_{n_0}(k)|^p < \frac{\varepsilon^p}{3m}, \quad \text{for all } l \neq s,$$

$$\left(1 - \left(\sum_{k\in J_s} |x_{n_0}(k)|^p\right)^{1/p}\right)^p < \frac{\varepsilon^p}{3}, \quad \text{and}$$

$$\sum_{k\in\mathbb{N}\backslash J} x_{n_0}^*(k)x_{n_0}(k) < \min\left\{\frac{\varepsilon^p}{3}, \frac{\varepsilon^q}{3}\right\}.$$

For simplicity, and using Lemma 2.3.4, for each $1 \leqslant l \leqslant m$, denote

$$A_l := \sum_{k\in J_l} x_{n_0}(k)x_{n_0}^*(k) = \sum_{k\in J_l} |x_{n_0}(k)|^p = \sum_{k\in J_l} |x_{n_0}^*(k)|^q.$$

Now define $y_n \in S_{\ell_p}$ as the point such that $y_n(k) := \frac{x_n(k)}{A_s^{1/p}}$ if $k \in J_s$, and $y_n(k) = 0$ if $k \in \mathbb{N}\backslash J_s$, and define $y_n^* \in S_{\ell_q}$ as the point satisfying $y_n^*(k) := \frac{x_n^*(k)}{A_s^{1/q}}$ if $k \in J_s$, and $y_n^*(k) = 0$ if $k \in \mathbb{N}\backslash J_s$. By Lemma 2.3.4, the following identities hold:

$$\begin{aligned}
\|y_n\|_p^p &= \frac{\sum_{k\in J_s} |x_n(k)|^p}{A_s} = \|x_n\|_p^p = 1,\\
\|y_n^*\|_q^q &= \frac{\sum_{k\in J_s} |x_n^*(k)|^q}{A_s} = \|x_n^*\|_q^q = 1,\\
y_n^*(y_n) &= \frac{\sum_{k\in J_s} x_n^*(k)x_n(k)}{A_s} = 1,\\
|y_n^*(T(y_n))| &= \frac{1}{A_s}\left|\sum_{k\in J_s} \alpha_k x_n^*(k)x_n(k)\right| = 1.
\end{aligned}$$

Now, to obtain a contradiction, it suffices to note that

$$\|y_n - x_n\|_p^p \leqslant \left(\frac{1-A_s^{1/p}}{A_s^{1/p}}\right)^p \sum_{n\in J_s} |x_n(k)|^p + (m-1)\frac{\varepsilon^p}{3m} + \frac{\varepsilon^p}{3} < \varepsilon^p.$$

$$\|y_n^* - x_n^*\|_q^q = \left(\frac{1-A_s^{1/q}}{A_s^{1/q}}\right)^q \sum_{n\in J_s} |x_n^*(k)|^q + (m-1)\frac{\varepsilon^q}{3m} + \frac{\varepsilon^q}{3} < \varepsilon^q.$$

Finally, note that for the case $J = \mathbb{N}$, the same argument is still valid if we change

$$\sum_{n\in\mathbb{N}\setminus J} x_n^*(k)x_n(k) = 0.$$

■

One may wonder whether or not we can find a characterization for diagonal operators in the set $\mathcal{A}_{\|\cdot\|}$ when the domain is different from the range space. As announced earlier, there is one for certain choices of domain and range spaces. Similar techniques as in Theorem 2.3.3 and Theorem 2.3.5 yield the following result on operators from c_0 into ℓ_p

and from ℓ_p into c_0. Notice that in this case we cannot consider the set $\mathcal{A}_{\mathrm{nu}}(X)$.

Theorem 2.3.7. *Let $1 \leqslant p < \infty$ be given.*

(i) *Let $T \in \mathcal{L}(\ell_p, c_0)$ be a norm one operator defined as*

$$T(x) = (\alpha_n x(n))_{n=1}^{\infty} \quad (x = (x(n))_{n=1}^{\infty} \in \ell_p),$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a bounded sequence of scalars. Then, the following assertions are equivalent:

(a) $T \in \mathcal{A}_{\|\cdot\|}(\ell_p, c_0)$.

(b) If $J = \{n \in \mathbb{N} : |\alpha_n| = 1\}$, then J is non empty and

1. $J = \mathbb{N}$ *or*

2. $\sup_{n \in \mathbb{N} \setminus J} |\alpha_n| < 1$

(ii) *Let $T \in \mathcal{L}(c_0, \ell_p)$ be a norm one operator defined as*

$$T(x) = (\alpha_n x(n))_{n=1}^{\infty} \quad (x = (x(n))_{n=1}^{\infty} \in c_0),$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence of scalars with p-norm equal to 1. Then, the following assertions are equivalent:

(a) $T \in \mathcal{A}_{\|\cdot\|}(c_0, \ell_p)$.

(b) There is some $N \in \mathbb{N}$ such that $\alpha_n = 0$ for all $n > N$.

Proof. (i). Let us prove $(a) \Rightarrow (b)$ first. In this case, note that, like before, T is well defined and $\|T\| = \sup_{n \in \mathbb{N}} |\alpha_n| = 1$. As T attains its norm at some $x_\infty \in S_{\ell_p}$, we have

$$\|T(x_\infty)\| = \sup_{n \in \mathbb{N}} |\alpha_n x_\infty(n)| = \|T\| = 1.$$

Moreover, there exists $n_0 \in \mathbb{N}$ such that $|\alpha_{n_0}| = 1$ and $|x_\infty(n_0)| = 1$ (otherwise, we can extract a subsequence $\{n_k\}_{k=1}^\infty$ such that $|x_\infty(n_k)| \to 1$, a contradiction). This shows that J is not empty. It remains to prove that $\sup_{n\in\mathbb{N}\setminus J} |\alpha_n| < 1$ when $j \neq \mathbb{N}$. Assume to the contrary that this supremum is 1. Then, there exists $N \in \mathbb{N}\setminus J$ such that $|\alpha_N| > 1-\eta(\varepsilon, T)$. Therefore, $\|T(e_N)\| > 1 - \eta(\varepsilon, T)$. Thus, there exists $x_0 \in S_{\ell_p}$ such that $\|T(x_0)\| = 1$ and $\|x_0 - e_N\| < \varepsilon$. However, as we observed above, there exists $n_1 \in J$ such that $|\alpha_{n_1}| = 1$ and $|x_0(n_1)| = 1$. This contradicts the fact that $\|x_0 - e_N\| < \varepsilon$.

Let us prove $(b) \Rightarrow (a)$ now. Assume first that $J = \mathbb{N}$ and $\|T(x_0)\| > 1-\varepsilon$ for some $x_0 \in S_{\ell_p}$. Then, there exists $n_0 \in \mathbb{N}$ such that $|x(n_0)| > 1 - \varepsilon$. Define $x_1 \in S_{\ell_p}$ by $x_1(n_0) = \frac{x_0(n_0)}{|x_0(n_0)|}$, and $x_1(n) = 0$ for all $n \neq n_0$. It is clear that $\|T(x_1)\| = 1$. Observe that

$$\begin{aligned}
\|x_1 - x_0\|^p &= \sum_{j\neq n_0} |x_0(j)|^p + \left(1 - \frac{1}{|x_0(n_0)|}^p\right) |x_0(n_0)|^p \\
&= 1 - |x_0(n_0)|^p + (1 - |x_0(n_0)|)^p < 1 - (1-\varepsilon)^p + \varepsilon^p.
\end{aligned}$$

In other words, we have $\|T(x_1)\| = 1$ and $\|x_1 - x_0\| < (1-(1-\varepsilon)^p+\varepsilon^p)^{1/p}$. So $T \in \mathcal{A}_{\|\cdot\|}(\ell_p, c_0)$.

Now, assume that $J \neq \mathbb{N}$ and $\sup_{n\in\mathbb{N}\setminus J} |\alpha_n| < \alpha < 1$. We will proceed once more by contradiction. Assume to the contrary that $T \notin \mathcal{A}_{\|\cdot\|}(\ell_p, c_0)$. Then, given $\varepsilon > 0$, there is a sequence $\{x_n\}_{n=1}^\infty \subset S_{\ell_p}$ such that $\|T(x_n)\| \geqslant 1 - \frac{1}{n}$, and if $x \in S_{\ell_p}$ satisfies that $\|x_n - x\| < \varepsilon$, then $\|T(x)\| < 1$. Choose $n_0 \in \mathbb{N}$ such that $\|T(x_{n_0})\| > \max\{\alpha, \delta\}$, where $\delta \approx 1$ is such that

$$(1 - \delta^p + (1-\delta)^p)^{1/p} < \varepsilon.$$

Let $y \in S_{\ell_p}$ be defined as $y(n_1) = \frac{x_{n_0}(n_1)}{|x_{n_0}(n_1)|}$ and $y(n) = 0$ for all $n \neq n_1$. Then $\|T(y)\| = 1$, and

$$\|x_{n_0} - y\| < (1 - \delta^p + (1 - \delta)^p)^{1/p} < \varepsilon.$$

This is a contradiction, and the proof of this case is finished.

(ii). First of all, T is well defined if and only if $\{\alpha_n\}_{n=1}^{\infty} \in \ell_p$. Indeed, $(\Leftarrow)$ is clear, and for $(\Rightarrow)$ it suffices to evaluate T at the points

$$x_n := (1, \overset{(n)}{\ldots}, 1, 0, 0, \ldots), \quad \text{for } n \in \mathbb{N},$$

and use the fact that ℓ_p is a Banach space.

Also, $\|T\| = \|\{\alpha_n\}_{n=1}^{\infty}\|_p$. Indeed, for all $x \in S_{c_0}$, it is clear that

$$\|T(x)\|^p \leqslant \sum_{n=1}^{\infty} |\alpha_n x(n)|^p \leqslant \sum_{n=1}^{\infty} |\alpha_n|^p.$$

Now it suffices to evaluate at the same points as before.

Moreover, T attains its norm if and only if there is some $N \in \mathbb{N}$ such that $\alpha_n = 0$ for all $n > N$. Indeed, the implication $(\Leftarrow)$ is clear, and for $(\Rightarrow)$, we will argue by contradiction. Assume that no such N exists but that T attains its norm at some $x \in S_{c_0}$. Consider a subsequence $\{n_k\}_{k=1}^{\infty}$ of the natural numbers such that $\alpha_{n_k} \neq 0$ for all $k \in \mathbb{N}$. Since $x \in c_0$, there exists some $k_0 \in \mathbb{N}$ such that $|x(n_{k_0})| < 1$. Then,

$$1 = \|T(x)\|^p = \sum_{n \neq n_{k_0}} |\alpha_n x(n)|^p + |\alpha_{n_{k_0}} x(n_{k_0})|^p < \sum_{n \neq n_{k_0}} |\alpha_n|^p + |\alpha_{n_{k_0}}|^p = 1,$$

which is a contradiction. Since every operator in $\mathcal{A}_{\|\cdot\|}(c_0, \ell_p)$ is norm-attaining by definition, this proves that $(a) \Rightarrow (b)$. Let us prove the other implication now.

First of all, if $M = \{k_1, \ldots, k_r\} \subset \mathbb{N}$ is the set of indexes such that $\alpha_n \neq 0$ if and only if $n \in M$, then a similar argument than before shows that the only points $x \in S_{c_0}$ such that $\|T(x)\| = 1$ are those such that $|x(n)| = 1$ for all $n \in \mathbb{N}$.

We will proceed now by contradiction. Assume that $T \notin \mathcal{A}_{\|\cdot\|}(c_0, \ell_p)$. Then there is some $\varepsilon_0 \in (0,1)$ such that for each $n \in \mathbb{N}$, there is some $x_n \in S_{c_0}$ such that $1 \geqslant \|T(x_n)\|^p \geqslant 1 - \frac{1}{n}$, and whenever $x \in S_{c_0}$ satisfies $\|x - x_n\| < \varepsilon_0$, we have that $\|T(x)\|^p < 1$. Then, for each $n \in \mathbb{N}$, we have

$$1 - \frac{1}{n} \leqslant \sum_{i=1}^{r} |\alpha_{k_i} x_n(k_i)|^p \leqslant \sum_{i=1}^{r} |\alpha_{k_i}|^p = 1.$$

Given $n \in \mathbb{N}$, assume now that there is some $1 \leqslant i_n \leqslant r$ such that $|x_n(k_{i_n})| \leqslant 1 - \varepsilon_0$. Then, we have

$$1 - \frac{1}{n} \leqslant \left(\sum_{i \in \{1,\ldots,r\} \setminus \{i_n\}} |\alpha_{k_i}|^p \right) + (1-\varepsilon_0)^p |\alpha_{k_{i_n}}| < \sum_{i=1}^{r} |\alpha_{k_i}|^p = 1.$$

From here, we would get that $(1 - (1-\varepsilon_0)^p)|\alpha_{k_{i_n}}|^p \leqslant \frac{1}{n}$. Therefore, if $n_0 \in \mathbb{N}$ is such that $\frac{1}{n_0} < (1-(1-\varepsilon_0)^p)m$, where $m := \min\{|\alpha_k| : k \in M\}$, then for all $n \geqslant n_0$, it is clear that for all $k \in M$, $|x_n(k)| > 1-\varepsilon_0$. Finally define $y \in S_{c_0}$ as the point such that $y(k) = \frac{x_{n_0}(k)}{|x_{n_0}(k)|}$ when $k \in M$ and $y(k) = x(k)$ when $k \notin M$. It is clear that $\|T(y)\| = 1$ and $\|x_{n_0} - y\| < \varepsilon_0$. This is a contradiction and completes the proof. ∎

The previous theorems provide a wide class of operators that belong to our sets. For instance, the *canonical projections* $P_N \in \mathcal{L}(X,X)$, $N \in \mathbb{N}$, belong to both $\mathcal{A}_{\|\cdot\|}(X,X)$ and $\mathcal{A}_{\mathrm{nu}}(X)$ for the Banach spaces $X = c_0$ or ℓ_p, with $1 \leqslant p < \infty$, and to $\mathcal{A}_{\|\cdot\|}(X,X)$ when $X = \ell_\infty$.

Corollary 2.3.8. *Let $N \in \mathbb{N}$ be given.*

(i) $P_N \in \mathcal{A}_{\|\cdot\|}(X, X)$ *if* $X = c_0$ *or* $X = \ell_p$, $1 \leqslant p \leqslant \infty$.

(ii) $P_N \in \mathcal{A}_{\text{nu}}(X)$ *if* $X = c_0$ *or* $X = \ell_p$, $1 \leqslant p < \infty$.

2.4 Connecting the sets via direct sums

In this section, we introduce a natural approach to connect the sets $\mathcal{A}_{\|\cdot\|}(W, Z)$ and $\mathcal{A}_{\text{nu}}(W \oplus Z)$ for some choices of direct sums and of the Banach spaces W and Z. Throughout the section, we will be using the following notation. Given two Banach spaces X_1 and X_2, consider the mappings $P_i \in \mathcal{L}(X_1 \oplus X_2, X_i)$ such that $P_i(x_1, x_2) := x_i$, $i = 1, 2$, and $\iota_j \in \mathcal{L}(X_j, X_1 \oplus X_2)$ such that $\iota_i(x) := xe_i$, where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. For Banach spaces W and Z, if we have an operator $T \in \mathcal{L}(W, Z)$, then there is a simple way to define $\widetilde{T} \in \mathcal{L}(W \oplus Z, W \oplus Z)$: consider $\widetilde{T} := \iota_2 \circ T \circ P_1$, that is, $\widetilde{T}(w, z) = (0, T(w))$ for every $(w, z) \in W \oplus Z$. Conversely, we can define a pseudo-inverse process as follows: if we have an operator $S \in \mathcal{L}(W \oplus Z, W \oplus Z)$, then we can consider $\breve{S} \in \mathcal{L}(W, Z)$ defined as $\breve{S} := P_2 \circ S \circ \iota_1$, that is, $\breve{S}(w) = (P_2 \circ S)(w, 0)$ for every $w \in W$. We start with the following result, which establishes a bond between the assertions $T \in \mathcal{A}_{\|\cdot\|}(W, Z)$ and $\widetilde{T} \in \mathcal{A}_{\text{nu}}(W \oplus_1 Z)$ under some assumptions on the spaces.

Proposition 2.4.1. *Let W and Z be two Banach spaces, and let $T \in S_{\mathcal{L}(W,Z)}$. Then,*

(i) *If $\widetilde{T} \in \mathcal{A}_{\text{nu}}(W \oplus_1 Z)$, then $T \in \mathcal{A}_{\|\cdot\|}(W, Z)$.*

(ii) *Suppose that W and Z are uniformly smooth Banach spaces. If $T \in \mathcal{A}_{\|\cdot\|}(W, Z)$, then $\widetilde{T} \in \mathcal{A}_{\text{nu}}(W \oplus_1 Z)$.*

Proof. (i). Assume $\widetilde{T} \in \mathcal{A}_{\text{nu}}(W \oplus_1 Z)$ and for a given $\varepsilon > 0$, set $\eta(\varepsilon, T) := \eta(\varepsilon, \widetilde{T}) > 0$. Pick $w_0 \in S_W$ to be such that $\|T(w_0)\| > 1 - \eta(\varepsilon, T)$. Let

$z_0^* \in S_{Z^*}$ be such that $|z_0^*(T(w_0))| = \|T(w_0)\| > 1 - \eta(\varepsilon, T)$. Let $w_0^* \in S_{W^*}$ be such that $w_0^*(w_0) = 1$ and consider the point $((w_0^*, z_0^*)((w_0, 0)) \in \Pi(W \oplus_1 Z)$. Since $\widetilde{T} \in \mathcal{A}_{\mathrm{nu}}(W \oplus_1 Z)$ and

$$|(w_0^*, z_0^*)(\widetilde{T}(w_0, 0))| = |z_0^*(T(w_0))| > 1 - \eta(\varepsilon, T) = 1 - \eta(\varepsilon, \widetilde{T}),$$

there is $(w_1^*, z_1^*)((w_1, z_1)) \in \Pi(W \oplus_1 Z)$ such that

$$\nu(\widetilde{T}) = |(w_1^*, z_1^*)(\widetilde{T}(w_1, z_1))|, \quad \|(w_1, z_1) - (w_0, 0)\|_1 < \varepsilon$$

and

$$\|(w_1^*, z_1^*) - (w_0^*, z_0^*)\|_\infty < \varepsilon.$$

So $1 = |(w_1^*, z_1^*)(\widetilde{T}(w_1, z_1))| = |z_1^*(T(w_1))| \leqslant \|z_1^*\|\|T(w_1)\| \leqslant 1$. Then, $\|T(w_1)\| = 1$ and $z_1 = 0$. So $\|w_1 - w_0\| < \varepsilon$. This proves that $T \in \mathcal{A}_{\|\cdot\|}(W, Z)$.

(ii). Suppose $T \in \mathcal{A}_{\|\cdot\|}(W, Z)$. It is not hard to check that $\widetilde{T}$ attains its numerical radius and $\nu(\widetilde{T}) = 1$. Indeed, note that $\nu(\widetilde{T}) \leqslant \|\widetilde{T}\| = \|T\| = 1$. On the other hand, if $w_0 \in S_W$ is such that $\|T(w_0)\| = \|T\| = 1$, then we take $z_0^* \in S_{Z^*}$ to be such that $|z_0^*(T(w_0))| = \|T(w_0)\| = 1$ and we consider the point $((w_0, 0), (w_0^*, z_0^*)) \in \Pi(W \oplus_1 Z)$, where $w_0^* \in S_{W^*}$ is such that $w_0^*(w_0) = 1$. So $1 = |z_0^*(T(w_0))| = |(w_0^*, z_0^*)(\widetilde{T}((w_0, 0)))|$. Given $\varepsilon \in (0, 1)$, we set the positive number

$$\eta(\varepsilon, \widetilde{T}) := \\ \min \left\{ \eta \left(\min \left\{ \frac{\delta_{W^*}(\varepsilon)}{2}, \frac{\delta_{Z^*}(\varepsilon)}{2}, \frac{\varepsilon}{2} \right\}, T \right), \frac{\delta_{W^*}(\varepsilon)}{2}, \frac{\delta_{Z^*}(\varepsilon)}{2}, \frac{\varepsilon}{2} \right\},$$

where $\varepsilon \mapsto \delta_{W^*}(\varepsilon)$ and $\varepsilon \mapsto \delta_{Z^*}(\varepsilon)$ are the modulus of convexity of W^* and Z^*, respectively. Let $((w_1, z_1), (w_1^*, z_1^*)) \in \Pi(W \oplus_1 Z)$ be such that

$$|z_1^*(T(w_1))| = \left|(w_1^*(z_1^*), \widetilde{T}(w_1, z_1))\right| > 1 - \eta(\varepsilon, \widetilde{T}).$$

As we have

$$\|T(w_1)\| \geqslant |z_1^*(T(w_1))| > 1 - \eta\left(\min\left\{\frac{\delta_{W^*}(\varepsilon)}{2}, \frac{\delta_{Z^*}(\varepsilon)}{2}, \frac{\varepsilon}{2}\right\}, T\right),$$

there is $w_2 \in S_W$ such that

$$\|T(w_2)\| = 1 \quad \text{and} \quad \|w_2 - w_1\| < \min\left\{\frac{\delta_{W^*}(\varepsilon)}{2}, \frac{\delta_{Z^*}(\varepsilon)}{2}, \frac{\varepsilon}{2}\right\}.$$

Since $\|w_1\| \geqslant |z_1^*(T(w_1))| > 1 - \eta(\varepsilon, \widetilde{T})$, we have that $\|z_1\| < \eta(\varepsilon, \widetilde{T})$. Let $w_2^* \in S_{W^*}$ be such that $w_2^*(w_2) = 1$, then

$$\begin{aligned}
\left\|\frac{w_1^* + w_2^*}{2}\right\| &\geqslant \left|\left(\frac{w_1^* + w_2^*}{2}\right)(w_2)\right| = \left|\frac{2 - z_1^*(z_1) + w_1^*(w_2 - w_1)}{2}\right| \\
&\geqslant \left|1 - \left(\frac{z_1^*(z_1) - w_1^*(w_2 - w_1)}{2}\right)\right| \\
&\geqslant 1 - \left|\left(\frac{z_1^*(z_1) - w_1^*(w_2 - w_1)}{2}\right)\right| \\
&> 1 - \left(\frac{\|z_1\| + \|w_2 - w_1\|}{2}\right) > 1 - \delta_{W^*}(\varepsilon),
\end{aligned}$$

which implies that $\|w_2^* - w_1^*\| < \varepsilon$.

Let $\theta \in \mathbb{R}$ be such that $z_1^*(T(w_2)) = e^{i\theta}|z_1^*(T(w_2))|$. Notice that

$$|z_1^*(T(w_2))| \geqslant |z_1^*(T(w_1))| - |z_1^*(T(w_2 - w_1))| \geqslant 1 - \delta_{Z^*}(\varepsilon).$$

Now, let $z_2^* \in S_{Z^*}$ be such that $z_2^*(T(w_2)) = e^{i\theta}$. Observe that

$$\left\|\frac{z_1^* + z_2^*}{2}\right\| \geqslant \left|\left(\frac{z_1^* + z_2^*}{2}\right)(T(w_2))\right| = \frac{1 + |z_1^*(T(w_2))|}{2} > 1 - \delta_{Z^*}(\varepsilon);$$

hence $\|z_2^* - z_1^*\| < \varepsilon$. Finally, considering the point $((w_2, 0), (w_2^*, z_2^*)) \in \Pi(W \oplus_1 Z)$, we conclude that $\widetilde{T} \in \mathcal{A}_{\mathrm{nu}}(W \oplus_1 Z)$. ■

Remark 2.4.2. Proposition 2.4.1.(ii) no longer holds in general if we consider arbitrary Banach spaces instead of uniformly smooth ones. Indeed, consider the real Banach space ℓ_1. In (2.2.2) we provided an operator that belongs to $\mathcal{A}_{\|\cdot\|}(\ell_1, \ell_1)$ but not to $\mathcal{A}_{\mathrm{nu}}(\ell_1)$. We will show that this operator does not satisfy the property stated in Proposition 2.4.1.(ii). Let $S \in \mathcal{L}(\ell_1, \ell_1)$ be the operator defined in (2.2.2). Note that if $((x, y), (x^*, y^*)) \in \Pi(\ell_1 \oplus_1 \ell_1)$ satisfies

$$|(x^*, y^*)(\widetilde{S}(x, y))| = |y^*(S(x))| = \left| \sum_{j=1}^{\infty} \frac{y^*(j)x(1)}{2^j} \right| = 1, \tag{2.4.1}$$

then, one gets easily that $y^*(j)x(1)$ has to be equal to either 1 or -1 for all $j \in \mathbb{N}$. From here, we get that the only possibilities have the form $x = se_1$, $y = 0$, $x^* = (s, x^*(2), x^*(3), \ldots)$, and $y^* = (r, r, r, \ldots)$ with $|x^*(j)| \leqslant 1$ for all $j > 1$, where $s, r \in \{-1, 1\}$. Now, we argue by contradiction. Suppose that for a given $\varepsilon \in (0, 1)$, there is $\eta(\varepsilon, \widetilde{S}) > 0$. Let $n_0 \in \mathbb{N}$ be such that $\sum_{j=1}^{n_0} \frac{1}{2^j} > 1 - \eta(\varepsilon, \widetilde{S})$, and set $w = e_1$, $z = 0$, $w^* = e_1^*$, and $z^* = e_1^* + \ldots + e_{n_0}^*$. It is immediate to check that $((w, z), (w^*, z^*)) \in \Pi(\ell_1 \oplus_1 \ell_1)$ and also that $|(w^*, z^*)(\widetilde{S}(w, z))| > 1 - \eta(\varepsilon, \widetilde{S})$. Then, there must be some $((x, y), (x^*, y^*)) \in \Pi(\ell_1 \oplus_1 \ell_1)$ satisfying (2.4.1) and such that $\|(w, z) - (x, y)\|_1 < \varepsilon$ and $\|(w^*, z^*) - (x^*, y^*)\|_\infty < \varepsilon$. But this is already a contradiction, since $\|(x^* - w^*, y^* - z^*)\|_\infty \geqslant \|y^* - z^*\|_\infty \geqslant 1$. Therefore $\widetilde{S} \notin \mathcal{A}_{\mathrm{nu}}(\ell_1 \oplus_1 \ell_1)$ as desired, even though $S \in \mathcal{A}_{\|\cdot\|}(\ell_1, \ell_1)$.

Notice that the situation for operators $\check{T}$ is different.

Remark 2.4.3. There exists an operator $S \in \mathcal{L}(W \oplus_1 Z, W \oplus_1 Z)$, with both W and Z being uniformly smooth Banach spaces, such that $S \in \mathcal{A}_{\mathrm{nu}}(W \oplus_1 Z)$ but $\check{S} \notin \mathcal{A}_{\|\cdot\|}(W, Z)$. Indeed, let $S \in \mathcal{L}(\ell_2 \oplus_1 \ell_2, \ell_2 \oplus_1 \ell_2)$

be defined as

$$S(x,y) = ((x(1),0,0,\cdots),(0,0,0,\cdots)), \quad \forall (x,y) \in \ell_2 \oplus_1 \ell_2,$$

where ℓ_2 is a real space. Note that $\nu(S) = 1$ and S attains its numerical radius. For $\varepsilon \in (0,1)$, suppose that $|(x^*,y^*)(S(x,y))| > 1-\varepsilon > 0$ for some $((x,y),(x^*,y^*)) \in \Pi(\ell_2 \oplus_1 \ell_2)$. Then $|x(1)| > 1-\varepsilon$, $|x^*(1)| > 1-\varepsilon$ and $\|y\| < \varepsilon$. Note also that

$$1 \geqslant |x(1)|^2 + \sum_{n \neq 1} |x(n)|^2 \geqslant |x(1)|^2 > (1-\varepsilon)^2$$

which implies that $(\sum_{n\neq 1} |x(n)|^2)^{1/2} < (2\varepsilon - \varepsilon^2)^{1/2}$.

On the other hand,

$$1 = \sum_n x(n)x^*(n) + \sum_n y(n)y^*(n) \leqslant \|x\|\|x^*\| + \|y\|\|y^*\| \leqslant \|x\| + \|y\| = 1$$

From this, we have $\|x^*\| = \|y^*\| = 1$. As above, we can see that

$$\left(\sum_{n \neq 1} |x^*(n)|^2 \right)^{1/2} < (2\varepsilon - \varepsilon^2)^{1/2}.$$

If we define pairs of vectors

$$(\widetilde{x},\widetilde{y}) = \left(\left(\frac{x(1)}{|x(1)|}, 0, 0, \cdots \right), 0 \right) \text{ and}$$
$$(\widetilde{x^*},\widetilde{y^*}) = \left(\left(\frac{x^*(1)}{|x^*(1)|}, 0, 0, \cdots \right), y^* \right),$$

then $\|(x,y) - (\widetilde{x},\widetilde{y})\| \leqslant \varepsilon + \sqrt{2\varepsilon}$ and $\|(x^*,y^*) - (\widetilde{x^*},\widetilde{y^*})\| \leqslant \sqrt{2\varepsilon}$.

It is clear that $((\widetilde{x},\widetilde{y}),(\widetilde{x^*},\widetilde{y^*})) \in \Pi(\ell_2 \oplus_1 \ell_2)$ and $|(\widetilde{x^*},\widetilde{y^*})(S(\widetilde{x},\widetilde{y}))| = 1$. This proves that S belongs to $\mathcal{A}_{\mathrm{nu}}(\ell_2 \oplus_1 \ell_2)$. However, $\check{S} \in \mathcal{L}(\ell_2,\ell_2)$ is the operator such that

$$\breve{S}x = (P_2 \circ S)(x,0) = P_2((x(1),0,0,\cdots),(0,0,0,\cdots)) = 0$$

for every $x \in \ell_2$; hence $\breve{S} = 0$, and the null operator cannot belong to $\mathcal{A}_{\|\cdot\|}(\ell_2, \ell_2)$.

We proceed now to prove the analogous results for ℓ_∞-sums but under different hypothesis on the underlying spaces.

Proposition 2.4.4. *Let W and Z be two Banach spaces, and let $T \in S_{\mathcal{L}(W,Z)}$. Then:*

(i) *If $\widetilde{T} \in \mathcal{A}_{\mathrm{nu}}(W \oplus_\infty Z)$, then $T \in \mathcal{A}_{\|\cdot\|}(W,Z)$.*

(ii) *Suppose that Z is uniformly convex and W is uniformly smooth. If $T \in \mathcal{A}_{\|\cdot\|}(W,Z)$, then $\widetilde{T} \in \mathcal{A}_{\mathrm{nu}}(W \oplus_\infty Z)$.*

Proof. (i). Suppose $\widetilde{T} \in \mathcal{A}_{\mathrm{nu}}(W \oplus_\infty Z)$. Given $\varepsilon \in (0,1)$, we set $\eta(\varepsilon, T) := \eta(\varepsilon, \widetilde{T}) > 0$. Let $w_0 \in S_W$ be such that $\|T(w_0)\| > 1 - \frac{\eta(\varepsilon,T)}{2}$. Take $\widetilde{z_0}^* \in S_{Z^*}$ to be such that

$$|\widetilde{z_0}^*(T(w_0))| = \|T(w_0)\| > 1 - \frac{\eta(\varepsilon, T)}{2}.$$

By the Bishop-Phelps theorem, there is $z_0^* \in S_{Z^*}$ and $\widetilde{z_0} \in S_Z$ such that $|z_0^*(\widetilde{z_0})| = 1$ and $\|z_0^* - \widetilde{z_0}^*\| < \frac{\eta(\varepsilon,T)}{2}$. Since $z_0^*(\widetilde{z_0}) = e^{i\theta}$ for some $\theta \in [0, 2\pi)$, we take $z_0 := e^{-i\theta}\widetilde{z_0} \in S_Z$ which satisfies $z_0^*(z_0) = 1$ and

$$\begin{aligned} |z_0^*(T(w_0))| &= |\widetilde{z_0}^*(T(w_0)) + (z_0^* - \widetilde{z_0}^*)(T(w_0))| \\ &\geqslant |\widetilde{z_0}^*(T(w_0))| - \|z_0^* - \widetilde{z_0}^*\| \\ &> 1 - \eta(\varepsilon, T). \end{aligned}$$

Consider the point $((w_0, z_0), (0, z_0^*)) \in \Pi(W \oplus_\infty Z)$. Then, since $\nu(\widetilde{T}) = 1$ and

$$|(0, z_0^*)(\widetilde{T}(w_0, z_0))| = |z_0^*(T(w_0))| > 1 - \eta(\varepsilon, T) = 1 - \eta(\varepsilon, \widetilde{T}),$$

there exists a state $((w_1, z_1), (w_1^*, z_1^*)) \in \Pi(W \oplus_\infty Z)$ satisfying that $|(w_1^*, z_1^*)(\widetilde{T}(w_1, z_1))| = 1$, $\|(w_1, z_1) - (w_0, z_0)\|_\infty < \varepsilon$ and $\|(w_1^*, z_1^*) - (0, z_0^*)\|_1 < \varepsilon$. So, since $1 = |(w_1^*, z_1^*)(\widetilde{T}(w_1, z_1))| = |z_1^*(T(w_1))| \leqslant \|T(w_1)\| \leqslant 1$, we get that $\|T(w_1)\| = \|w_1\| = 1$. Finally, $\|w_1 - w_0\| \leqslant \|(w_1, z_1) - (w_0, z_0)\|_\infty < \varepsilon$. This shows that $T \in \mathcal{A}_{\|\cdot\|}(W, Z)$.

(ii). Suppose $T \in \mathcal{A}_{\|\cdot\|}(W, Z)$. It is not difficult to see that $\nu(\widetilde{T}) = 1$ and that $\widetilde{T}$ attains its numerical radius. Indeed, we always have $\nu(\widetilde{T}) \leqslant \|\widetilde{T}\| = \|T\| = 1$. Now take $w_0 \in S_W$ to be such that $\|T(w_0)\| = \|T\| = 1$. Let $z_0^* \in S_{Z^*}$ be such that $|z_0^*(T(w_0))| = \|T(w_0)\| = 1$. Since Z is reflexive, there is $z_0 \in S_Z$ such that $z_0^*(z_0) = 1$. Consider the point $((w_0, z_0), (0, z_0^*)) \in \Pi(W \oplus_\infty Z)$. Then $|(0, z_0^*)(\widetilde{T}((w_0, z_0)))| = |z_0^*(T(w_0))| = 1$.

Now let $\varepsilon \in (0, 1)$ be given and set $\eta(\varepsilon, \widetilde{T})$ as the positive real number $\eta(\varepsilon, \widetilde{T}) := \min\{\varepsilon_0, \eta(\varepsilon_0, T)\}$, where

$$\varepsilon_0 = \min\left\{\frac{1}{2}\delta_{Z^*}\left(\min\left\{\frac{\delta_Z(\varepsilon)}{2}, \frac{\varepsilon}{2}\right\}\right), \frac{\delta_Z(\varepsilon)}{2}, \frac{\varepsilon}{2}\right\}.$$

Let $((w_1, z_1), (w_1^*, z_1^*)) \in \Pi(W \oplus_\infty Z)$ be such that

$$|z_1^*(T(w_1))| = \left|(w_1^*, z_1^*)(\widetilde{T}(w_1, z_1))\right| > 1 - \eta(\varepsilon, \widetilde{T}).$$

Since $\|T(w_1)\| \geqslant |z_1^*(T(w_1))| > 1 - \eta(\varepsilon, \widetilde{T})$, there is $w_2 \in S_W$ such that $\|T(w_2)\| = 1$ and $\|w_2 - w_1\| < \varepsilon_0$. Since $\|z_1^*\| \geqslant |z_1^*(T(w_1))| > 1 - \eta(\varepsilon, \widetilde{T})$, we get that $\|w_1^*\| < \eta(\varepsilon, \widetilde{T}) \leqslant \frac{\varepsilon}{2}$. Let $\theta \in \mathbb{R}$ be such that $z_1^*(T(w_2)) = |z_1^*(T(w_2))|e^{i\theta}$. Pick $z_2^* \in S_{Z^*}$ to be such that $z_2^*(T(w_2)) = e^{i\theta}$ and notice that $|z_1^*(T(w_2))| > 1 - 2\varepsilon_0 > 1 - \delta_{Z^*}\left(\min\left\{\frac{\delta_Z(\varepsilon)}{2}, \frac{\varepsilon}{2}\right\}\right)$. Thus,

$$\begin{aligned}\left\|\frac{z_1^* + z_2^*}{2}\right\| \geqslant \left|\frac{z_1^* + z_2^*}{2}(T(w_2))\right| &= \frac{|z_1^*(T(w_2))| + 1}{2} \\ &> 1 - \delta_{Z^*}\left(\min\left\{\frac{\delta_Z(\varepsilon)}{2}, \frac{\varepsilon}{2}\right\}\right). \qquad (2.4.2)\end{aligned}$$

This implies that $\|z_2^* - z_1^*\| < \min\left\{\frac{\delta_Z(\varepsilon)}{2}, \frac{\varepsilon}{2}\right\}$.

By using the above estimates,

$$\begin{aligned}\left\|\frac{T(e^{-i\theta}w_2) + z_1}{2}\right\| &\geqslant \left|z_1^*\left(\frac{T(e^{-i\theta}w_2) + z_1}{2}\right)\right| \\ &= \left|\frac{|z_1^*(T(w_2))| + 1 - w_1^*(w_1)}{2}\right| \\ &\geqslant \left|\frac{|z_1^*(T(w_2))| + 1}{2}\right| - \left|\frac{w_1^*(w_1)}{2}\right| \geqslant 1 - \delta_Z(\varepsilon)\end{aligned}$$

and so $\|T(e^{-i\theta}w_2) - z_1\| < \varepsilon$. Finally, we conclude that $\widetilde{T}$ attains its numerical radius at the point $\big((w_2, T(e^{-i\theta}w_2)), (0, z_2^*)\big) \in \Pi(W \oplus_\infty Z)$ which is close to $((w_1, z_1), (w_1^*, z_1^*))$; hence $\widetilde{T} \in \mathcal{A}_{\mathrm{nu}}(W \oplus_\infty Z)$. ■

Remark 2.4.5. Similar to what happened on Proposition 2.4.1, the item (ii) from Proposition 2.4.4 is not true in general for arbitrary Banach spaces. Indeed, consider the real Banach space ℓ_1. Like we did in Remark 2.4.2, we will show that the operator introduced in (2.2.2) does not satisfy the property stated in Proposition 2.4.4.(ii): Let $S \in \mathcal{L}(\ell_1, \ell_1)$ be the operator defined in (2.2.2) and let $\widetilde{S} \in \mathcal{L}(\ell_1 \oplus_\infty \ell_1, \ell_1 \oplus_\infty \ell_1)$ be defined accordingly. Notice as before that if $((x, y), (x^*, y^*)) \in \Pi(\ell_1 \oplus_\infty \ell_1)$ satisfies $|(x^*, y^*)(\widetilde{S}(x, y))| = |y^*(S(x))| = \left|\sum_{j=1}^\infty \frac{y^*(j)x(1)}{2^j}\right| = 1$, then $y^*(j)x(1)$ has to be equal to either 1 or -1 for all $j \in \mathbb{N}$. From here, we get that the only possibilities have the form $x = se_1$, $y =$

$(y(1), y(2), y(3), \ldots)$ with $\sum_{j=1}^{\infty} y(j) = r$, $x^* = 0$, and $y^* = (r, r, r, \ldots)$, where $s, r \in \{-1, 1\}$. Assuming that for $\varepsilon \in (0,1)$, there exists $\eta(\varepsilon, \widetilde{S}) > 0$, we get a contradiction in the same manner as in Remark 2.4.2.

Once again, notice that the situation for operators $\check{T}$ is different.

Remark 2.4.6. There exists an operator $S \in \mathcal{L}(W \oplus_1 Z, W \oplus_1 Z)$, with W uniformly smooth and Z uniformly convex, such that $S \in \mathcal{A}_{\mathrm{nu}}(W \oplus_\infty Z)$ but $\check{S} \notin \mathcal{A}_{\|\cdot\|}(W, Z)$. Indeed, the same argument used in Remark 2.4.3 shows that $S \in \mathcal{L}(\ell_2 \oplus_\infty \ell_2, \ell_2 \oplus_\infty \ell_2)$, which is defined as

$$S(x, y) = ((x(1), 0, 0, \cdots), (0, 0, 0, \cdots)), \quad \forall (x, y) \in \ell_2 \oplus_\infty \ell_2,$$

where ℓ_2 is a real space, belongs to $\mathcal{A}_{\mathrm{nu}}(\ell_2 \oplus_\infty \ell_2)$. However, $\check{S} = 0$ cannot belong to $\mathcal{A}_{\|\cdot\|}(\ell_2, \ell_2)$.

We finish the chapter by noting that Propositions 2.4.1.(ii) and 2.4.4.(ii) are no longer true for p-sums with $1 < p < \infty$. Indeed, let X be a uniformly convex and uniformly smooth Banach space and consider the identity operator $\mathrm{Id}_X \in \mathcal{L}(X, X)$. Clearly, Id_X belongs to $\mathcal{A}_{\|\cdot\|}(X, X)$. On the other hand, $\widetilde{\mathrm{Id}}_X \in \mathcal{L}(X \oplus_p X, X \oplus_p X)$ is defined as $\widetilde{\mathrm{Id}}_X(x_1, x_2) = (0, x_1)$ for all $x_1, x_2 \in X$. Then $\nu(\widetilde{\mathrm{Id}}_X) \leqslant \|\widetilde{\mathrm{Id}}_X\| = \|\mathrm{Id}_X\| = 1$. If $|(x_1^*, x_2^*)(\widetilde{\mathrm{Id}}_X(x_1, x_2))| = 1$ for some $((x_1, x_2), (x_1^*, x_2^*)) \in \Pi(X \oplus_p X)$, we would have $|x_2^*(x_1)| = 1$, which would imply $\|x_2^*\| = \|x_1\| = 1$. Because of this, we would have $x_1^* = x_2 = 0$ since $\|x_1^*\|^q + \|x_2^*\|^q = 1 = \|x_1\|^p + \|x_2\|^p$ with $\frac{1}{p} + \frac{1}{q} = 1$, contradicting the assumption $x_1^*(x_1) + x_2^*(x_2) = 1$. So, $\widetilde{\mathrm{Id}}_X$ cannot attain its numerical radius; hence it cannot belong to $\mathcal{A}_{\mathrm{nu}}(X \oplus_p X)$.

Chapter 3

The Bishop-Phelps-Bollobás property for numerical radius and compact operators

3.1 Introduction and motivation

Let X be a Banach space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Recall that the *numerical index* of X is defined as

$$n(X) := \inf_{T \in S_{\mathcal{L}(X,X)}} \nu(T) = \max_{T \in \mathcal{L}(X,X)} \{k \geqslant 0\colon\ k\|T\| \leqslant \nu(T)\}.$$

For the purposes of this chapter, where we will mainly deal with compact operators, we need a compact version of $n(X)$. Following [26], we define the *compact numerical index* of X as

$$n_{\mathcal{K}}(X) := \inf_{T \in S_{\mathcal{K}(X,X)}} \nu(T) = \max_{T \in \mathcal{K}(X,X)} \{k \geqslant 0\colon\ k\|T\| \leqslant \nu(T)\}.$$

We refer to [26], [82], [81], [89, Subsection 1.1], and references therein for more information and background on both of these terms.

In his 1972 PhD dissertation, Sims asked when the set of numerical radius attaining operators on a Banach space X is dense (see [111]). Many authors have contributed to this question ever since (see for instance Acosta's PhD dissertation, where this question is studied systematically, [1]). With this in mind, and inspired by [5], Guirao and Kozhushkina introduced and studied in 2013 a numerical radius version of the Bishop-Phelps-Bollobás property (see [69]). We recall its definition.

Definition 3.1.1 (Combining [69, Definition 1.2] and [87, Definition 5])**.** A Banach space X has the *weak Bishop-Phelps-Bollobás property for the numerical radius* (weak BPBp-nu, for short) if given $\varepsilon > 0$, there exists $\eta(\varepsilon) > 0$ such that, whenever $T \in \mathcal{L}(X, X)$ with $\nu(T) = 1$ and $(x, x^*) \in \Pi(X)$ satisfy $|x^*(T(x))| > 1 - \eta(\varepsilon)$, there exist $S \in \mathcal{L}(X, X)$ and $(y, y^*) \in \Pi(X)$ such that

$$\nu(S) = |y^*(S(y))|, \quad \|x - y\| < \varepsilon, \quad \|x^* - y^*\| < \varepsilon, \quad \|T - S\| < \varepsilon.$$

If, moreover, S can be chosen so that $\nu(S) = 1$, we say that X has the *Bishop-Phelps-Bollobás property for the numerical radius* (abbreviated BPBp-nu, although some authors use the notation BPBp-ν as well).

Since then, several works have been done in order to study what spaces satisfy that property. We summarize next some of the most important results on the matter:

(i) All the spaces $c_0(\Gamma)$ and $\ell_1(\Gamma)$ have the BPBp-nu (see [69]).

(ii) $L_1(\mathbb{R})$ has the BPBp-nu (see [55]).

(iii) Finite-dimensional spaces have the BPBp-nu (see [87]).

(iv) The real or complex space $L_p(\mu)$ has the BPBp-nu for every measure μ when $1 \leqslant p < \infty$ ([87, Example 8] except for the real case with $p = 2$, which is covered in [89, Corollary 3.3]).

(v) Any uniformly convex and uniformly smooth Banach space X with $n(X) > 0$ has the BPBp-nu (see [87]).

(vi) Every separable infinite-dimensional Banach space can be renormed to fail the BPBp-nu, even though the set of numerical radius attaining operators is always dense in spaces with the Radon-Nikodým property (see [87]).

(vii) The real space $C(K)$ has the BPBp-nu if the compact Hausdorff space K has local compensation, which happens for instance if K is metrizable ([13]).

We refer to the cited papers [13, 55, 69, 87, 89], the papers [7, 34, 88], the surveys [3, 40], and references therein for more information and background.

In 2018, Dantas, García, Maestre and Martín [39] studied the BPBp for compact operators. They presented some abstract techniques (based on results about norm-attaining compact operators by Johnson and Wolfe [77]) which allow to carry the BPBp for compact operators from sequence spaces (such as c_0 and ℓ_p) to function spaces (such as $C_0(L, Y)$ and $L_p(\mu, Y)$). As one of the main results, it is shown in [39] that the BPBp for compact operators of the pair (c_0, Y) is sufficient to get the BPBp for compact operators of all the pairs $(C_0(L), Y)$ regardless of the locally compact Hausdorff topological space L.

The numerical radius attaining compact operators have also been studied in the recent years (see [22]). Our aim in this chapter is to study the following property, inspired both by the BPBp for compact operators and by the BPBp for numerical radius.

Definition 3.1.2. A Banach space X is said to have the *weak BPBp-nu for compact operators* if for every $0 < \varepsilon < 1$, there exists $\eta(\varepsilon) > 0$ such that whenever $T \in \mathcal{K}(X, X)$ and $(x, x^*) \in \Pi(X)$ satisfy $\nu(T) = 1$ and $|x^*(T(x))| > 1 - \eta(\varepsilon)$, there exist $S \in \mathcal{K}(X, X)$ and $(y, y^*) \in \Pi(X)$ such that

$$\nu(S) = |y^*(S(y))|, \quad \|T - S\| < \varepsilon, \quad \|x - y\| < \varepsilon, \quad \|x^* - y^*\| < \varepsilon.$$

If, moreover, S can be chosen so that $\nu(S) = 1$, we say that X has the *BPBp-nu for compact operators.*

The first work where a somewhat similar property was introduced is [7], where the BPBp-nu was considered for subspaces of $\mathcal{L}(X, X)$ and studied in the case of $L_1(\mu)$, with μ a finite measure.

In this chapter, we will study the BPBp-nu for compact operators, and we will provide an extensive list of Banach spaces that satisfy it. The rest of the chapter is structured as follows. In Section 3.2, we will show that a wide collection of classical Banach spaces have the property, including finite-dimensional spaces, $c_0(\Gamma)$ and $\ell_1(\Gamma)$ spaces, and $L_p(\mu)$ spaces ($1 \leqslant p < \infty$, μ any measure). At the end of the chapter, we will show that all $C_0(L)$ spaces also have the property (see Theorem 3.4.7). In order to prove that, we need two kinds of ingredients. First, we provide in Section 3.3 some abstract results that will allow us to carry the BPBp-nu for compact operators from some spaces (for example, some sequence spaces) into some other spaces (for example, some function spaces). The most general result we have of this kind is Lemma 3.3.1, which will be the first ingredient for the proof of Theorem 3.4.7. This lemma extends [39, Lemma 2.1], but it needs more restrictive hypothesis in order to deal with the numerical radius instead of with the norm of the operators. We also provide in that section some useful particular cases and applications of Lemma 3.3.1 which allow to show, for instance, that every predual of ℓ_1

has the BPBp-nu for compact operators (see Corollary 3.3.6). The second ingredient for the proof of Theorem 3.4.7 is some strong approximation property of $C_0(L)$ and its dual which will be provided in Section 3.4 (see Theorem 3.4.5) and which will allow us to apply Lemma 3.3.1 in this case, thus providing the proof of Theorem 3.4.7. Let us also comment that Theorem 3.4.5 gives a much stronger approximation property of $C_0(L)$ and its dual space than [39, Lemma 3.4].

3.2 First results

In this section, we will provide an extensive list of Banach spaces that have the BPBp-nu for compact operators. First, by carefully examining the known proofs of spaces that have the BPBp-nu, we can already show that many of those spaces also have the BPBp-nu for compact operators with some adaptations.

Examples 3.2.1. The following spaces have the BPBp-nu for compact operators:

(i) Finite-dimensional spaces.

(ii) $c_0(\Gamma)$ and $\ell_1(\Gamma)$ for every index set Γ.

(iii) $L_1(\mu)$ for every measure μ.

Proof.

(i) It follows from [87, Proposition 2] and the fact that every operator $T : X \to X$ is compact if X is finite-dimensional.

(ii) This follows from [69, Corollary 3.3] and [69, Corollary 4.2], since in the proof of those results, if the original operator T is compact, the new operator they build is also compact.

(iii) The case of finite measures was already studied in [7, Corollary 2.1]. From there, it suffices to follow the reasoning in the proof of [87, Theorem 9], keeping in mind that if the original operator T_0 is compact, so is the new one they build. ■

Adapting now the results from [87] and [89], one also has that the $L_p(\mu)$ spaces have the BPBp-nu for compact operators when $1 < p < \infty$, as we will see in Proposition 3.2.8. We need some preparatory work to do so. It is shown in [87, Proposition 4] that uniformly convex and uniformly smooth Banach spaces have the weak BPBp-nu. This result also holds for the compact operators version by an easy adaptation of the proof.

Proposition 3.2.2. *If a Banach space is uniformly convex and uniformly smooth, then it has the weak BPBp-nu for compact operators.*

Proof. We can follow the proof of [87, Proposition 4], just keeping in mind that if the original operator T_0 is compact, then the rest of operators T_n from that proof are also compact, and so, S is compact too. ■

Later, in [87, Proposition 6], it is proven that in Banach spaces with positive numerical index, the BPBp-nu and the weak BPBp-nu are equivalent. This claim is also true for the compact operators versions of the properties if we use the compact numerical index.

Proposition 3.2.3. *Let X be a Banach space such that $n_{\mathcal{K}}(X) > 0$. Then X has the BPBp-nu if and only if it has the weak BPBp-nu.*

Proof. It suffices to follow the proof from [87, Proposition 6] but with both T and S being now compact operators, and using $n_{\mathcal{K}}(X)$ instead of $n(X)$ in all instances. ■

As a consequence of these two results, similarly to [87], we get that all $L_p(\mu)$ spaces have the BPBp-nu for compact operators when $1 < p < \infty$

in the complex case and when $1 < p < \infty$, $p \neq 2$, in the real case. This is so because, on the one hand, in the real case,

$$n_{\mathcal{K}}(L_p(\mu)) \geqslant n(L_p(\mu)) > 0 \qquad \big(1 < p < \infty,\ p \neq 2\big)$$

by [99] and, on the other hand, $n_{\mathcal{K}}(X) \geqslant 1/\mathrm{e} > 0$ for every complex Banach space (see [82, Eq. (1) in p. 156], for instance).

This provides the proof of Proposition 3.2.8 except for the real space $L_2(\mu)$. Our next aim is to show that real Hilbert spaces also have the BPBp-nu for compact operators, by adapting the ideas from [89].

First, given a real Banach space X, we consider the following subset of $\mathcal{K}(X,X)$:

$$\mathcal{Z}_{\mathcal{K}}(X) := \big\{T \in \mathcal{K}(X,X) \colon \nu(T) = 0\big\},$$

which is the set of all skew-hermitian compact operators on X. Observe that $\mathcal{Z}_{\mathcal{K}}(X) = \mathcal{K}(X,X) \cap \mathcal{Z}(X)$, where $\mathcal{Z}(X) = \{T \in \mathcal{L}(X,X) \colon \nu(T) = 0\}$. Adapting the concept of second numerical index given in [89], we define the *second numerical index for compact operators* of a Banach space X as the constant

$$\begin{aligned} n'_{\mathcal{K}}(X) &:= \inf\big\{\nu(T) \colon T \in \mathcal{K}(X,X), \|T + \mathcal{Z}_{\mathcal{K}}(X)\| = 1\big\} \\ &= \max\big\{M \geqslant 0 \colon M\|T + \mathcal{Z}_{\mathcal{K}}(X)\| \leqslant \nu(T) \text{ for all } T \in \mathcal{K}(X,X)\big\}, \end{aligned}$$

where $\|T + \mathcal{Z}_{\mathcal{K}}(X)\|$ is the quotient norm in $\mathcal{K}(X,X)/\mathcal{Z}_{\mathcal{K}}(X)$.

The next result is a version for compact operators of [89, Theorem 3.2].

Proposition 3.2.4. *Let X be a real Banach space with $n'_{\mathcal{K}}(X) > 0$. Then, the BPBp-nu for compact operators and the weak BPBp-nu for compact operators are equivalent in X.*

Proof. It suffices to adapt the steps from the proof of [89, Theorem 3.2] to the case of compact operators. That is: all the involved operators T, S, S_1 and S_2 are now compact, the set $\mathcal{Z}(X)$ is replaced by $\mathcal{Z}_{\mathcal{K}}(X)$, and the index $n'(X)$ is replaced by $n'_{\mathcal{K}}(X)$. ■

We are going to see next that the second numerical index for compact operators of a real Hilbert space equals one.

Proposition 3.2.5. *Let H be a real Hilbert space. Then, $n'_{\mathcal{K}}(H) = 1$.*

The proof of this result will be an adaptation of the one of [89, Theorem 2.3]. Recall that in a real Hilbert space endowed with an inner product $\langle \cdot, \cdot \rangle$, H^* is identified with H by the isometric isomorphism $x \longmapsto \langle \cdot, x \rangle$. Therefore, $\Pi(X) = \{(x, x) \in H \times H : x \in S_H\}$, and so, for every $T \in \mathcal{L}(H, H)$, one has $\nu(T) = \sup\{|\langle T(x), x \rangle| : x \in S_H\}$. We first need to give the compact operators version of [89, Lemma 2.4] whose proof is an obvious adaptation of the proof of that result.

Lemma 3.2.6. *Let H be a real Hilbert space.*

(i) $\mathcal{Z}_{\mathcal{K}}(H) = \{T \in \mathcal{K}(H, H) : T = -T^*\}$.

(ii) *If $T \in \mathcal{K}(H, H)$ is self-adjoint (i.e. $T = T^*$), then $\|T\| = \nu(T)$.*

We are now ready to present the proof of Proposition 3.2.5.

Proof of Proposition 3.2.5. It suffices to adapt the proof of [89, Theorem 2.3] to the compact operators case, that is: the involved operators T and S are now compact, and the set $\mathcal{Z}(X)$ is replaced by $\mathcal{Z}_{\mathcal{K}}(X)$. ■

As a consequence of Propositions 3.2.2, 3.2.4, and 3.2.5, we get the following result which finishes the proof of Proposition 3.2.8.

Corollary 3.2.7. *If H is a real Hilbert space, then it has the BPBp-nu for compact operators.*

As a consequece of this all, we get the result we wanted.

Proposition 3.2.8. *$L_p(\mu)$ has the BPBp-nu for compact operators, for every measure μ and $1 < p < \infty$.*

3.3 Technical tools

In this section, we will provide an abstract result that will allow us later to carry the BPBp-nu for compact operators from some sequence spaces to function spaces, or from some projections of a space to the space itself. The most general version that we are able to prove is Lemma 3.3.1 below, which is inspired by [39, Lemma 2.1], but it needs more requirements. We recall some needed notation first. An *absolute norm* $|\cdot|_a$ is a norm in $\mathbb{R}^2$ such that $|(1,0)|_a = |(0,1)|_a = 1$ and $|(s,t)|_a = |(|s|,|t|)|_a$ for every $(s,t) \in \mathbb{R}^2$. Given a Banach space X, we say that a projection P on X is an *absolute projection* if there is an absolute norm $|\cdot|_a$ such that $\|x\| = \big|(\|P(x)\|, \|x - P(x)\|)\big|_a$ for every $x \in X$. Examples of absolute projections are the M-projections, the L-projections and, more in general, the ℓ_p-projections. We refer the reader to [39] for the use of absolute norms with the Bishop-Phelps-Bollobás type properties and to the references therein for more information on absolute norms.

Lemma 3.3.1. *Let X be a Banach space satisfying that $n_{\mathcal{K}}(X) > 0$. Suppose that there is a mapping $\eta\colon (0,1) \longrightarrow (0,1)$ such that given $\delta > 0$, $x_1^*, \ldots, x_n^* \in B_{X^*}$ and $x_1, \ldots, x_\ell \in B_X$, we can find norm one operators $\widetilde{P}\colon X \longrightarrow \widetilde{P}(X)$, $i\colon \widetilde{P}(X) \longrightarrow X$ such that for $P := i \circ \widetilde{P}\colon X \longrightarrow X$, the following conditions are satisfied:*

(i) $\|P^*(x_j^*) - x_j^*\| < \delta$, *for* $j = 1, \ldots, n$.

(ii) $\|P(x_j) - x_j\| < \delta$, *for* $j = 1, \ldots, \ell$.

(iii) $\widetilde{P} \circ i = \mathrm{Id}_{\widetilde{P}(X)}$.

(iv) $\widetilde{P}(X)$ *satisfies the Bishop-Phelps-Bollobás property for numerical radius for compact operators with the mapping* η.

(v) *Either* P *is an absolute projection and* i *is the natural inclusion, or* $n_{\mathcal{K}}(\widetilde{P}(X)) = n_{\mathcal{K}}(X) = 1$.

Then, X *satisfies the BPBp-nu for compact operators.*

Let us comment on the differences between the lemma above and [39, Lemma 2.1]. First, condition (ii) is more restrictive here than in that lemma that only dealt with one point. Second, the requirements of item (v) on the compact numerical index or on the absoluteness of the projections did not appear in [39, Lemma 2.1], but they are needed here as numerical radius does not behave well in general with respect to extensions of operators.

Proof. Given $\varepsilon \in (0, 1)$, let $\varepsilon_0(\varepsilon)$ be the unique number with $0 < \varepsilon_0(\varepsilon) < 1$ such that

$$\varepsilon_0(\varepsilon)\left(\frac{2}{3} + \frac{1}{(1-\varepsilon_0(\varepsilon))\, n_{\mathcal{K}}(X)}\right) = \varepsilon,$$

which, in particular, exists and satisfies that $\varepsilon_0(\varepsilon) < \varepsilon$. From now on, we write ε_0 instead of $\varepsilon_0(\varepsilon)$. We define next for each $\varepsilon \in (0,1)$

$$\eta'(\varepsilon) := \min\left\{\frac{\varepsilon_0^2(n_{\mathcal{K}}(X))^2}{72}, \frac{\left(\eta\left(\frac{\varepsilon_0}{3}\right)\right)^2(n_{\mathcal{K}}(X))^2}{72}\right\}, \tag{3.3.1}$$

where η is the function appearing in the hypotheses of the lemma. We fix $T \in \mathcal{K}(X,X)$ with $\nu(T) = 1$ (thus, $\|T\| \leqslant \frac{1}{n_{\mathcal{K}}(X)}$) and $(x_1, x_1^*) \in \Pi(X)$

such that

$$|x_1^*(T(x_1))| > 1 - \eta'(\varepsilon).$$

Since $T^*(B_{X^*})$ is relatively compact, we can find $x_2^*, \ldots, x_n^* \in B_{X^*}$ such that

$$\min_{2 \leqslant j \leqslant n} \|T^*(x^*) - x_j^*\| < \eta'(\varepsilon) \quad \text{for all } x^* \in B_{X^*}.$$

Similarly, since $T(B_X)$ is relatively compact, we can find $x_2, \ldots, x_\ell \in B_X$ such that

$$\min_{2 \leqslant j \leqslant \ell} \|T(x) - x_j\| < \eta'(\varepsilon) \quad \text{for all } x \in B_X.$$

Let $\widetilde{P}\colon X \longrightarrow \widetilde{P}(X)$, $i\colon \widetilde{P}(X) \longrightarrow X$ and $P := i \circ \widetilde{P}\colon X \longrightarrow X$ satisfying the conditions (i)-(v) for $x_1, \ldots, x_\ell \in B_X$, $x_1^*, \ldots, x_n^* \in B_{X^*}$ and $\delta = \eta'(\varepsilon)$.

Now, for every $x^* \in B_{X^*}$, we have

$$\begin{aligned}
&\|T^*(x^*) - P^*(T^*(x^*))\| \\
&\quad \leqslant \min_{2 \leqslant j \leqslant n} \left\{\|T^*(x^*) - x_j^*\| + \|x_j^* - P^*(x_j^*)\| + \|P^*(x_j^*) - P^*(T^*(x^*))\|\right\} \\
&\quad < 3\eta'(\varepsilon),
\end{aligned}$$

and hence, $\|T - TP\| = \|T^* - P^*T^*\| \leqslant 3\eta'(\varepsilon)$. On the other hand, for each $x \in B_X$, we have

$$\begin{aligned}
&\|T(x) - P(T(x))\| \\
&\quad \leqslant \min_{2 \leqslant j \leqslant \ell} \{\|T(x) - x_j\| + \|x_j - P(x_j)\| + \|P(x_j) - P(T(x))\|\} \\
&\quad < 3\eta'(\varepsilon),
\end{aligned}$$

and then, $\|T - PT\| \leqslant 3\eta'(\varepsilon)$. Therefore,

$$\|PTP - T\| \leqslant \|PTP - PT\| + \|PT - T\| \leqslant \|TP - T\| + \|PT - T\| \leqslant 6\eta'(\varepsilon).$$

Consider $(\widetilde{P}(x_1),\, i^*(x_1^*)) \in \widetilde{P}(X) \times (\widetilde{P}(X))^*$. Note that it is not true in general that $(\widetilde{P}(x_1),\, i^*(x_1^*)) \in \Pi(\widetilde{P}(X))$, but we have that $\|\widetilde{P}(x_1)\| \leqslant 1$, $\|i^*(x_1^*)\| \leqslant 1$, and also, that

$$x_1^*(i(\widetilde{P}(x_1))) = \underbrace{x_1^*(x_1)}_{=1} - \underbrace{x_1^*(i(\widetilde{P}(x_1)) - x_1)}_{\|Px_1 - x_1\| < \eta'(\varepsilon)},$$

and so, $\mathrm{Re}(x_1^*(i(\widetilde{P}(x_1)))) \geqslant 1 - \eta'(\varepsilon)$.

By the Bishop-Phelps-Bollobás Theorem (in particular, the sharp version from [25, Corollary 2.4.b]), there is $(y,\, y^*) \in \Pi(\widetilde{P}(X))$ satisfying that

$$\max\left\{\|y - \widetilde{P}(x_1)\|,\, \|y^* - i^*(x_1^*)\|\right\} \leqslant \sqrt{2\eta'(\varepsilon)} \leqslant \frac{\varepsilon_0}{3}.$$

Next, we observe that the following two inequalities hold:

$$\begin{aligned} \|\widetilde{P}^*(y^*) - x_1^*\| &\leqslant \|\widetilde{P}^*(y^*) - \widetilde{P}^*(i^*(x_1^*))\| + \|\widetilde{P}^*(i^*(x_1^*)) - x_1^*\| \\ &\leqslant \sqrt{2\eta'(\varepsilon)} + \eta'(\varepsilon) \leqslant \frac{2}{3}\varepsilon_0, \end{aligned} \tag{3.3.2}$$

$$\begin{aligned} \|i(y) - x_1\| &\leqslant \|i(y) - i(\widetilde{P}(x_1))\| + \|i(\widetilde{P}(x_1)) - x_1\| \\ &\leqslant \sqrt{2\eta'(\varepsilon)} + \eta'(\varepsilon) \leqslant \frac{2}{3}\varepsilon_0. \end{aligned} \tag{3.3.3}$$

Let $T_1 := \widetilde{P} \circ T \circ i \colon \widetilde{P}(X) \longrightarrow \widetilde{P}(X)$.

Claim: we have that

$$|y^*(T_1(y))| > 1 - \eta\left(\frac{\varepsilon_0}{3}\right) \quad \text{and} \quad |y^*(T_1(y))| > 1 - \varepsilon_0.$$

Indeed, from equations (3.3.2) and (3.3.3), we obtain that

$$\begin{aligned}
&\left|x^*(T(x_1)) - \widetilde{P}^*(y^*(T(i(y))))\right| \\
&\quad \leqslant |x_1^*(T(x_1)) - x_1^*(T(i(y)))| + |x_1^*(T(i(y))) - \widetilde{P}^*(y^*(T(i(y))))| \\
&\quad \leqslant \|T\|\|x_1 - i(y)\| + \|T\|\|x_1^* - \widetilde{P}^*(y^*)\| \\
&\quad \leqslant 2\|T\| \left(\sqrt{2\eta'(\varepsilon)} + \eta'(\varepsilon)\right).
\end{aligned}$$

Now, we can estimate $|y^*(T_1(y))|$ as follows:

$$\begin{aligned}
|y^*(T_1(y))| &= \left|\widetilde{P}^*(y^*(T(i(y))))\right| \\
&\geqslant \left|x_1^*(T(x_1))\right| - \left|x_1^*(T(x_1)) - \widetilde{P}^*(y^*(T(i(y))))\right| \\
&\geqslant 1 - \eta'(\varepsilon) - 2\|T\|\sqrt{2\eta'(\varepsilon)} - 2\|T\|\eta'(\varepsilon).
\end{aligned}$$

From here, using the definition of $\eta'(\varepsilon)$ given in Eq. (3.3.1) and the fact that $\|T\| \leqslant 1/n_{\mathcal{K}}(X)$, we get both assertions of the claim.

In particular, we get that $\nu(T_1) \geqslant 1 - \varepsilon_0 > 0$. On the other hand, we also have that $\nu(T_1) \leqslant 1$. Indeed, if there were some $(q, q^*) \in \Pi(\widetilde{P}(X))$ with $|q^*(T_1(q))| > 1$, we would get

$$|q^*(T_1(q))| = |q^*(\widetilde{P}(T(i(q))))| = |(\widetilde{P}^*(q^*))(T(i(q)))| > 1,$$

but $\nu(T) = 1$, and

$$(\widetilde{P}^*(q^*))(i(q)) = q^*(\widetilde{P}(i(q))) = q^*(q) = 1.$$

Thus $(i(q), \widetilde{P}^*(q^*)) \in \Pi(X)$, and that is a contradiction.

We define now the operator $\widetilde{T} := \frac{T_1}{\nu(T_1)}$. Clearly, $\widetilde{T}$ is a compact operator such that $\nu(\widetilde{T}) = 1$. From the claim, we get that

$$\left|y^*(\widetilde{T}(y))\right| = \frac{1}{\nu(T_1)}|y^*(T_1(y))| \geqslant |y^*(T_1(y))| > 1 - \eta\left(\frac{\varepsilon_0}{3}\right).$$

Now, since $\widetilde{P}(X)$ has the BPBp-nu for compact operators with the mapping η, there exist a compact operator $\widetilde{S}\colon \widetilde{P}(X) \longrightarrow \widetilde{P}(X)$ with $\nu(\widetilde{S}) = 1$ and $(z, z^*) \in \Pi(\widetilde{P}(X))$ such that

$$\nu(\widetilde{S}) = \left|z^*(\widetilde{S}(z))\right| = 1, \quad \|z-y\| < \frac{\varepsilon_0}{3}, \quad \|z^*-y^*\| < \frac{\varepsilon_0}{3}, \quad \|\widetilde{S}-\widetilde{T}\| < \frac{\varepsilon_0}{3}.$$

Let $t = i(z) \in B_X$ and $t^* = \widetilde{P}^*(z^*) \in B_{X^*}$. We have that

$$t^*(t) = z^*(\widetilde{P}(i(z))) = z^*(z) = 1.$$

Thus $(t, t^*) \in \Pi(X)$, and also, by (3.3.2) and (3.3.3),

$$\|t-x_1\| \leqslant \|t-i(y)\|+\|i(y)-x_1\| = \|i(z)-i(y)\|+\|i(y)-x_1\| < \frac{\varepsilon_0}{3}+\frac{2\varepsilon_0}{3} \leqslant \varepsilon,$$

$$\|t^* - x_1^*\| \leqslant \|\widetilde{P}^*(z^*) - \widetilde{P}^*(y^*)\| + \|\widetilde{P}^*(y^*) - x_1^*\| < \frac{\varepsilon_0}{3} + \frac{2\varepsilon_0}{3} \leqslant \varepsilon.$$

We define $S = i \circ \widetilde{S} \circ \widetilde{P}\colon X \longrightarrow X$, which is a compact operator. It is clear that $\nu(S) \geqslant 1$ since

$$|t^*(S(t))| = |z^*(\widetilde{P}(i(\widetilde{S}(\widetilde{P}(i(z))))))| = |z^*(\widetilde{S}(z))| = 1.$$

Also,

$$\begin{aligned}
\|S - T\| &= \|i\widetilde{S}\widetilde{P} - T\| \\
&\leqslant \|i\widetilde{S}\widetilde{P} - i\widetilde{T}\widetilde{P}\| + \|i\widetilde{T}\widetilde{P} - PTP\| + \|PTP - T\| \\
&= \|i\widetilde{S}\widetilde{P} - i\widetilde{T}\widetilde{P}\| + \left\|\frac{PTP}{\nu(T_1)} - PTP\right\| + \|PTP - T\| \\
&\leqslant \|\widetilde{S} - \widetilde{T}\| + \|T\| \cdot \left|\frac{1}{\nu(T_1)} - 1\right| + \|PTP - T\|
\end{aligned}$$

and, since $\|T\| \leqslant \frac{1}{n_{\mathcal{K}}(X)}$, $1-\varepsilon_0 \leqslant \nu(T_1) \leqslant 1$, and $6\eta'(\varepsilon) \leqslant \frac{\varepsilon_0}{3}$, we continue as:

$$\leqslant \frac{\varepsilon_0}{3} + \frac{\varepsilon_0}{(1-\varepsilon_0)n_{\mathcal{K}}(X)} + 6\eta'(\varepsilon) \leqslant \varepsilon_0 \left(\frac{2}{3} + \frac{1}{(1-\varepsilon_0)n_{\mathcal{K}}(X)} \right) < \varepsilon.$$

We finish the proof if we prove that $\nu(S) \leqslant 1$. We consider the following cases:

- Case 1: if $\widetilde{P}$ is an absolute projection and i is the natural inclusion, as a consequence of [26, Lemma 3.3], we get that

$$\nu(S) = \nu(i \circ \widetilde{S} \circ \widetilde{P}) = \nu(\widetilde{S}) = 1.$$

- Case 2: if $n_{\mathcal{K}}(X) = n_{\mathcal{K}}(\widetilde{P}(X)) = 1$, then

$$\nu(S) = \|S\| \leqslant \|\widetilde{S}\| = \nu(\widetilde{S}) = 1.$$

Hence, the result follows in both cases. ■

We will now provide some applications and consequences of the previous lemma. Given a continuous projection $P\colon X \longrightarrow X$, if we set $\widetilde{P}\colon X \longrightarrow \widetilde{P}(X) = P(X) \subset X$ to be the operator P with a restricted codomain and $i\colon P(X) \longrightarrow X$ is the natural inclusion, then, trivially, we have that $P = i \circ \widetilde{P}$ and that $\widetilde{P} \circ i = \mathrm{Id}_{\widetilde{P}(X)}$. This easy observation allows to get the following particular case of Lemma 3.3.1.

Proposition 3.3.2. *Let X be a Banach space with $n_{\mathcal{K}}(X) > 0$. Suppose that there exists a net $\{P_\alpha\}_{\alpha\in\Lambda}$ of norm-one projections on X satisfying that $\{P_\alpha(x)\} \longrightarrow x$ for all $x \in X$ and $\{P_\alpha^*(x^*)\} \longrightarrow x^*$ for all $x^* \in X^*$, and that there exists a function $\eta\colon (0,1) \longrightarrow (0,1)$ such that all the spaces $P_\alpha(X)$ with $\alpha \in \Lambda$ have the BPBp-nu for compact operators with* *the function η. Suppose, moreover, that for each $\alpha \in \Lambda$, at least one of the following conditions is satisfied:*

(i) *the projection P_α is absolute,*

(ii) $n_{\mathcal{K}}(P_\alpha(X)) = n_{\mathcal{K}}(X) = 1$.

Then, the space X has the BPBp-nu for compact operators.

We can now obtain the following consequence of the above result. Given a Banach space X and $m \in \mathbb{N}$, the space $\ell_\infty^m(X)$ represents the ℓ_∞-sum of m copies of X, and we will write $\ell_\infty(X)$ for the ℓ_∞-sum of countably infinitely many copies of X. Similarly, $c_0(X)$ is the c_0-sum of countably infinitely many copies of X. When $X = \mathbb{K}$, we just write ℓ_∞^m for $\ell_\infty^m(\mathbb{K})$.

Corollary 3.3.3. *Let X be a Banach space with $n_{\mathcal{K}}(X) > 0$. Then, the following statements are equivalent:*

(i) *The space $c_0(X)$ has the BPBp-nu for compact operators.*

(ii) *There is a function $\eta\colon (0,1) \longrightarrow (0,1)$ such that all the spaces $\ell_\infty^n(X)$, with $n \in \mathbb{N}$, have the BPBp-nu for compact operators with the function η.*

Moreover, if X is finite-dimensional, these properties hold whenever $c_0(X)$ or $\ell_\infty(X)$ have the BPBp-nu.

Proof. That (ii) implies (i) is a consequence of Proposition 3.3.2 since for every $n \in \mathbb{N}$, the canonical projection on $c_0(X)$ which is the identity on the first n coordinates and 0 elsewhere is an absolute projection whose image is isometrically isomorphic to $\ell_\infty^n(X)$.

(i) implies (ii) is a consequence of [34, Proposition 4.3], as one can easily see $\ell_\infty^n(X)$ as an ℓ_∞-summand of $c_0(X)$. Let us comment that the function η valid for all $\ell_\infty^n(X)$ is the function valid for $c_0(X)$ and this actually follows from the proof of [34, Theorem 4.1] (from which [34, Proposition 4.3] actually follows).

Finally, when X has finite dimension, if $c_0(X)$ or $\ell_\infty(X)$ has the BPBp-nu, then condition (ii) holds by using [34, Theorem 4.1] and the fact that $\ell_\infty^n(X)$ is finite-dimensional and so, every operator from $\ell_\infty^n(X)$ to itself is compact. ■

As stated in Examples 3.2.1, that c_0 and the spaces ℓ_∞^n for $n \in \mathbb{N}$ have the BPBp-nu for compact operators is a consequence of [69, Corollary 4.2] and [87, Proposition 2]. Actually, the fact that all the spaces ℓ_∞^n have the BPBp-nu with the same function η follows from [69, Corollary 4.2] and (the proof of) [34, Theorem 4.1]. However, let us note that we can also get this result as a consequence of our previous corollary.

Corollary 3.3.4. *There is a function $\eta\colon (0,1) \longrightarrow (0,1)$ such that the space c_0 and the spaces ℓ_∞^n with $n \in \mathbb{N}$, have the BPBp-nu for compact operators with the function η.*

Additionally, [34, Proposition 4.3] also implies that whenever $\ell_\infty^n(X)$ has the BPBp-nu for compact operators for some $n \in \mathbb{N}$, then so does X, although the converse remains unknown in general (even for $n = 2$).

Another consequence of Proposition 3.3.2 is the following:

Corollary 3.3.5. *Let X be a Banach space with $n_{\mathcal{K}}(X) > 0$. Suppose that there exists a net $\{P_\alpha\}_{\alpha\in\Lambda}$ of norm-one projections on X such that $\alpha \leq \beta$ implies $P_\alpha(X) \subset P_\beta(X)$, that $\{P_\alpha^*(x^*)\} \longrightarrow x^*$ for all $x^* \in X^*$, and that there exists a function $\eta\colon (0,1) \longrightarrow (0,1)$ such that all the spaces $P_\alpha(X)$ with $\alpha \in \Lambda$ have the BPBp-nu for compact operators with the function η. Suppose, moreover, that for each $\alpha \in \Lambda$, at least one of the following conditions is satisfied:*

(i) *the projection* P_α *is absolute,*

(ii) $n_{\mathcal{K}}(P_\alpha(X)) = n_{\mathcal{K}}(X) = 1$.

Then, the space X *has the BPBp-nu for compact operators.*

Proof. Observe that in order to apply Proposition 3.3.2 we only need that $\{P_\alpha(x)\} \longrightarrow x$ in norm for all $x \in X$. But this is proven in [39, Corollary 2.4], so we are done. ■

The previous result can be used to show that all the isometric preduals of ℓ_1 have the BPBp-nu for compact operators.

Corollary 3.3.6. *Let* X *be a Banach space such that* X^* *is isometrically isomorphic to* ℓ_1*. Then* X *has the BPBp-nu for compact operators.*

Proof. By a deep result due to Gasparis [63], it is shown in the proof of [39, Theorem 3.6] that there exists a sequence of norm-one projections $P_n\colon X \longrightarrow X$ satisfiying that $P_{n+1}P_n = P_n$ (and so, $P_n(X) \subset P_{n+1}(X)$), that $P_n(X)$ is isometrically isomorphic to ℓ_∞^n, and also that $P_n^*(x^*) \longrightarrow x^*$ for all $x^* \in X^*$ (this claim holds since the sets Y_n defined on that proof satisfy that their union is dense in $X^* = \ell_1$).

Next, as $P_n(X)$ is isometrically isomorphic to ℓ_∞^n, on the one hand we have that all the spaces $P_n(X)$ have the BPBp-nu for compact operators with the same function η as a consequence of Corollary 3.3.4. On the other hand, $n(X) = n(P_n(X)) = 1$ for all $n \in \mathbb{N}$ (see [82], for instance) so, in particular, $n_{\mathcal{K}}(X) = n_{\mathcal{K}}(P_n(X)) = 1$ for all $n \in \mathbb{N}$. Finally, by Corollary 3.3.5, we get the desired result. ■

3.4 $C_0(L)$ spaces

The aim of this section is to provide some strong approximation property of $C_0(L)$ and its dual (see Theorem 3.4.5) which allow to use Lemma 3.3.1 (or Proposition 3.3.2) to give a proof of Theorem 3.4.7. Before we get into the results, we will attempt to visualize some of the key ideas that will be used in the proofs.

In order to prove Theorem 3.4.7, given $X = C_0(L)$ for some locally compact Hausdorff space L, $\varepsilon > 0$, $f_1, \ldots, f_\ell \in B_X$ and $\mu_1, \ldots, \mu_n \in B_{X^*}$, we will find in Theorem 3.4.5 a projection $P \in S_{\mathcal{L}(X,X)}$ satisfying $\|P(f_j) - f_j\| < \varepsilon$, for $j = 1, \ldots, \ell$, and $\|P^*(\mu_j) - \mu_j\| < \varepsilon$, for $j = 1, \ldots, n$, and such that $P(C_0(L))$ is isometrically isomorphic to some ℓ_∞^p (for some $p \in \mathbb{N}$). Now, since ℓ_∞^p has the BPBp-nu for compact operators, and since $n(C_0(L)) = n(\ell_\infty^p) = 1$, we have that $n_{\mathcal{K}}(C_0(L)) = n_{\mathcal{K}}(\ell_\infty^p) = 1$, and so, we can now apply the following simplified version of Lemma 3.3.1:

Lemma 3.4.1. *Let X be a Banach space with $n_{\mathcal{K}}(X) = 1$. Suppose that there is a mapping $\eta\colon (0,1) \longrightarrow (0,1)$ such that given any $\varepsilon > 0$, $f_1, \ldots, f_\ell \in B_X$ and $\mu_1, \ldots, \mu_n \in B_{X^*}$, we can find a projection $P \in S_{\mathcal{L}(X,X)}$ s.t.:*

(i) $\|P(f_j) - f_j\| < \varepsilon$, *for* $j = 1, \ldots, \ell$.

(ii) $\|P^*(\mu_j) - \mu_j\| < \varepsilon$, *for* $j = 1, \ldots, n$.

(iii) $P(X)$ *has BPBp-nu-K with the mapping* η.

(iv) $n_{\mathcal{K}}(P(X)) = 1$.

Then X has the BPBp-nu for compact operators.

In order to find such a projection P, first we need to cover L with some conveniently chosen sets where functions and measures have small variation. Then, we will find a partition of the unity subordinated to those sets, and define P in terms of those functions (recall that a *partition of the unity* of a topological space A is a set of continuous functions $\{\phi_\gamma\}_{\gamma\in\Gamma}$ from A to $[0,1]$ such that $\sum_{\gamma\in\Gamma}\phi_\gamma(x) = 1$ for all $x \in A$). The following diagrams may help to visualize the upcoming proofs.

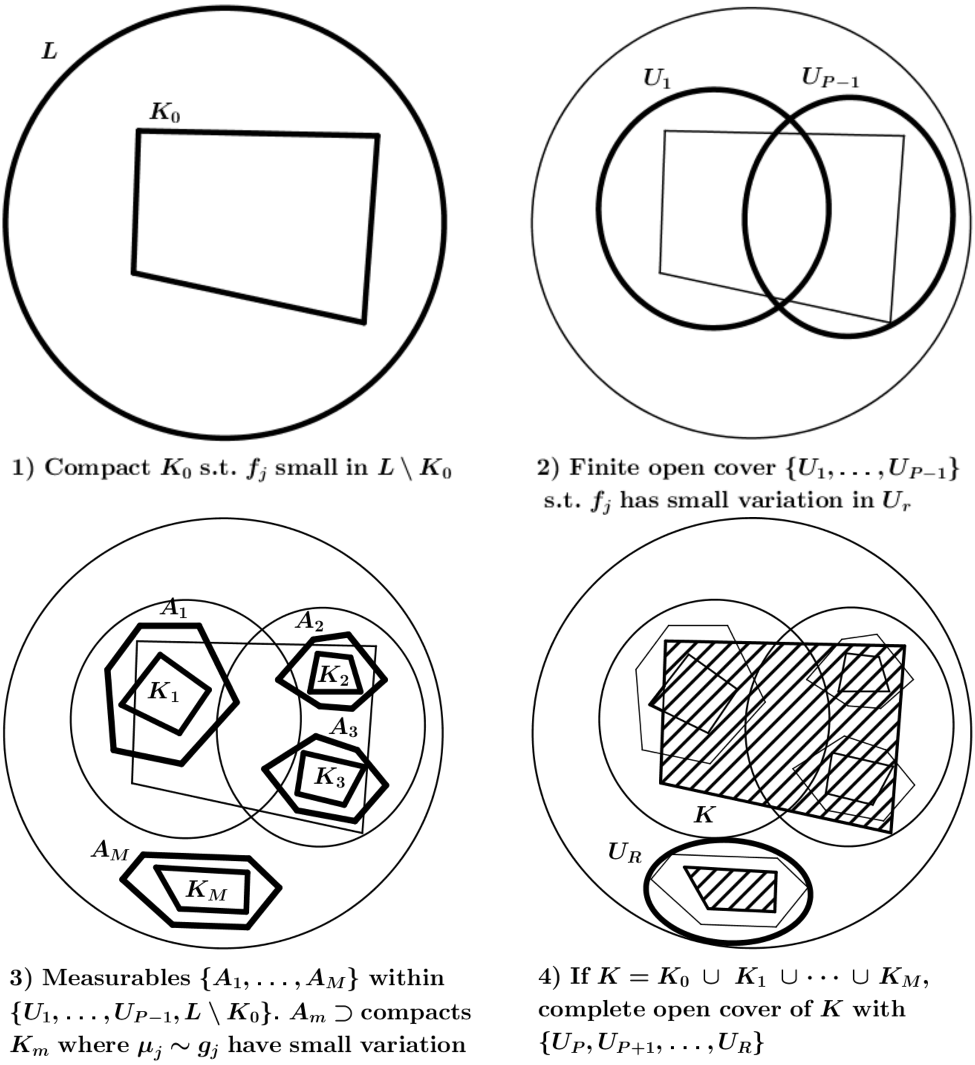


Figure 3.1. **Step 1**: Finding the sets $\{U_r\}_{r=1}^{R}$, $\{K_m\}_{m=1}^{M}$, and K.

Now we need to refine the open cover conveniently.

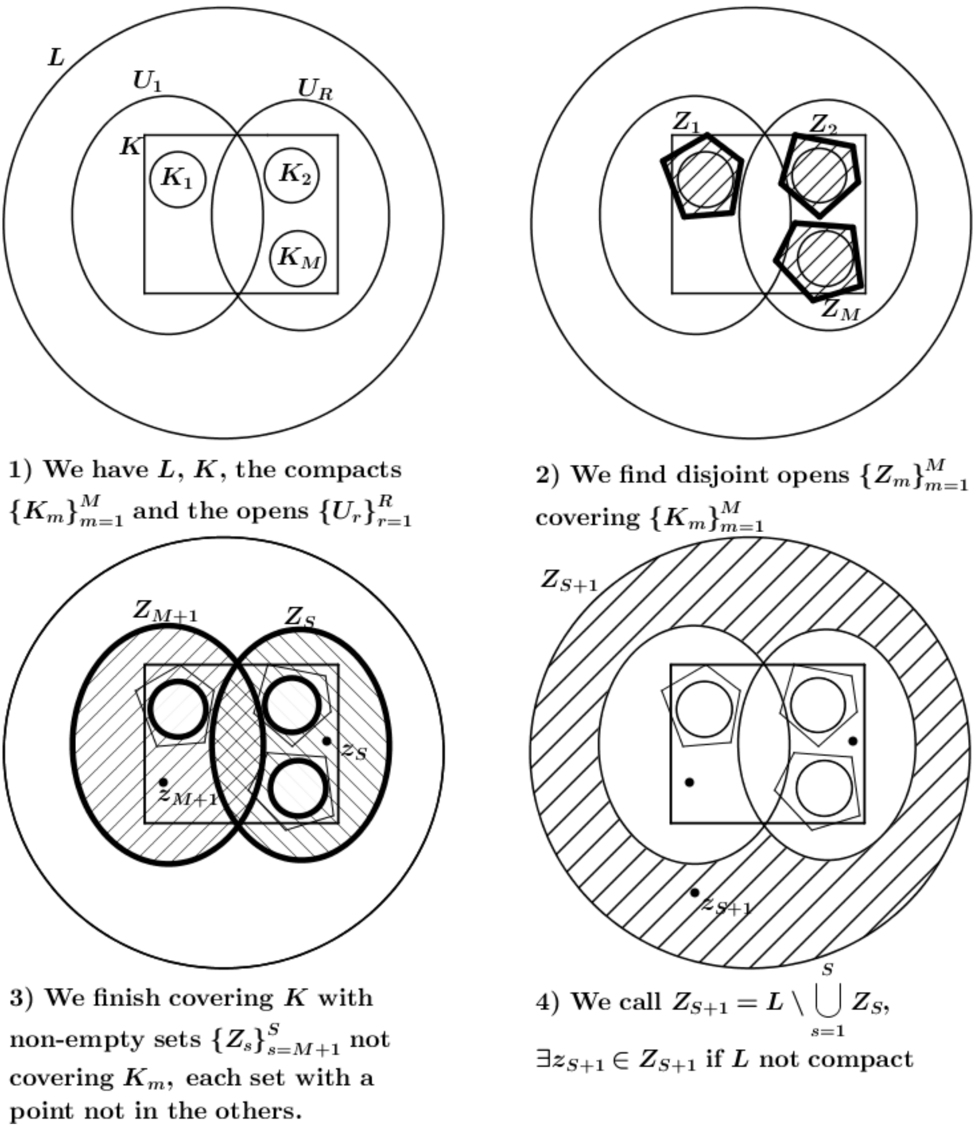


Figure 3.2. **Step 2**: Finding the sets $\{Z_s\}_{s=1}^{S+1}$.

Finally, we use Urysohn's lemma to find a partition of the unity $\{\varphi_s\}_{s=1}^{S+1}$ such that $\varphi_s(x) = 0$ for all $x \notin Z_s$ $(1 \leqslant s \leqslant S)$, $\varphi_m \equiv 1$ on K_m

$(1 \leqslant m \leqslant M)$, $\varphi_s(z_s) = 1$ $(M+1 \leqslant s \leqslant S+1)$, and $\varphi_1 + \ldots + \varphi_S \equiv 1$ on K. We will define P in terms of those functions (see the proof of Theorem 3.4.5).

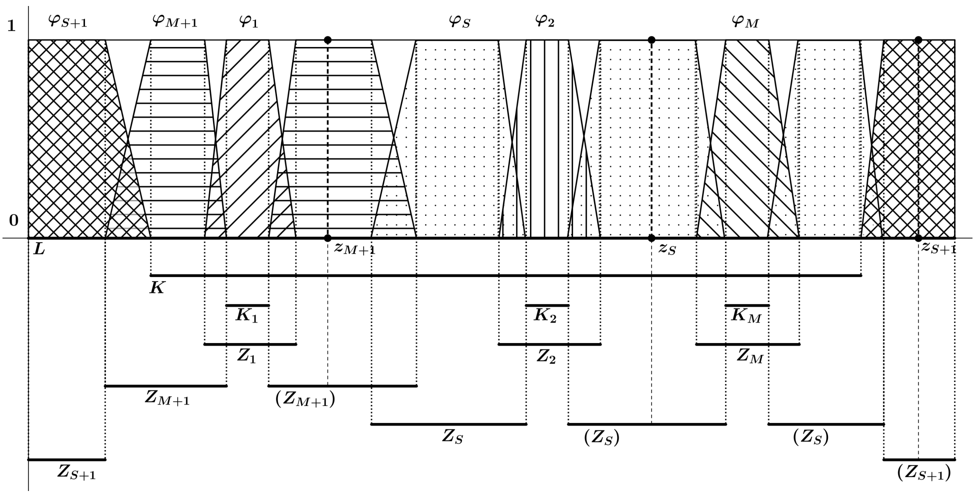


Figure 3.3. **Step 3**: Finding the partition of the unity $\{\varphi_s\}_{s=1}^{S+1}$. Each function is shaded with a unique texture.

We will now present the formal statements and proofs of all the results needed in this section.

Lemma 3.4.2. *Let L be a locally compact space, let $\{K_1, \ldots, K_M\}$ be a family of pairwise disjoint non-empty compact subsets of L, and let $K \subset L$ be a compact set with $\bigcup_{m=1}^{M} K_m \subset K$. If $\{U_1, \ldots, U_R\}$ is a family of relatively compact open subsets of L covering K such that for each m there is an $r(m)$ with $K_m \subset U_{r(m)}$, $m = 1, \ldots, M$, then there exists an open refinement $\{Z_1, \ldots, Z_S\}$, $M \leqslant S \leqslant R + M$ with $Z_1, \ldots, Z_M$ pairwise disjoint, satisfying:*

(1) *For $m = 1, \ldots, M$, $K_m \subset Z_m$, and $K_m \cap Z_s = \varnothing$ for all $s \in \{1, \ldots, S\} \backslash \{m\}$.*

(2) *For all* $s_0 > M$, *there exists* $z_{s_0} \in Z_{s_0} \backslash \left(\bigcup_{s \neq s_0} Z_s \right)$.

Proof. As $\{K_1, \ldots, K_M\}$ are pairwise disjoint, there exist $\{V_1, \ldots V_M\}$ pairwise disjoint open subsets of L with $K_m \subset V_m \subset U_{r(m)}$, $m = 1, \ldots, M$.

The family $\left\{V_1, \ldots, V_M, U_1 \backslash \left(\bigcup_{m=1}^M K_m \right), \ldots, U_R \backslash \left(\bigcup_{m=1}^M K_m \right) \right\}$ is another cover of K by open subsets of L subordinated to $\{U_r\}_{r=1}^R$. We define the sets $Z_m := V_m$ for $m = 1, \ldots, M$, and $W_r := U_r \backslash \left(\bigcup_{m=1}^M K_m \right)$ for $r = 1, \ldots, R$.

If $W_1 \subset V_1 \cup \ldots \cup V_M$, then $\{V_1, \ldots, V_M, W_2, \ldots, W_R\}$ is again a cover of K. If that happens again and again until W_R, we have that $\{Z_1, \ldots, Z_M\}$ is the cover we were looking for. In other case, let $r_1 \geqslant 1$ be the first natural number such that there exists $w_{r_1} \in W_{r_1} \backslash \left(\bigcup_{m=1}^M V_m \right)$, and denote $Z_{M+1} := W_{r_1}$. The family $\{V_1, \ldots, V_M, W_{r_1}, W_{r_1+1}, \ldots, W_R\}$ is a cover of K by open sets, and then, so is the family

$$\left\{V_1, \ldots, V_M, W_{r_1}, W_{r_1+1} \backslash \{w_{r_1}\}, \ldots, W_R \backslash \{w_{r_1}\} \right\}.$$

Consider now $r_2 > r_1$ the first natural number such that there exists $w_{r_2} \in W_{r_2} \backslash \{w_{r_1}\}$ and $w_{r_2} \notin V_1 \cup \ldots \cup V_M \cup W_{r_1}$. Let $Z_{M+2} := W_{r_2} \backslash \{w_{r_1}\}$ and proceed as before. In at most R steps, we get $\{Z_1, \ldots, Z_S\}$, $M \leqslant S \leqslant R + M$, such that

- $K_m \subset Z_m$ for $m = 1, \ldots M$.
- $\left(\bigcup_{m=1}^M K_m \right) \cap Z_s = \varnothing$ for $s > M$.
- For all $s_0 > M$, there exists $w_{r_{s_0 - M}} \in Z_{s_0} \backslash \left(\bigcup_{s \neq s_0} Z_s \right)$. ■

We next provide a result showing the existence of certain partitions of the unity. We separate the non-compact case (Lemma 3.4.3) and the compact case (Lemma 3.4.4) for the sake of clarity. We start with the non-compact case.

Lemma 3.4.3. *Let L be a non-compact locally compact space. Let $K \subset L$ be a compact set and $\{K_1, \ldots, K_M\}$ a family of pairwise disjoint non-empty compact subsets of K. Given a family $\{U_1, \ldots, U_R\}$ of relatively compact open subsets of L that cover K, let $\{Z_1, \ldots, Z_S\}$ be a family of open subsets of L covering K such that they satisfy the thesis of Lemma 3.4.2, and denote by Z_{S+1} the set $L \setminus \left(\bigcup_{s=1}^{S} Z_s \right)$. Then, there exists a partition of the unity subordinated to $\{Z_s\}_{s=1}^{S+1}$, $\{\varphi_s\}_{s=1}^{S+1}$, such that:*

(1) $\{\varphi_1, \ldots, \varphi_M\}$ *have disjoint support.*

(2) $\varphi_m(K_m) \equiv 1$, *for $m = 1, \ldots, M$.*

(3) *For all $M < s \leqslant S+1$, there exists $z_s \in Z_s$ such that $\varphi_s(z_s) = 1$.*

(4) *For $s = 1, \ldots, S$, $\varphi_s(x) = 0$ for all $x \notin Z_s$.*

(5) $(\varphi_1 + \cdots + \varphi_S)(x) = 1$, *for all $x \in K$.*

Proof. By hypothesis, there exists some $z_{S+1} \in L \setminus \left(\bigcup_{s=1}^{S} Z_s \right)$, since for all s, $\overline{Z_s} \subset \overline{\bigcup_{r=1}^{R} U_r}$, which is a compact set. Now, we follow the argument from the proof of [106, Theorem 2.13], but adapted to our case.

As $K \subset Z_1 \cup \ldots \cup Z_S$, for each $x \in K$, there exists a neighbourhood of x, Y_x, with compact closure $\overline{Y_x} \subset Z_s$ for some s. Consider $x_1, \ldots, x_p$ such that $K \subset Y_{x_1} \cup \ldots \cup Y_{x_p}$. For each $1 \leqslant s \leqslant S$, let H_s be the union of those $\overline{Y_{x_j}}$ which lie in Z_s, and if $M < s_0 \leqslant S$, we take $H_{s_0} \cup$

$\{z_{s_0}\}$, with $z_{s_0} \in Z_{s_0} \backslash \left(\bigcup_{s \neq s_0} Z_s \right)$. Note that the sets $H_1, \ldots, H_M$ and $H_{M+1} \cup \{z_{M+1}\}, \ldots, H_S \cup \{z_S\}$ are non-empty. By Urysohn's Lemma, there are continuous functions $g_s \colon L \longrightarrow [0,1]$ such that $g_s(H_s) \equiv 1$ and $g_s\big|_{L \backslash Z_s} \equiv 0$, for $1 \leqslant s \leqslant M$, and $g_{s_0}(H_{s_0} \cup \{z_0\}) \equiv 1$ and $g_{s_0}\big|_{L \backslash Z_{s_0}} \equiv 0$ for $M < s_0 \leqslant S$. Define

$$
\begin{aligned}
\varphi_1 &:= g_1, \\
\varphi_2 &:= (1 - g_1) g_2, \\
&\ \vdots \\
\varphi_S &:= (1 - g_1)(1 - g_2) \cdots (1 - g_{S-1}) g_S
\end{aligned}
$$

Clearly, $\varphi_s(x) = 0$ for all $x \notin Z_s$ for all $s = 1, \ldots, S$, and we have that

$$\varphi_1 + \cdots + \varphi_S = 1 - (1 - g_1) \cdots (1 - g_S).$$

Since $K \subset H_1 \cup \ldots \cup H_S$, for each $x \in K$, there exists $s = s(x)$ with $g_s(x) = 1$, and also, for all $s = 1, \ldots, M$, we have that

$$\{x \in L \colon \varphi_s(x) \neq 0\} \subset \{x \in L \colon g_s(x) \neq 0\} \subset Z_s.$$

Therefore, the functions $\{\varphi_1, \ldots, \varphi_M\}$ have disjoint support, and $\varphi_1 + \cdots + \varphi_S \equiv 1$ on K.

We define $\varphi_{S+1} := 1 - (\varphi_1 + \cdots + \varphi_S) = (1 - g_1) \cdots (1 - g_S)$. Moreover, $K_m \subset Z_m$ for $m = 1, \ldots, M$, and $K_m \cap Z_s = \varnothing$ for $m \neq s$, $m = 1, \ldots, M$, $s = 1, \ldots, S$. Hence,

$$\varphi_m(x) = \sum_{s=1}^{S} \varphi_s(x) = 1, \quad \forall x \in K_m,\ m = 1, \ldots, M.$$

On the other hand, if $M < s_0 \leqslant S$, let $z_{s_0} \in Z_{s_0} \setminus \left(\bigcup_{s \neq s_0} Z_s \right)$. We have that

$$\varphi_{s_0}(z_{s_0}) = \sum_{s=1}^{S} \varphi_s(z_{s_0}) = 1,$$

and

$$z_{S+1} \notin \bigcup_{s=1}^{S} Z_s, \text{ thus } \varphi_{S+1}(z_{S+1}) = 1. \qquad \blacksquare$$

The next result is the version of the previous lemma for compact topological spaces.

Lemma 3.4.4. *Let L be a compact space. Let $\{K_1, \ldots, K_M\}$ be a family of pairwise disjoint non-empty compact subsets of L. Given a family $\{U_1, \ldots, U_R\}$ of relatively compact open subsets of L that cover it, let $\{Z_1, \ldots, Z_S\}$ be a family of open subsets of L covering K such that they satisfy the thesis of Lemma 3.4.2. Then, there exists a partition of the unity subordinated to $\{Z_s\}_{s=1}^{S}$, $\{\varphi_s\}_{s=1}^{S}$, such that:*

(1) $\{\varphi_1, \ldots, \varphi_M\}$ *have disjoint support.*

(2) $\varphi_m(K_m) \equiv 1$, *for* $m = 1, \ldots, M$.

(3) *For all* $M < s \leqslant S$, *there exists* $z_s \in Z_s$ *such that* $\varphi_s(z_s) = 1$.

(4) *For* $s = 1, \ldots, S$, $\varphi_s(x) = 0$ *for all* $x \notin Z_s$.

(5) $(\varphi_1 + \cdots + \varphi_S)(x) = 1$, *for all* $x \in K$.

Proof. We can follow the proof of Lemma 3.4.3 taking $K = L$ and adapting the steps from that proof, keeping in mind that now $Z_{S+1} = \varnothing$ (and hence there is not such a point z_{S+1}), and that the mapping φ_{S+1} is identically 0, and hence, it can be omitted. $\blacksquare$

The following result provides the promised approximation property of $C_0(L)$ and its dual.

Theorem 3.4.5. *Let L be a locally compact space. Given $\{f_1, \ldots, f_\ell\} \subset C_0(L)$ such that $\|f_j\| \leqslant 1$ for $j = 1, \ldots, \ell$, and given $\{\mu_1, \ldots, \mu_n\} \subset C_0(L)^*$ with $\|\mu_j\| \leqslant 1$ for $j = 1, \ldots, n$, for each $\varepsilon > 0$ there exists a norm one projection $P\colon C_0(L) \longrightarrow C_0(L)$ satisfying:*

(1) $\|P^*(\mu_j) - \mu_j\| < \varepsilon$, *for* $j = 1, \ldots, n$,

(2) $\|P(f_j) - f_j\| < \varepsilon$, *for* $j = 1, \ldots, \ell$,

(3) $P(C_0(L))$ *is isometrically isomorphic to* ℓ_∞^p *for some* $p \in \mathbb{N}$.

Let us comment that this result extends [39, Lemma 3.4] (which, actually, was itself an extension of [6, Proposition 3.2] and [77, Proposition 3.2]). The main difference is that here we are able to deal with an arbitrary number of functions of $C_0(L)$ in (2), while in that lemma only one function is controlled, and besides, this was done with the help of an inclusion operator which is not the canonical one. However, this difference is crucial in order to apply Lemma 3.3.1 (or its consequence Proposition 3.3.2).

The following observation on the theorem is worth mentioning.

Remark 3.4.6. Let us observe that by just conveniently ordering the obtained projections in Theorem 3.4.5, we actually get the following: *given a Hausdorff locally compact topological space L, there is a net $\{P_\alpha\}_{\alpha\in\Lambda}$ of norm-one projections on $C_0(L)$, converging in the strong operator topology to the identity operator, such that $\{P_\alpha^*\}_{\alpha\in\Lambda}$ converges in the strong operator topology to the identity on $C_0(L)^*$, and such that $P_\alpha(C_0(L))$ is isometrically isomorphic to a finite-dimensional ℓ_∞ space.*

Proof of Theorem 3.4.5. We will assume first that L is not compact. Since $f_j \in C_0(L)$, $j = 1, \ldots, \ell$, there exists a compact set $K_0 \subset L$ such that

$$\sup_{j=1,\ldots,\ell} \{|f_j(x)| \colon x \in L \backslash K_0\} < \frac{\varepsilon}{4}.$$

For each $x \in K_0$, there exists a relatively compact open subset U_x of L containing x and such that

$$|f_j(x) - f_j(y)| < \frac{\varepsilon}{4} \quad \text{ for } y \in U_x \text{ and } j = 1, \ldots, \ell.$$

Therefore, $\{U_x\}_{x \in K_0}$ is a cover of K_0, and so, there exist a finite subcover $\{U_1, \ldots, U_{R-1}\}$ such that $K_0 \subseteq U_1 \cup \ldots \cup U_{R-1}$, and if $x, y \in U_r$ for some r, then $|f_j(x) - f_j(y)| < \frac{\varepsilon}{2}$, for $j = 1, \ldots, \ell$.

We define $\mu := \sum_{j=1}^{n} |\mu_j| \in C_0(L)^*$. Since for each $j \in \{1, \ldots, n\}$ μ_j is absolutely continuous with respect to μ, by the Radon-Nikodým Theorem, there exists $g_j \in L_1(\mu)$ such that $\mu_j = g_j \mu$, that is,

$$\mu_j(f) := \int_L f \mathrm{d}\mu_j = \int_L f(x) g_j(x) \mathrm{d}\mu(x) \quad \text{ for all } f \in C_0(L).$$

Since the set of simple functions is dense in $L_1(\mu)$, we may choose a set of simple functions $\{s_j \colon j = 1, \ldots, n\}$ such that $\|g_j - s_j\|_1 < \frac{\varepsilon}{4}$ for $j = 1, \ldots, n$.

Next, we consider a family $\{A_m\}_{m=1}^{M}$ of pairwise disjoint measurable sets with $\mu(A_m) > 0$ for all m, such that each A_m is contained in one of the elements of the following cover of L: $\{U_1, \ldots, U_{R-1}, L \backslash K_0\}$, and also $\{\alpha_{m,j} \colon m = 1, \ldots, M,\ j = 1, \ldots, n\}$ such that $s_j = \sum_{m=1}^{M} \alpha_{m,j} \chi_{A_m}$. This cover satisfies that if $x, y \in L \backslash K_0$, or if $x, y \in U_r$, then $|f_j(x) - f_j(y)| < \frac{\varepsilon}{2}$ for all $j = 1, \ldots, \ell$ and all $r = 1, \ldots, R-1$. Let $C > \max\{|\alpha_{m,j}| \colon m = 1, \ldots, M,\ j = 1, \ldots, n\}$.

Since μ is regular, for each $1 \leqslant m \leqslant M$, we can find a compact set $K_m \subset A_m$ such that $\mu(A_m \backslash K_m) < \frac{\varepsilon}{4MC}$ and $\mu(K_m) > 0$ for all $m = 1, \ldots, M$. Let $K = K_0 \cup K_1 \cup \ldots \cup K_M$. As $K \backslash \left(\bigcup_{r=1}^{R-1} U_r \right)$ is a compact subset of L, we can cover it with finitely many relatively compact open subsets of $L \backslash K_0$ that we will denote $U_R, U_{R+1}, \ldots, U_P$. If we now apply Lemmas 3.4.2 and 3.4.3 to the family $\{U_1, \ldots, U_P\}$ and the compacts $\{K_1, \ldots, K_M\}$ and K, we obtain a refinement of relatively compact open subsets of L, $\{Z_1, \ldots, Z_S\}$ with $K_m \subset Z_m$ for $m = 1, \ldots, M$ and $\{Z_1, \ldots Z_M\}$ pairwise disjoint, and defining Z_{S+1} to be the set $L \backslash \left(\bigcup_{s=1}^{S} Z_s \right)$, we also have a partition of the unity subordinated to $\{Z_s\}_{s=1}^{S}$, $\{\varphi_s\}_{s=1}^{S+1}$, such that:

(i) $\{\varphi_1, \ldots, \varphi_M\}$ have disjoint support.

(ii) $\varphi_m(K_m) \equiv 1$ for $m = 1, \ldots, M$.

(iii) For all $M < s \leqslant S+1$, there exists $z_s \in Z_s$ such that $\varphi_s(z_s) = 1$.

(iv) For $s = 1, \ldots, S$, $\varphi_s(x) = 0$ for all $x \notin Z_s$.

(v) $(\varphi_1 + \ldots + \varphi_S)(K) \equiv 1$.

Now, we define $P\colon C_0(L) \longrightarrow C_0(L)$ by

$$P(f) := \sum_{m=1}^{M} \frac{1}{\mu(K_m)} \left(\int_{K_m} f \,\mathrm{d}\mu \right) \varphi_m + \sum_{s=M+1}^{S+1} f(z_s)\varphi_s, \quad \text{for all } f \in C_0(L).$$

Let us first check that (2) holds, that is, that $\|P(f_j) - f_j\| < \varepsilon$ for all $j = 1, \ldots, \ell$. Let $x \in L$. We will distinguish two cases:

- Case 1: if $x \in \bigcup_{m=1}^{M} Z_m$, then there exists exactly one m_0 such that $x \in Z_{m_0}$. Then, for each $j = 1, \dots, \ell$, we have:

$$\begin{aligned}
&|P(f_j)(x) - f_j(x)| \\
&\qquad = \left| P(f_j)(x) - \sum_{m=1}^{M} f_j(x)\varphi_m(x) - \sum_{s=M+1}^{S+1} f_j(x)\varphi_s(x) \right| \\
&\qquad \leqslant \underbrace{\left| \frac{1}{\mu(K_{m_0})} \left(\int_{K_{m_0}} f_j(y)\, \mathrm{d}\mu(y) \right) - f_j(x) \right| \varphi_{m_0}(x)}_{\text{(I)}} + \\
&\qquad\quad + \underbrace{\sum_{s=M+1}^{S+1} |f_j(x) - f_j(z_s)| \varphi_s(x)}_{\text{(II)}} .
\end{aligned}$$

For (I), we have

$$\begin{aligned}
\text{(I)} &= \left| \frac{1}{\mu(K_{m_0})} \left(\int_{K_{m_0}} (f_j(y) - f_j(x))\, \mathrm{d}\mu(y) \right) \right| \varphi_{m_0}(x) \\
&\leqslant \frac{1}{\mu(K_{m_0})} \int_{K_{m_0}} \frac{\varepsilon}{2}\, \mathrm{d}\mu(y) = \frac{\varepsilon}{2}.
\end{aligned}$$

Now, for (II), let $s \in \{M+1, \dots, S+1\}$. Note that if $x \notin Z_s$, then $\varphi_s(x) = 0$, and if $x \in Z_s$, we have that $|f_j(x) - f_j(z_s)| < \frac{\varepsilon}{2}$ and $\sum_{s=M+1}^{S+1} \varphi_s(x) \leqslant 1$, and so, (II) $< \frac{\varepsilon}{2}$. Therefore, $|P(f_j)(x) - f_j(x)| < \varepsilon$ for all $x \in \bigcup_{m=1}^{M} Z_m$, for all $j = 1, \dots, \ell$.

- Case 2: if $x \notin \bigcup_{m=1}^{M} Z_m$, then for each $j = 1, \dots, \ell$, we have

$$|P(f_j)(x) - f_j(x)| = \left| \sum_{s=M+1}^{S+1} (f_j(x) - f_j(z_s))\varphi_s(x) \right| < \frac{\varepsilon}{2}$$

as in item (II) of the previous case.

Summarizing, we get $\|P(f_j) - f_j\| < \varepsilon$ for $j = 1, \ldots, \ell$, getting thus (2).

Now we check (1), that is, that $\|P^*(\mu_j) - \mu_j\| < \varepsilon$ for all $j = 1, \ldots, n$. Indeed, first observe that if ν is a regular Borel (real or complex) measure on L, its associated $x_\nu^* \in C_0(L)^*$ is defined as

$$x_\nu^*(f) := \int_L f(x)\,\mathrm{d}\nu(x), \quad \forall f \in C_0(L),$$

and we identify $x_\nu^* \equiv \nu$. In our case, we have that

$$\begin{aligned} P^*(\nu)(f) &= \int_L P(f)(x)\,\mathrm{d}\nu(x) \\ &= \int_L \left(\sum_{m=1}^{M} \frac{1}{\mu(K_m)} \left(\int_{K_m} f\,\mathrm{d}\mu \right) \varphi_m(x) \right) \mathrm{d}\nu(x) + \\ &\quad + \int_L \left(\sum_{s=M+1}^{S+1} f(z_s)\varphi_s(x) \right) \mathrm{d}\nu(x) \\ &= \sum_{m=1}^{M} \frac{1}{\mu(K_m)} \left(\int_{K_m} f\,\mathrm{d}\mu \right) \int_L \varphi_m(x)\,\mathrm{d}\nu(x) + \\ &\quad + \sum_{s=M+1}^{S+1} f(z_s) \int_L \varphi_s(x)\,\mathrm{d}\nu(x). \end{aligned}$$

In particular, if $\mathrm{supp}(\nu) \subset \bigcup_{m=1}^{M} K_m$, then by Lemma 3.4.2.(1)

$$\sum_{s=M+1}^{S+1} f(z_s) \int_L \varphi_s(x)\,\mathrm{d}\nu(x) \equiv 0, \quad \forall f \in C_0(L).$$

Let now $\nu_j := t_j\mu$, where $t_j := \sum_{m=1}^{M} \alpha_{m,j}\chi_{K_m}$, for all $j = 1, \ldots, n$, that is,

$$\nu_j(f) = \int_L f(x) \left(\sum_{m=1}^{M} \alpha_{m,j}\chi_{K_m}(x) \right) \mathrm{d}\mu(x), \quad \forall f \in C_0(L).$$

It holds that $P^*(\nu_j) = \nu_j$ for $j = 1, \ldots, n$. Indeed, as $\operatorname{supp}(\nu_j) \subset \bigcup_{m=1}^{M} K_m$, we have

$$\begin{aligned} P^*(\nu_j)(f) &= \sum_{m=1}^{M} \frac{1}{\mu(K_m)} \left(\int_{K_m} f \,\mathrm{d}\mu \right) \int_L \varphi_m(x) \left(\sum_{l=1}^{M} \alpha_{l,j}\chi_{K_l}(x) \right) \mathrm{d}\mu(x) \\ &= \sum_{m=1}^{M} \frac{1}{\mu(K_m)} \left(\int_{K_m} f \,\mathrm{d}\mu \right) \underbrace{\int_L \alpha_{m,j}\chi_{K_m}(x) \,\mathrm{d}\mu(x)}_{\alpha_{m,j}\mu(K_m)} \\ &= \int_L f(x) \left(\sum_{m=1}^{M} \alpha_{m,j}\chi_{K_m}(x) \right) \mathrm{d}\mu(x) = \nu_j(f) \end{aligned}$$

for all $f \in C_0(L)$ and all $j = 1, \ldots, n$.

Now, we know that $\|P^*\| = \|P\| \leqslant 1$ and, since $P(\varphi_j) = \varphi_j$ for $j = 1, \ldots, n$, we get that $\|P^*\| = 1$. Therefore, since $P^*(\nu_j) = \nu_j$, we get

$$\begin{aligned} \|P^*(\mu_j) - \mu_j\| &\leqslant \|P^*(\mu_j - \nu_j)\| + \|\nu_j - \mu_j\| \\ &\leqslant \|P^*\| \cdot \|\mu_j - \nu_j\| + \|\mu_j - \nu_j\| \leqslant 2\|\mu_j - \nu_j\|. \end{aligned}$$

But we have

$$\begin{aligned} \|\mu_j - \nu_j\| &= \|g_j\mu - t_j\mu\| \leqslant \|g_j\mu - s_j\mu\| + \|s_j\mu - t_j\mu\| \\ &= \|g_j - s_j\|_1 + \|s_j - t_j\|_1 < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}, \end{aligned}$$

since

$$\|s_j - t_j\|_1 = \int_L \left| \sum_{m=1}^{M} \alpha_{m,j}\chi_{A_m} - \sum_{m=1}^{M} \alpha_{m,j}\chi_{K_m} \right| \mathrm{d}\mu$$
$$\leqslant \sum_{m=1}^{M} \underbrace{|\alpha_{m,j}|}_{\leqslant C} \mu(A_m \backslash K_m) < \frac{MC\varepsilon}{4MC} = \frac{\varepsilon}{4},$$

for all $j = 1, \ldots, n$. Hence,

$$\|P^*(\mu_j) - \mu_j\| \leqslant 2\|\mu_j - \nu_j\| < 2\frac{\varepsilon}{2} = \varepsilon \quad \text{for } j = 1, \ldots, n.$$

Let us finish the proof by checking (3). As $\mu(K_m) > 0$, we have $K_m \neq \varnothing$, $m = 1, \ldots, M$. Hence, we have that $z_s \in Z_s$ for $s = 1, \ldots, S+1$ and that $z_{s_0} \notin \bigcup_{s \neq s_0} Z_s$ for all $s_0 = 1, \ldots, S+1$. By the definition of P, we have that $P(C_0(L)) = \operatorname{span}\{\varphi_s \colon s = 1, \ldots, S+1\}$ and we will be done by proving the following equality:

$$\left\|a_1\varphi_1 + \cdots + a_{S+1}\varphi_{S+1}\right\|_\infty = \max\{|a_1|, \ldots, |a_{S+1}|\} = \|a\|_\infty$$

for every $a = (a_1, \ldots, a_{S+1})$. Indeed, for $x \in L$

$$\left|a_1\varphi_1(x) + \cdots + a_{S+1}\varphi_{S+1}(x)\right| \leqslant \|a\|_\infty \sum_{s=1}^{S+1} \varphi_s(x) = \|a\|_\infty.$$

But for each s,

$$\left|a_1\varphi_1(z_s) + \cdots + a_{S+1}\varphi_{S+1}(z_s)\right| = |a_s|,$$

and then,

$$\left\|a_1\varphi_1 + \ldots + a_{S+1}\varphi_{S+1}\right\|_\infty \geqslant \|a\|_\infty.$$

Hence, the mapping $\rho\colon \ell_\infty^{S+1} \longrightarrow C_0(L)$ given by

$$(a_1, \ldots, a_{S+1}) \longmapsto a_1\varphi_1 + \ldots + a_{S+1}\varphi_{S+1}$$

is an isometry, and hence, $P(C_0(L))$ is isometrically isomorphic to ℓ_∞^{S+1}.

Now, for the case when L is compact, by taking $K_0 = L$ and using Lemma 3.4.4 instead of Lemma 3.4.3, a similar proof is valid, except that now all the elements depending on $S+1$ will vanish in the proof: here we get $Z_{S+1} = \varnothing$ (hence z_{S+1} does not exist), $\varphi_{S+1} \equiv 0$ (and hence it can be omitted), and so, the vector a will only have S components; therefore $P(C_0(L))$ is isometrically isomorphic to ℓ_∞^S in this case. ■

We are ready now to state and prove the main result of this section and chapter.

Theorem 3.4.7. *If L is a locally compact Hausdorff space, then $C_0(L)$ has the BPBp-nu for compact operators.*

Proof of Theorem 3.4.7. Let $f_1, \ldots, f_\ell \in B_{C_0(L)}$, $\mu_1, \ldots, \mu_n \in B_{(C_0(L))^*}$ and $\varepsilon > 0$ be given. Let $P\colon C_0(L) \longrightarrow C_0(L)$ be the same projection from Theorem 3.4.5, which satisfies that $P(C_0(L))$ is isometrically isomorphic to ℓ_∞^p for some $p \in \mathbb{N}$. Let $\widetilde{P}\colon X \longrightarrow \widetilde{P}(C_0(L))$ be the operator such that $\widetilde{P}(f) = P(f)$ for all $f \in C_0(L)$, and let $i\colon \widetilde{P}(C_0(L)) \longrightarrow C_0(L)$ be the natural inclusion. Let η be the mapping with which all ℓ_∞^n spaces has the BPBp-nu for compact operators (see Corollary 3.3.4). Since $n(C_0(L)) = 1$ and $n(\ell_p^n) = 1$ for all $n \in \mathbb{N}$ (see [89, Proposition 1.11] for instance), in particular, $n_k(P(C_0(L))) = n_k(C_0(L)) = 1$. Therefore, we are in the conditions to apply Lemma 3.3.1 (in fact, by the simplified version Lemma 3.4.1) and get that $C_0(L)$ has the BPBp-nu for compact operators, as desired. ■

Alternatively, by Remark 3.4.6, we may prove Theorem 3.4.7 applying Proposition 3.3.2 instead of Lemma 3.3.1.

As a direct consequence of Theorem 3.4.7, $L_\infty(\mu)$ spaces also have the BPBp-nu for compact operators, thus completing the claims from Example 3.2.1.(iii) and Proposition 3.2.8.

Corollary 3.4.8. *$L_\infty(\mu)$ has the BPBp-nu for compact operators for every measure μ.*

Finally, let us point out that it is shown in [13] that the real space $C(K)$ has the BPBp-nu for some compact Hausdorff spaces K, but the general case, as well as the complex case, remain open as far as we know. However, Theorem 3.4.7 gives a complete answer for these spaces in the case of compact operators: all the real and complex $C(K)$ spaces have the BPBp-nu for compact operators.

Chapter 4

Norm-attaining projective tensors and nuclear operators

4.1 Introduction and motivation

Ever since the works of James ([74, 75]), Bishop and Phelps ([17]), and Lindenstrauss ([93]), many authors have contributed to the theory of norm-attaining mappings. Just to name some, Bollobás, Bourgain, Huff, Johnson, Schachermayer, Uhl, Wolfe, and Zizler continued the study about the set of all linear operators which attain their norms ([18, 21, 72, 77, 110, 113, 115]); Acosta, Aron, Aguirre, Choi, and Payá tackled problems about norm-attaining bilinear mappings ([4, 11, 31]); García and Maestre studied norm-attaining homogeneous polynomials ([8, 12]); and many more authors have studied norm-attainment questions for other types of mappings, such as multilinear mappings, holomorphic functions, compact operators (see Chapter 3) and Lipschitz mappings (see Chapter 5).

In 2014, Miguel Martín solved in the negative an open problem from the 1970s (posed explicitly by Diestel and Uhl in [51] and by Johnson and Wolfe in [77]) on whether or not every compact operator can be approximated by norm-attaining operators (see [97, Theorem 1]). On the other hand, the main open problem in the theory of norm-attaining operators nowadays seems to be if every finite-rank operator can be approximated by norm-attaining operators (see [97, Question 9]). Since every nuclear operator is compact and is a limit of a sequence of finite-rank operators, it seems natural to introduce and study norm attainment questions for nuclear operators (we will see all the necessary definitions and background below). On account of clear relations between nuclear operators and projective tensor products, we focus also on a concept of norm-attainment in projective tensor products (see Definition 4.1.1). The study of these questions has strong and deep connections with different open problems from the theory of norm-attaining operators (check for example Remark 4.2.14 to see how our topic is connected to the main open problem about norm-attaining operators).

4.1.1 Tensor Products and Nuclear Operators

We use essentially the notations and terminology from [107]. Let X and Y be two Banach spaces over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We are going to define now their projective tensor product, and its associated projective norm following [107, Chapter 2]. First, note that it is natural to require that a norm in $X \otimes Y$ satisfies that $\|x \otimes y\| \leqslant \|x\|\|y\|$ for all $x \in X$ and $y \in Y$. From here, it is natural to define the *projective norm* of a tensor $z \in X \otimes Y$ as

$$\|z\|_\pi := \inf\left\{ \sum_{n=1}^{N} \|x_n\|\|y_n\| \,:\, z = \sum_{n=1}^{N} x_n \otimes y_n,\ N \in \mathbb{N} \right\}.$$

This is, indeed, a norm, and we refer again to [107, Chapter 2] to see its most basic properties. If $X \otimes_\pi Y$ denotes the space $X \otimes Y$ endowed with the projective norm, then the *projective tensor product* of X and Y, denoted by $X\widehat{\otimes}_\pi Y$, is the completion $X \otimes_\pi Y$. By [107, Proposition 2.8], for each $z \in X\widehat{\otimes}_\pi Y$ and for each $\varepsilon > 0$, there exist sequences $\{x_n\}_{n=1}^\infty \subset X$ and $\{y_n\}_{n=1}^\infty \subset Y$ such that $z = \sum_{n=1}^\infty x_n \otimes y_n$ and

$$\sum_{n=1}^\infty \|x_n\|\|y_n\| < \|z\|_\pi + \varepsilon.$$

Because of that, we get the following equality, which gives us the projective norm of any tensor $z \in X\widehat{\otimes}_\pi Y$:

$$\begin{aligned}\|z\|_\pi &= \inf\left\{\sum_{n=1}^\infty \|x_n\|\|y_n\| : \sum_{n=1}^\infty \|x_n\|\|y_n\| < \infty, z = \sum_{n=1}^\infty x_n \otimes y_n\right\} \\ &= \inf\left\{\sum_{n=1}^\infty |\lambda_n| : z = \sum_{n=1}^\infty \lambda_n x_n \otimes y_n, \sum_{n=1}^\infty |\lambda_n| < \infty, \|x_n\| = \|y_n\| = 1\right\},\end{aligned}$$

where the infimum is taken over all such representations of z. It is well-known that $\|x \otimes y\|_\pi = \|x\|\|y\|$ for every $x \in X$, $y \in Y$, and the closed unit ball of $X\widehat{\otimes}_\pi Y$ is the closed convex hull of the set $B_X \otimes B_Y = \{x \otimes y : x \in B_X, y \in B_Y\}$. Throughout the chapter, we will be using both formulas indistinctly, without any explicit reference. The canonical identification $\mathcal{B}(X \times Y, Z) = \mathcal{L}(X\widehat{\otimes}_\pi Y, Z)$ allows us to obtain the isometrical identification $\mathcal{B}(X \times Y, \mathbb{K}) = (X\widehat{\otimes}_\pi Y)^*$. Using the fact that the spaces $\mathcal{B}(X \times Y, \mathbb{K})$ and $\mathcal{L}(X, Y^*)$ are isometrically isomorphic, we also have the identification $(X\widehat{\otimes}_\pi Y)^* = \mathcal{L}(X, Y^*)$, where the action of an operator $G : X \longrightarrow Y^*$ as a linear functional on $X\widehat{\otimes}_\pi Y$ is given by

$$G\left(\sum_{n=1}^\infty x_n \otimes y_n\right) = \sum_{n=1}^\infty G(x_n)(y_n),$$

for every $\sum_{n=1}^{\infty} x_n \otimes y_n \in X \widehat{\otimes}_\pi Y$. Let us recall also that there is a canonical operator $Q : X^* \widehat{\otimes}_\pi Y \longrightarrow \mathcal{L}(X, Y)$ with $\|Q\| = 1$ defined by $z = \sum_{n=1}^{\infty} \varphi_n \otimes y_n \mapsto L_z$, where $L_z : X \longrightarrow Y$ is given by

$$L_z(x) = \sum_{n=1}^{\infty} \varphi_n(x) y_n \quad (x \in X).$$

The operators that arise in this way are called *nuclear operators*. We denote the set of such operators by $\mathcal{N}(X, Y)$ endowed with the *nuclear norm*

$$\|T\|_{\mathcal{N}} = \inf \left\{ \sum_{n=1}^{\infty} \|x_n^*\| \|y_n\| : T(x) = \sum_{n=1}^{\infty} x_n^*(x) y_n \right\},$$

where the infimum is taken over all possible representations of T of the form $T(x) = \sum_{n=1}^{\infty} x_n^*(x) y_n$ for bounded sequences $\{x_n^*\}_{n=1}^{\infty} \subseteq X^*$ and $\{y_n\}_{n=1}^{\infty} \subseteq Y$ such that $\sum_{n=1}^{\infty} \|x_n^*\| \|y_n\| < \infty$. Notice that every nuclear operator is compact since it is the limit in the operator norm of a sequence of finite-rank operators. Using the function Q, we can identify the space $\mathcal{N}(X, Y)$ with $X^* \widehat{\otimes}_\pi Y / \ker Q$ isometrically. In order to clarify the relations between the set of nuclear operators, the quotient space of the projective tensor product and their respective duals, we consider the following diagram:

$$\begin{tikzcd}
(\ker Q)^\perp \arrow[r, "\delta"] & \left(X^* \widehat{\otimes}_\pi Y / \ker Q\right)^* \arrow[d, dash, dashed] & \mathcal{N}(X, Y)^* \arrow[l, "\widetilde{Q}^*"'] \arrow[d, dash, dashed] \\
& X^* \widehat{\otimes}_\pi Y / \ker Q \arrow[r, "\widetilde{Q}"] & \mathcal{N}(X, Y)
\end{tikzcd},$$

where $\widetilde{Q}$ and δ are isometric isomorphisms between $X^* \widehat{\otimes}_\pi Y / \ker Q$ and $\mathcal{N}(X, Y)$, and $(\ker Q)^\perp$ and $\left(X^* \widehat{\otimes}_\pi Y / \ker Q\right)^*$, respectively. If we consider a nuclear operator $T \in \mathcal{N}(X, Y)$ given by $T = \sum_{n=1}^{\infty} x_n^* \otimes y_n$ for some $\{x_n^*\}_{n=1}^\infty \subset X^*$ and $\{y_n\}_{n=1}^\infty \subset Y$ bounded with $\sum_{n=1}^\infty \|x_n^*\|\|y_n\| < \infty$, then for every $H \in \mathcal{N}(X,Y)^*$, we have

$$H(T) = \sum_{n=1}^{\infty} G(x_n^*)(y_n),$$

where $G = (\delta^{-1} \circ \widetilde{Q}^*)(H) \in (\ker Q)^\perp$.

Recall that a Banach space is said to have the *approximation property* if for every compact subset K of X and every $\varepsilon > 0$, there exists a finite-rank operator $T : X \longrightarrow X$ such that $\|T(x) - x\| \leqslant \varepsilon$ for every $x \in K$. Let us take into account that if X^* or Y has the approximation property, then $X^* \widehat{\otimes}_\pi Y = \mathcal{N}(X,Y)$ (see, for instance, [107, Corollary 4.8]). Recall also that the *injective norm* of $z \in X \otimes Y$ is defined by

$$\|z\|_\varepsilon = \sup \left\{ \left| \sum_{i=1}^{n} x^*(x_i) y^*(y_i) \right| : x^* \in B_{X^*}, y^* \in B_{Y^*} \right\},$$

where $\sum_{i=1}^n x_i \otimes y_i$ is any representation of z. We denote by $X \otimes_\varepsilon Y$ the tensor product $X \otimes Y$ with the injective norm and its completion, denoted by $X \widehat{\otimes}_\varepsilon Y$, is called the *injective tensor product* of X and Y.

For a complete background on tensor products in Banach spaces, we refer to the books [50, 107].

4.1.2 Norm-attaininment concepts

Recall that $T \in \mathcal{L}(X,Y)$ *attains its norm* (in the classical way) if there is $x_0 \in S_X$ such that $\|T(x_0)\| = \|T\| = \sup_{x \in S_X} \|T(x)\|$. In this case, we say that T is a *norm-attaining operator*, and the set of norm-attaining operators from X to Y is denoted $\operatorname{NA}(X,Y)$. Recall also that $B \in \mathcal{B}(X \times Y, Z)$ attains its norm if there is $(x_0, y_0) \in S_X \times S_Y$ such that

$\|B(x_0, y_0)\| = \|B\|_{\mathcal{B}} = \sup_{(x,y)\in S_X\times S_Y} \|B(x, y)\|$. In this case, we say that B is a *norm-attaining bilinear mapping*, and the set of norm-attaining bilinear mappings from $X \times Y$ to Z is denoted $\mathrm{NA}_{\mathcal{B}}(X \times Y, Z)$. In the next sections, we will be considering norm-attainment concepts on the Banach spaces $X\widehat{\otimes}_\pi Y$ and $\mathcal{N}(X, Y)$. We introduce next a natural approach for both scenarios.

Definition 4.1.1. Let X, Y be Banach spaces. We say that

(1) $z \in X\widehat{\otimes}_\pi Y$ *attains its projective norm* if there is a bounded sequence $\{(x_n, y_n)\}_{n=1}^\infty \subseteq X \times Y$ with $\sum_{n=1}^\infty \|x_n\|\|y_n\| < \infty$ such that we have $z = \sum_{n=1}^\infty x_n \otimes y_n$ and $\|z\|_\pi = \sum_{n=1}^\infty \|x_n\|\|y_n\|$. In this case, we say that z is a *norm-attaining tensor*. The set of norm-attaining projective tensors in $X\widehat{\otimes}_\pi Y$ is denoted $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)$.

(2) $T \in \mathcal{N}(X, Y)$ *attains its nuclear norm* if there is a bounded sequence $\{x_n^*, y_n\}_{n=1}^\infty \subseteq X^* \times Y$ with $\sum_{n=1}^\infty \|x_n^*\|\|y_n\| < \infty$ such that $T = \sum_{n=1}^\infty x_n^* \otimes y_n$ and that $\|T\|_{\mathcal{N}} = \sum_{n=1}^\infty \|x_n^*\|\|y_n\|$. In this case, we say that T is a *norm-attaining nuclear operator*. The set of norm-attaining nuclear operators in $\mathcal{N}(X, Y)$ is denoted $\mathrm{NA}_{\mathcal{N}}(X, Y)$.

If (1) (respectively, (2)) holds, then we say that $\sum_{n=1}^\infty x_n \otimes y_n$ (respectively, $\sum_{n=1}^\infty x_n^* \otimes y_n$) is a *norm-attaining representation*. Notice that, as we pointed out before, when X^* or Y has the approximation property then $X^*\widehat{\otimes}_\pi Y$ is isometrically isomorphic to $\mathcal{N}(X, Y)$. In such case, it is clear that both concepts of norm-attainment agree. Due to the connection between projective tensor products, bilinear mappings, and operators, we should also observe that the density of the set $\mathrm{NA}_{\mathcal{B}}(X \times Y, \mathbb{K})$ clearly implies the density of the set $\mathrm{NA}(X, Y^*)$, but the converse is not true in general: indeed, Choi proved in [31] that $\mathrm{NA}_{\mathcal{B}}(L_1([0, 1]) \times L_1([0, 1]), \mathbb{R})$ is not dense in $\mathcal{B}(L_1([0, 1]) \times L_1([0, 1]), \mathbb{R})$, but Finet and Payá showed in [56] that if μ is any σ-finite measure, then $\mathrm{NA}(L_1(\mu), L_\infty([0,1]))$ is dense in $\mathcal{L}(L_1(\mu), L_\infty([0,1]))$ (in fact, that pair of spaces has the BPBp, as shown in [9]).

Since several norms are being considered in these spaces of mappings, let us clarify what we mean by approximating elements from $X\widehat{\otimes}_\pi Y$ or $\mathcal{N}(X,Y)$ by norm-attaining ones. When working with $X\widehat{\otimes}_\pi Y$, it is natural to make the approximation of an element $z \in X\widehat{\otimes}_\pi Y$ by an element $z' \in \mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)$ using the projective norm $\|\cdot\|_\pi$. Similarly, we shall be dealing with the nuclear operator norm $\|\cdot\|_{\mathcal{N}}$ whenever we approximate a given nuclear operator T by a norm-attaining nuclear operator T'. Density results will also use these respective norms.

Finally, to end this section, we will briefly discuss the structure of this chapter. Section 4.2 is devoted to find the first examples of nuclear operators and tensors which attain their norms. We give a characterization for these kind of elements, which will be very helpful during the entire chapter (see Theorems 4.2.1 and 4.2.2). We provide a list of pairs of spaces where every projective tensor and every nuclear operator attains its norm. Nevertheless, we also show that this cannot hold in general, since if every tensor from $X\widehat{\otimes}_\pi Y$ attains its projective norm, then the set of norm-attaining operators from X to Y^* must be dense, and there are known examples of pairs of spaces that do not satisfy that (see Corollary 4.2.11). Therefore, there also exist projective tensors and nuclear operators which do not attain their respective norms. In Section 4.3, we show that the set of all norm-attaining projective tensors is dense in the projective tensor product whenever the involved Banach spaces X and Y satisfy certain conditions (for instance, when $(X \times Y, \mathbb{K})$ has the $\mathbf{L}_{o,o}$ for bilinear forms, or when both X and Y have the metric π-property; see Proposition 4.3.5 and Theorem 4.3.8), and analogous results are also obtained for nuclear operators. As a consequence, a wide list of spaces for which the density holds are found. Finally, in Section

4.4, inspired by [97], we present an example of two Banach spaces X and Y, both failing the approximation property, for which the set of norm-attaining tensors is not dense in the projective tensor product space.

4.2 Existence of norm-attaining elements

In this section, we provide the first examples of elements in $X\widehat{\otimes}_\pi Y$ and $\mathcal{N}(X,Y)$ which attain their norms. The first result gives us an important characterization that will be used abundantly and implicitly in the rest of the chapter.

Theorem 4.2.1. *Let X, Y be Banach spaces. Let $z \in X\widehat{\otimes}_\pi Y$ with*

$$z = \sum_{n=1}^{\infty} \lambda_n x_n \otimes y_n,$$

where $\lambda_n \in \mathbb{R}^+$, $x_n \in S_X$, and $y_n \in S_Y$ for every $n \in \mathbb{N}$. Then, the following assertions are equivalent:

(1) $\|z\|_\pi = \sum_{n=1}^{\infty} \lambda_n$*; in other words,* $z \in \mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)$.

(2) *There is $G \in \mathcal{L}(X, Y^*)$ with $\|G\| = 1$ such that $G(x_n)(y_n) = 1$ for every $n \in \mathbb{N}$.*

(3) *Every norm one $G \in \mathcal{L}(X, Y^*)$ such that $G(z) = \|z\|_\pi$ satisfies that $G(x_n)(y_n) = 1$ for every $n \in \mathbb{N}$.*

Proof. Suppose that $\|z\|_\pi = \sum_{n=1}^{\infty} \lambda_n$ with $z = \sum_{n=1}^{\infty} \lambda_n x_n \otimes y_n$, where $\{\lambda_n\}_{n=1}^{\infty} \subseteq \mathbb{R}^+$, $\{x_n\}_{n=1}^{\infty} \subseteq S_X$, and $\{y_n\}_{n=1}^{\infty} \subseteq S_Y$. Now pick any $G \in$

$(X\widehat{\otimes}_\pi Y)^* = \mathcal{L}(X, Y^*)$ such that $\|G\| = 1$ and $G(z) = \|z\|_\pi$. Since we have

$$\sum_{n=1}^{\infty} \lambda_n = \|z\|_\pi = G(z) = \sum_{n=1}^{\infty} \lambda_n G(x_n)(y_n),$$

it follows that $G(x_n)(y_n) = 1$ for each $n \in \mathbb{N}$, which proves that (1) implies (3). It is obvious that (3) implies (2). Finally, assume that there exists $G \in \mathcal{L}(X, Y^*)$ with $\|G\| = 1$ such that $G(x_n)(y_n) = 1$ for every $n \in \mathbb{N}$. Then,

$$\sum_{n=1}^{\infty} \lambda_n = \sum_{n=1}^{\infty} \lambda_n G(x_n)(y_n) = G(z) \leqslant \|z\|_\pi \leqslant \sum_{n=1}^{\infty} \lambda_n.$$

This completes the proof. ■

Taking into account the isometric isomorphism between $\mathcal{N}(X, Y)$ and $X^*\widehat{\otimes}_\pi Y/\ker(Q)$, we can take advantage of the previous estimates to prove a nuclear operator version of Theorem 4.2.1 as follows.

Theorem 4.2.2. *Let X, Y be Banach spaces. Let $T \in \mathcal{N}(X, Y)$ with*

$$T = \sum_{n=1}^{\infty} \lambda_n x_n^* \otimes y_n,$$

where $\lambda_n \in \mathbb{R}^+$, $x_n \in S_X$, and $y_n \in S_Y$ for every $n \in \mathbb{N}$. Then, the following assertions are equivalent:

(1) $\|T\|_{\mathcal{N}} = \sum_{n=1}^{\infty} \lambda_n$; *in other words, $T \in \mathrm{NA}_{\mathcal{N}}(X, Y)$.*

(2) *There is $G \in (\ker Q)^\perp$ with $\|G\| = 1$ such that $G(x_n^*)(y_n) = 1$ for every $n \in \mathbb{N}$.*

(3) *For any $G \in (\ker Q)^\perp$ with $\|G\| = 1$ and $G(T) = \|T\|_{\mathcal{N}}$ we get that $G(x_n^*)(y_n) = 1$ holds for every $n \in \mathbb{N}$.*

Proof. Let $\widetilde{Q} : X^*\widehat{\otimes}_\pi Y/\ker Q \longrightarrow \mathcal{N}(X,Y)$ be an isometric isomorphism which maps, according to the notation of Subsection 4.1.1, $z + \ker Q$ to L_z. If we let $z_0 := \sum_{n=1}^{\infty} \lambda_n x_n^* \otimes y_n \in X^*\widehat{\otimes}_\pi Y$, then $Q(z_0) = T$ and $\|T\|_{\mathcal{N}} = \|z_0 + \ker Q\|$. Now assume (1) and let us prove (3). To this end, pick any $G \in (\ker Q)^\perp$ with $\|G\| = 1$ and $G(z_0 + \ker Q) = \|z_0 + \ker Q\|$. Then,

$$\begin{aligned}\sum_{n=1}^{\infty} \lambda_n = \|T\|_{\mathcal{N}} = \|z_0 + \ker Q\| = |G(z_0)| = \left| G\left(\sum_{n=1}^{\infty} \lambda_n x_n^* \otimes y_n\right)\right| \\ \leqslant \sum_{n=1}^{\infty} \lambda_n |G(x_n^*)(y_n)| \leqslant \sum_{n=1}^{\infty} \lambda_n.\end{aligned}$$

Then, we have $|G(x_n^*)(y_n)| = 1$ for each $n \in \mathbb{N}$. Using a convexity argument, we get that $G(x_n^*)(y_n) = 1$ for every $n \in \mathbb{N}$. The other implications can be proved as in Theorem 4.2.1. ■

With Theorems 4.2.1 and 4.2.2 in mind, we can now exhibit examples of nuclear operators which attain their nuclear norms.

Example 4.2.3. Let X, Y be two reflexive Banach spaces such that X^* or Y has the approximation property (recall that, in this case, we have $X^*\widehat{\otimes}_\pi Y = \mathcal{N}(X,Y)$). Assume further that X^* is isometrically isomorphic to a subspace of Y^*. Take $G : X^* \longrightarrow Y^*$ to be a linear isometry and pick $\{x_n^*\}_{n=1}^{\infty} \subseteq S_{X^*}$. Now, for any $n \in \mathbb{N}$, notice that $\|G(x_n^*)\| = \|x_n^*\| = 1$. Since Y is reflexive, by using the James' theorem, we have that $G(x_n^*) \in S_{Y^*}$ attains its norm, so there exists $y_n \in S_Y$ so that $G(x_n^*)(y_n) = 1$. Now, Theorem 4.2.1(or Theorem 4.2.2) implies that, given any sequence $\{\lambda_n\}_{n=1}^{\infty} \subseteq (0,1]$ with $\sum_{n=1}^{\infty} \lambda_n < \infty$, the nuclear operator

$$T := \sum_{n=1}^{\infty} \lambda_n x_n^* \otimes y_n \in \mathcal{N}(X,Y)$$

attains its nuclear norm.

One may think that a norm-attaining nuclear operator should attain its norm (in the classical way). This is not true in general as observed below.

Remark 4.2.4. Let Y be an infinite-dimensional strictly convex Banach space. Then, there is $T \in \mathrm{NA}_{\mathcal{N}}(c_0, Y)$ such that $T \notin \mathrm{NA}(c_0, Y)$. Indeed, let $\{y_n\}_{n=1}^{\infty} \subseteq S_Y$ be linearly independent. For every $n \in \mathbb{N}$, find $y_n^* \in S_{Y^*}$ such that $y_n^*(y_n) = 1$. Define $\phi : Y \longrightarrow \ell_\infty$ by $\phi(y) := \{y_j^*(y)\}_{j=1}^{\infty} \in \ell_\infty$ for every $y \in Y$. Given $n \in \mathbb{N}$ we get that $|y_n^*(y)| \leqslant \|y\|$ since $\|y_n^*\| = 1$ holds for every $n \in \mathbb{N}$. This implies that $\sup_{n \in \mathbb{N}} |y_n^*(y)| \leqslant \|y\|$, which proves that $\phi(y) \in \ell_\infty$ for every y (i.e., ϕ is well defined) . In view of the linearity, we have that ϕ is continuous and $\|\phi\| \leqslant 1$. Furthermore, notice that $\phi(y_n)(e_n) = 1$ holds for every $n \in \mathbb{N}$, where $\{e_n\}_{n=1}^{\infty}$ is the basis of ℓ_1. This proves that the nuclear operator $T : c_0 \longrightarrow Y$ defined by

$$T = \sum_{n=1}^{\infty} \frac{1}{2^n} e_n \otimes y_n \in \ell_1 \widehat{\otimes}_\pi Y$$

attains its nuclear norm by Theorem 4.2.2. Nevertheless, notice that T is not a finite-rank operator and, consequently, T does not belong to $\mathrm{NA}(c_0, Y)$ (see [97, Lemma 2.2] or the proof of [93, Proposition 4]).

We prove next that on the the finite-dimensional setting, *every* tensor is norm-attaining. Before presenting the proof, let us notice that since the convex hull of a compact set is compact when X and Y are both finite-dimensional spaces, we have that $\overline{\mathrm{conv}}(B_X \otimes B_Y) = \mathrm{conv}\,(B_X \otimes B_Y)$ (where $\mathrm{conv}(A)$ denotes the convex hull of A, and $\overline{\mathrm{conv}}(A)$ is its closure), which is a consequence of Minkowski-Carathéodory theorem (see, for instance, [53, Exercises 1.57 and 1.58]).

Proposition 4.2.5. *Let X, Y be finite-dimensional Banach spaces. Then, every tensor attains its projective tensor norm. In other words,* $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y) = X\widehat{\otimes}_\pi Y$.

Proof. Let $z \in X\widehat{\otimes}_\pi Y$ with $\|z\|_\pi = 1$ be given and let us prove that $z \in \mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)$. As we have mentioned before, since X and Y are finite-dimensional Banach spaces, $B_X \otimes B_Y$ is compact in $X\widehat{\otimes}_\pi Y$ and this implies that $B_{X\widehat{\otimes}_\pi Y} = \overline{\mathrm{conv}}(B_X \otimes B_Y) = \mathrm{conv}(B_X \otimes B_Y)$. Therefore, z can be written as a finite convex combination of elements in $B_X \otimes B_Y$, i.e.,

$$z = \sum_{j=1}^{n} \lambda_j x_j \otimes y_j \quad \text{with} \quad \sum_{j=1}^{n} \lambda_j = 1,$$

where $\lambda_j \in \mathbb{R}^+$, $x_j \in B_X$, and $y_j \in B_Y$ for $j = 1, \ldots, n$, that is, z is norm-attaining. ■

Let us notice that in Remark 4.2.4, we have constructed by hand a nuclear operator from c_0 into a particular Y which attains its nuclear norm. It turns out that *every* nuclear operator from c_0 into *any* Banach space Y attains its nuclear norm. This should be compared to the fact that, in the classical theory, whenever X is a Banach space such that $\mathrm{NA}(X, Y) = \mathcal{L}(X, Y)$ for some $Y \neq \{0\}$, X must be reflexive, by James' theorem. However, by the following proposition, this result is no longer true in the context of nuclear operators.

Proposition 4.2.6. *Let Y be a Banach space. Then,*

(a) every $T \in \mathcal{N}(c_0, Y)$ attains its nuclear norm. Equivalently,

(b) every element in $\ell_1\widehat{\otimes}_\pi Y$ attains its projective norm.

Proof. Indeed, in the last part of [107, Lemma 2.6], it is proved that $\Phi : \ell_1(Y) \longrightarrow \ell_1 \widehat{\otimes}_\pi Y$ given by

$$\Phi(\{x_n\}_{n=1}^\infty) = \sum_{n=1}^\infty e_n \otimes x_n$$

is an onto linear isometry, where $\{e_n\}_{n=1}^\infty$ is the basis of ℓ_1 (in fact, $\Phi = Q^{-1}$ in the proof given there). Let $T \in \mathcal{N}(c_0, Y) = \ell_1 \widehat{\otimes}_\pi Y$ be given. By the surjectivity of Φ, we can find an element $\{x_n\}_{n=1}^\infty \in \ell_1(Y)$ such that $\Phi(\{x_n\}_{n=1}^\infty) = T$. Consequently, $T = \sum_{n=1}^\infty e_n \otimes x_n$. Then,

$$\|T\|_{\mathcal{N}} = \|\Phi(\{x_n\}_{n=1}^\infty)\| = \|\{x_n\}_{n=1}^\infty\| = \sum_{n=1}^\infty \|x_n\| = \sum_{n=1}^\infty \|e_n\|\|x_n\|.$$

This proves that T attains its nuclear norm, as desired. ■

Remark 4.2.7. Notice that Proposition 4.2.6 is also true for $c_0(\Gamma)$ and $\ell_1(\Gamma)$ for any arbitrary index set Γ (see [107, Example 2.6]).

In the infinite-dimensional case, besides the nuclear operators from c_0 into an arbitrary Banach space Y, we have that every nuclear operator on a complex Hilbert space attains its nuclear norm. Although we prove this result for nuclear operators (justified by the fact that we will be dealing with eigenvalues and Schatten classes), we also get that every tensor in $H \widehat{\otimes}_\pi H$ attains its projective norm, as every Hilbert space H has the approximation property.

Proposition 4.2.8. *Let H be a complex Hilbert space. Then, every nuclear operator $T \in \mathcal{N}(H, H)$ attains its nuclear norm.*

Proof. Note that $T \in \mathcal{N}(H, H)$ can be written as

$$T = \sum_{j=1}^{n_0} \lambda_j \langle \cdot, x_j \rangle y_j,$$

where $n_0 \in \mathbb{N} \cup \{\infty\}$, $\{\lambda_j\}_{j=1}^{\infty}$ is the sequence of nonzero eigenvalues of $|T| = (T^*T)^{\frac{1}{2}}$, and $\{x_j\}_{j=1}^{\infty}$ and $\{y_j\}_{j=1}^{\infty}$ are orthonormal systems in H (see [67, Theorem 2.1]). On the other hand, it is well-known that $\|T\|_{\mathcal{N}} = \sigma_1(T) = \sum_{j=1}^{n_0} \lambda_j$, where $\sigma_1(\cdot)$ is the Schatten 1st norm (see, for example, [67, pages 96-97]). This completes the proof. ■

Taking into account Propositions 4.2.5, 4.2.6 and 4.2.8, it is natural to ask whether or not the equalities $\mathrm{NA}_{\mathcal{N}}(X,Y) = \mathcal{N}(X,Y)$ or $\mathrm{NA}_{\pi}(X\widehat{\otimes}_\pi Y) = X\widehat{\otimes}_\pi Y$ hold for all Banach spaces X and Y. We will give a negative answer to this question by proving that if this happens, then the set of norm-attaining bilinear forms which attain their norms is dense in $\mathcal{B}(X \times Y, \mathbb{K})$, something which is known to fail in many spaces.

Lemma 4.2.9. *Let X, Y be Banach spaces. If $B \in \mathcal{B}(X \times Y, \mathbb{K}) = (X\widehat{\otimes}_\pi Y)^*$ attains its functional norm at an element of $\mathrm{NA}_{\pi}(X\widehat{\otimes}_\pi Y)$, then $B \in \mathrm{NA}_{\mathcal{B}}(X \times Y, \mathbb{K})$.*

Proof. Let $B \in \mathcal{B}(X \times Y, \mathbb{K}) = (X\widehat{\otimes}_\pi Y)^*$ and let $z \in S_{X\widehat{\otimes}_\pi Y}$ such that $z = \sum_{n=1}^{\infty} \lambda_n x_n \otimes y_n \in \mathrm{NA}_{\pi}(X\widehat{\otimes}_\pi Y)$ be such that $B(z) = 1$, where $\lambda_n \in \mathbb{R}^+$, $x_n \in S_X$, and $y_n \in S_Y$. By Theorem 4.2.1, $B(x_n, y_n) = 1$ for every $n \in \mathbb{N}$. In particular, $B \in \mathrm{NA}_{\mathcal{B}}(X \times Y, \mathbb{K})$. ■

Proposition 4.2.10. *Let X, Y be Banach spaces. If every element in $X\widehat{\otimes}_\pi Y$ attains its projective norm, then the set of all bilinear forms on $X \times Y$ which attain their norms is dense in $\mathcal{B}(X \times Y, \mathbb{K})$. In other words, if $\mathrm{NA}_{\pi}(X\widehat{\otimes}_\pi Y) = X\widehat{\otimes}_\pi Y$, then*

$$\overline{\mathrm{NA}_{\mathcal{B}}(X \times Y, \mathbb{K})}^{\|\cdot\|_{\mathcal{B}}} = \mathcal{B}(X \times Y, \mathbb{K}).$$

Proof. Let $\varepsilon > 0$. Let $B \in \mathcal{B}(X \times Y, \mathbb{K}) = (X\widehat{\otimes}_\pi Y)^*$ with $\|B\|_{\mathcal{B}} = 1$. By the Bishop-Phelps theorem, for $X\widehat{\otimes}_\pi Y$, there are $B_0 \in (X\widehat{\otimes}_\pi Y)^*$ with $\|B_0\|_{\mathcal{B}} = 1$ and $z_0 \in S_{X\widehat{\otimes}_\pi Y}$ such that $B_0(z_0) = 1$ and $\|B_0 - B\|_{\mathcal{B}} < \varepsilon$. By hypothesis, $z_0 \in \mathrm{NA}_\pi(X, Y)$ attains its projective norm and by Lemma 4.2.9 we have that $B_0 \in \mathrm{NA}_{\mathcal{B}}(X \times Y, \mathbb{K})$. Since $\|B_0 - B\|_{\mathcal{B}} < \varepsilon$, we are done. ∎

Proposition 4.2.10 yields the following immediate consequence.

Corollary 4.2.11. *Let X, Y be Banach spaces. Suppose that every element in $X\widehat{\otimes}_\pi Y$ attains its projective norm. Then, the set of norm-attaining operators from X into Y^* is dense in $\mathcal{L}(X, Y^*)$. In other words, if $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y) = X\widehat{\otimes}_\pi Y$, then*

$$\overline{\mathrm{NA}(X, Y^*)}^{\|\cdot\|} = \mathcal{L}(X, Y^*).$$

Now, by using Lemma 4.2.9, Proposition 4.2.10, and Corollary 4.2.11, we can get examples of pairs of Banach spaces (X, Y) such that there are elements in the projective tensor product $X\widehat{\otimes}_\pi Y$ which *do not* attain their projective norms.

Examples 4.2.12. In the following cases, there are elements $z \in X\widehat{\otimes}_\pi Y$ such that $z \notin \mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)$.

(a) When $X = L_1(\mathbb{T})$, where the unit circle $\mathbb{T}$ is equipped with the Haar measure m, and Y is the two-dimensional Hilbert space. Indeed, it is shown in [65, Remark 5.7.(2)] that there is $T \in \mathcal{B}(X \times Y, \mathbb{K})$ which attains its norm as a linear functional on $X\widehat{\otimes}_\pi Y$ but not as an operator from X into Y^* (nor the more as a bilinear form on $X \times Y$). By Lemma 4.2.9, it follows that $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y) \neq X\widehat{\otimes}_\pi Y$.

(b) When X is $L_1[0,1]$ and Y^* is a strictly convex Banach space without the Radon-Nikodým property. Indeed, by [113, Theorem 3], the set $\mathrm{NA}(L_1[0,1], Y^*)$ is not dense in $\mathcal{L}(L_1[0,1], Y^*)$. Let us notice that this also shows that Proposition 4.2.6 is no longer true if we consider an $L_1(\mu)$-space for a non-purely atomic measure μ.

(c) When $Y = \ell_p$ for $1 < p < \infty$ and X is the Banach space constructed by Gowers. Indeed, there is a Banach space G such that $\mathrm{NA}_{\mathcal{B}}(G \times \ell_p, \mathbb{K})$ is not dense in $\mathcal{B}(G \times \ell_p, \mathbb{K})$ (see [68, Theorem, page 149]). We should notice that the unit ball of G lacks extreme points. This result should be compared to the fact that, if X is reflexive and Y is any Banach space, then $\mathcal{K}(X, Y) \subseteq \mathrm{NA}(X, Y)$.

(d) When X and Y are both $L_1[0, 1]$. Indeed, [31, Theorem 3] shows that the set $\mathrm{NA}_{\mathcal{B}}(L_1[0, 1] \times L_1[0, 1], \mathbb{K})$ is not dense in $\mathcal{B}(L_1[0, 1] \times L_1[0, 1], \mathbb{K})$.

Let us finish this section by highlighting two observations.

Remark 4.2.13. Notice that if we weaken the hypothesis in Proposition 4.2.10 and just assume that $\mathrm{NA}_\pi(X \widehat{\otimes}_\pi Y)$ is dense in $X \widehat{\otimes}_\pi Y$, the result does *not* remain true. Indeed, by Example 4.2.12.(c), we know that $\mathrm{NA}_{\mathcal{B}}(L_1[0, 1] \times L_1[0, 1], \mathbb{K})$ is not dense in $\mathcal{B}(L_1[0, 1] \times L_1[0, 1], \mathbb{K})$, but we will see in Section 4.3 that the set of all tensors which attain their projective norm on $L_1[0, 1] \widehat{\otimes}_\pi L_1[0, 1]$ is dense in $L_1[0, 1] \widehat{\otimes}_\pi L_1[0, 1]$ (see Theorem 4.3.8 and Example 4.3.12.(b)). Nevertheless, we will always have that $\mathrm{NA}(X, Y^*) \cap B_{\mathcal{L}(X,Y^*)}$ is w^*-dense in $B_{\mathcal{L}(X,Y^*)}$ under this hypothesis (see Remark 4.4.4).

Remark 4.2.14. In Proposition 4.2.5, we saw that if both Banach spaces are finite-dimensional, then every tensor attains its projective norm, and every nuclear operator attains its nuclear norm. It is natural to wonder if the same holds by just considering one of the spaces to be finite-dimensional. Let Y be a finite-dimensional Banach space. Then, $\mathrm{NA}(Y, Z) = \mathcal{L}(Y, Z)$ for every Banach space Z, by the compactness of the unit ball of Y. Let us suppose for a second that the same holds for nuclear operators. Then, $\mathrm{NA}_{\mathcal{N}}(Y,Z) = \mathcal{N}(Y,Z)$ for every Banach space Z. Since Y is finite-dimensional, it has the approximation property and then we would have that $\mathrm{NA}_\pi(Y^*\widehat{\otimes}_\pi Z) = Y^*\widehat{\otimes}_\pi Z$, and so, that $\mathrm{NA}_\pi(Z\widehat{\otimes}_\pi Y^*) = Z\widehat{\otimes}_\pi Y^*$, for every Banach space Z. By Corollary 4.2.11, we would then have that the set $\mathrm{NA}(Z,Y)$ is dense in $\mathcal{L}(Z,Y)$ for every Banach space Z, which would imply that Y has property B of Lindenstrauss (positively solving the main open question on norm attainment, [97, Question 9]). Therefore, it is natural to wonder whether every nuclear operator $T : Y \longrightarrow Z$ attain its nuclear norm for every Banach space Z whenever Y is finite-dimensional. However, this is actually *not* the case, as shown in Example 4.2.12.(a), by taking $Z = L_1(\mathbb{T})$ and $Y = \ell_2^2$, the Euclidean plane (see [65, Remark 5.7.(2)]).

4.3 First density results

In this section we will be focusing on examples of Banach spaces X and Y such that the sets $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)$ and $\mathrm{NA}_{\mathcal{N}}(X,Y)$ are dense in $X\widehat{\otimes}_\pi Y$ and $\mathcal{N}(X,Y)$, respectively. As we have seen in the previous section, there are many examples of projective tensor products where we can guarantee the existence of elements which do not attain their projective norms even when one of the factors is reflexive (see Example 4.2.12.(b)). In spite of the existence of such non-norm-attaining tensors, it is natural to ask if the set of elements in a tensor product space which attain their projective norms is dense in the whole space.

Let us start by explaining where the difficulty comes from when one tries to get such a property. Assume that $z \in \mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)$ is a norm-attaining tensor in $X\widehat{\otimes}_\pi Y$. This implies that there are bounded sequences $\{x_n\}_{n=1}^\infty \subseteq X$ and $\{y_n\}_{n=1}^\infty \subseteq Y$ such that $z = \sum_{n=1}^\infty x_n \otimes y_n$ with

$\|z\|_\pi = \sum_{n=1}^{\infty} \|x_n\|\|y_n\|$. It is clear that the task of choosing the optimal representation for z as a series of basic tensors is the most difficult part. In order to avoid this inconvenience, let us make use of Theorem 4.2.1. By applying it, for any bilinear mapping $B \in S_{\mathcal{B}(X\times Y,\mathbb{K})} = S_{(X\widehat{\otimes}_\pi Y)^*}$ such that $B(z) = \|z\|_\pi$, we have that $B(x_n)(y_n) = \|x_n\|\|y_n\|$ for every $n \in \mathbb{N}$. In other words, B attains its bilinear norm at the pair $\left(\frac{x_n}{\|x_n\|}, \frac{y_n}{\|y_n\|}\right)$ for every $n \in \mathbb{N}$. Because of this, in order to get examples of Banach spaces X and Y where the set $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)$ is dense in $X\widehat{\otimes}_\pi Y$, we need somehow that the space $\mathcal{B}(X \times Y, \mathbb{K})$ contains many bilinear forms which attain their bilinear norm at many elements of $S_X \times S_Y$. This motivates us to make use of the following definitions, which can be found in [38] and [48].

Definition 4.3.1. Let X, Y and Z be Banach spaces.

(a) We say that (X, Y) has the $\mathbf{L}_{o,o}$ for operators if given $\varepsilon > 0$ and $T \in \mathcal{L}(X, Y)$ with $\|T\| = 1$, there is $\eta(\varepsilon, T) > 0$ such that whenever $x \in S_X$ satisfies $\|T(x)\| > 1 - \eta(\varepsilon, T)$, there is $x_0 \in S_X$ such that $\|T(x_0)\| = 1$ and $\|x_0 - x\| < \varepsilon$.

(b) We say that $(X \times Y, Z)$ satisfies the $\mathbf{L}_{o,o}$ for bilinear mappings (or just $\mathbf{L}_{o,o,\mathcal{B}}$) if given $\varepsilon > 0$ and $B \in \mathcal{B}(X \times Y, Z)$ with $\|B\|_{\mathcal{B}} = 1$, there exists $\eta(\varepsilon, B) > 0$ such that whenever $(x, y) \in S_X \times S_Y$ satisfies $\|B(x, y)\| > 1 - \eta(\varepsilon, B)$, there is $(x_0, y_0) \in S_X \times S_Y$ such that $\|B(x_0, y_0)\| = 1$, $\|x - x_0\| < \varepsilon$, and $\|y - y_0\| < \varepsilon$.

Example 4.3.2. Let us highlight some examples and results related to the properties just defined.

(a) If $\dim(X), \dim(Y) < \infty$, then $(X \times Y, Z)$ has the $\mathbf{L}_{o,o,\mathcal{B}}$ for every Banach space Z (see [48, Proposition 2.2]).

(b) $(X \times Y, \mathbb{K})$ has the $\mathbf{L}_{o,o}$ for bilinear mappings if and only if (X, Y^*) has the $\mathbf{L}_{o,o}$ for operators, whenever Y is uniformly convex (see [48, Lemma 2.6]). In particular, if X is finite-dimensional and Y is uniformly convex, then $(X \times Y, \mathbb{K})$ has the $\mathbf{L}_{o,o}$ for bilinear forms (see [38, Theorem 2.4]).

(c) If $1 < p, q < \infty$, then $(\ell_p \times \ell_q, \mathbb{K})$ has the $\mathbf{L}_{o,o,\mathcal{B}}$ if and only if $p > q'$, where q' is the conjugate of q (see [48, Theorem 2.7.(b)]).

(d) As a consequence of (c), there are reflexive spaces X and Y such that $(X \times Y, \mathbb{K})$ fails the $\mathbf{L}_{o,o,\mathcal{B}}$ (see also [38, Theorem 2.21.(ii)]).

Using this property, we will find our first positive density results about norm attainment in our context.

Proposition 4.3.3. *Let X, Y be Banach spaces. Suppose that $(X^* \times Y, \mathbb{K})$ has $\boldsymbol{L}_{o,o}$ for bilinear forms. Then, every nuclear operator from X into Y can be approximated (in the nuclear norm) by nuclear operators which attain their nuclear norm. In other words,*

$$\overline{\mathrm{NA}_{\mathcal{N}}(X, Y)}^{\|\cdot\|_{\mathcal{N}}} = \mathcal{N}(X, Y).$$

We get the following particular case by combining Proposition 4.3.3 with Example 4.3.2.

Corollary 4.3.4. *Let X be finite-dimensional Banach space. If Y is uniformly convex, then*

$$\overline{\mathrm{NA}_{\mathcal{N}}(X, Y)}^{\|\cdot\|_{\mathcal{N}}} = \mathcal{N}(X, Y).$$

Now, we prove Proposition 4.3.3.

Proof of Proposition 4.3.3. Let $T \in \mathcal{N}(X,Y)$ and $\varepsilon > 0$ be given. There exists $H \in \mathcal{N}(X,Y)^*$ with $\|H\| = 1$ such that $H(T) = \|T\|_{\mathcal{N}}$. Consider $G := (\delta^{-1} \circ \widetilde{Q}^*)(H) \in (\ker Q)^{\perp}$ (see Subsection 4.1.1). Let A_G be the bilinear form on $X^* \times Y$ defined by $A_G(x^*, y) = G(x^*)(y)$ for every $x^* \in X^*$ and $y \in Y$. Then $\|A_G\|_{\mathcal{B}} = \|G\| = 1$. Consider the positive value $\eta(\varepsilon, A_G) > 0$ from the assumption that $(X^* \times Y, \mathbb{K})$ has $\mathbf{L}_{o,o}$ for bilinear forms. Now, choose $\{\lambda_n\}_{n=1}^{\infty} \subseteq \mathbb{R}^+$, $\{x_n^*\}_{n=1}^{\infty} \subseteq S_{X^*}$, and $\{y_n\}_{n=1}^{\infty} \subseteq S_Y$ so that $T = \sum_{n=1}^{\infty} \lambda_n x_n^* \otimes y_n$ with

$$\sum_{n=1}^{\infty} \lambda_n < \|T\|_{\mathcal{N}} + \eta(\varepsilon, A_G)^2.$$

We get that

$$\begin{aligned} \|T\|_{\mathcal{N}} = H(T) = \operatorname{Re} H(T) &= \sum_{n=1}^{\infty} \lambda_n \operatorname{Re}\left(G(x_n^*)(y_n)\right) \\ &\leqslant \sum_{n=1}^{\infty} \lambda_n |G(x_n^*)(y_n)| \\ &\leqslant \sum_{n=1}^{\infty} \lambda_n < \|T\|_{\mathcal{N}} + \eta(\varepsilon, A_G)^2. \end{aligned}$$

In particular,

$$\sum_{n=1}^{\infty} \lambda_n \left(1 - \operatorname{Re}\left(G(x_n^*)(y_n)\right)\right) < \eta(\varepsilon, A_G)^2 \tag{4.3.1}$$

Consider the following set

$$I = \{n \in \mathbb{N} : \operatorname{Re}\left(G(x_n^*)(y_n)\right) > 1 - \eta(\varepsilon, A_G)\}.$$

From (4.3.1), notice that

$$\eta(\varepsilon, A_G) \sum_{n \in I^c} \lambda_n \leqslant \sum_{n \in I^c} \lambda_n \left(1 - \operatorname{Re}\left(G(x_n^*)(y_n)\right)\right) < \eta(\varepsilon, A_G)^2,$$

which implies that $\sum_{n \in I^c} \lambda_n < \eta(\varepsilon, A_G)$. On the other hand, for each $n \in I$,

$$\operatorname{Re} A_G(x_n^*, y_n) = \operatorname{Re}\left(G(x_n^*)(y_n)\right) > 1 - \eta(\varepsilon, A_G).$$

Thus, there exist norm one vectors $\{\widetilde{x}_n^*\}_{n \in I}$ in X^* and $\{\widetilde{y}_n\}_{n \in I}$ in Y such that

$$|A_G(\widetilde{x}_n^*, \widetilde{y}_n)| = |G(\widetilde{x}_n^*)(\widetilde{y}_n)| = 1, \quad \|\widetilde{x}_n^* - x_n^*\| < \varepsilon, \quad \text{and} \quad \|\widetilde{y}_n - y_n\| < \varepsilon$$

for every $n \in I$. Let us write $G(\widetilde{x}_n^*)(\widetilde{y}_n) = e^{i\theta_n}$ with some $\theta_n \in \mathbb{R}$ for every $n \in I$. Notice that $|1 - e^{i\theta_n}| < \sqrt{2\eta(\varepsilon, A_G)}$ for every $n \in I$. Let us define

$$T' := \sum_{n \in I} \lambda_n e^{-i\theta_n} \widetilde{x}_n^* \otimes \widetilde{y}_n.$$

Then,

$$\begin{aligned}
\|T' - T\|_{\mathcal{N}} &\leqslant \left\| \sum_{n \in I} \lambda_n (e^{-i\theta_n} \widetilde{x}_n^* \otimes \widetilde{y}_n - x_n^* \otimes y_n) \right\|_{\mathcal{N}} + \sum_{n \in I^c} \lambda_n \\
&< \sum_{n \in I} \lambda_n |1 - e^{i\theta_n}| + \left\| \sum_{n \in I} \lambda_n (\widetilde{x}_n^* \otimes \widetilde{y}_n - x_n^* \otimes y_n) \right\|_{\mathcal{N}} + \eta(\varepsilon, A_G) \\
&< \sqrt{2\eta(\varepsilon, A_G)}(\|T\|_{\mathcal{N}} + \eta(\varepsilon, A_G)^2) + 2\varepsilon(\|T\|_{\mathcal{N}} + \eta(\varepsilon, A_G)^2) + \eta(\varepsilon, A_G) \\
&= (\sqrt{2\eta(\varepsilon, A_G)} + 2\varepsilon)(\|T\|_{\mathcal{N}} + \eta(\varepsilon, A_G)^2) + \eta(\varepsilon, A_G).
\end{aligned}$$

Finally, it is clear by definition that $\|T'\|_{\mathcal{N}} \leqslant \sum_{i\in I} \lambda_n$. On the other hand,

$$\|T'\|_{\mathcal{N}} \geqslant |H(T')| = \left|\sum_{n\in I} \lambda_n e^{-i\theta_n} G(\widetilde{x}_n^*)(\widetilde{y}_n)\right| = \sum_{n\in I} \lambda_n.$$

This shows that T' attains its nuclear norm and completes the proof. ■

Using very similar arguments to Proposition 4.3.3 and Corollary 4.3.4, we can obtain the following results.

Proposition 4.3.5. *Let X, Y be Banach spaces. Suppose that $(X \times Y, \mathbb{K})$ has $\boldsymbol{L}_{o,o}$ for bilinear forms. Then, every tensor in $X\widehat{\otimes}_\pi Y$ can be approximated by tensors which attain their projective norm. In other words,*

$$\overline{\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)}^{\|\cdot\|_\pi} = X\widehat{\otimes}_\pi Y.$$

Corollary 4.3.6. *Let X be a finite-dimensional Banach space. If Y is uniformly convex, then*

$$\overline{\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)}^{\|\cdot\|_\pi} = X\widehat{\otimes}_\pi Y.$$

Let us notice that, although we have the first examples of denseness by using Propositions 4.3.3 and 4.3.5, property $\mathbf{L}_{o,o,\mathcal{B}}$ seems to be very restrictive. Indeed, when a pair of Banach spaces satisfies this property, both of them must be reflexive since every bilinear mapping attains its norm. Moreover, even if both spaces are reflexive, sometimes $(X \times Y, \mathbb{K})$ fails to have this property (see Example 4.3.2.(d)). On the other hand, we could have used the previous results together with Example 4.3.2.(c) in order to get examples where the denseness holds for ℓ_p-spaces: for instance, if $1 < p, q < \infty$ and $p > q'$, then the set $\mathrm{NA}_\pi(\ell_p\widehat{\otimes}_\pi \ell_q)$ is dense in $\ell_p\widehat{\otimes}_\pi \ell_q$ by Proposition 4.3.5. Nevertheless, in what follows we will take advantage of the finite-dimensional case to obtain more general examples of Banach spaces where the density holds. The only problem here is the fact that in general the projective norm does not respect subspaces, but it does respect 1-complemented subspaces (recall that a subspace Y of X is called a *complemented subspace* if there exists a projection $P \in \mathcal{L}(X, X)$ such that $P(X) = Y$, and if P can be chosen to have norm 1, Y is a 1-*complemented subspace*). For this reason, intuitively, we need a property of Banach spaces which guarantees the existence of many 1-complemented subspaces. Motivated by this, we consider the following definition.

Definition 4.3.7. Let X be a Banach space. We will say that X has the *metric π-property* if given $\varepsilon > 0$ and $\{x_1, \ldots, x_n\} \subseteq S_X$ a finite collection in the sphere, then we can find a finite-dimensional 1-complemented subspace $M \subseteq X$ such that for each $i \in \{1, \ldots, n\}$ there exists $x_i' \in M$ with $\|x_i - x_i'\| < \varepsilon$.

Before proceeding, let us make a small observation. Let $\varepsilon > 0$ and $F = \{x_1, \ldots, x_n\} \subseteq S_X$ be given. Suppose that X has metric π-property as defined above and let M be a finite-dimensional subspace of X with $\|x_i' - x_i\| < \varepsilon$ for $x_i' \in M$ and $i = 1, \ldots, n$. Let $P_{\varepsilon,F}$ be the norm one projection onto M. Then, for each $i = 1, \ldots, n$, we have

$$\|P_{\varepsilon,F}(x_i) - x_i\| \leqslant \|P_{\varepsilon,F}(x_i) - P_{\varepsilon,F}(x_i')\| + \|P_{\varepsilon,F}(x_i') - x_i\| < 2\varepsilon.$$

Consider now the net $\{P_{\varepsilon,F} : \varepsilon > 0,\ F \subset S_X \text{ a finite set}\}$ with $(\varepsilon_1, F_1) \leqslant (\varepsilon_2, F_2)$ if and only if $\varepsilon_2 < \varepsilon_1$ and $F_1 \subseteq F_2$. Then, $\{P_{\varepsilon,F}\}_{(\varepsilon,F)}$ strongly converges to the identity on S_X and hence on X with $\|P_{\varepsilon,F}\| \leqslant 1$ for every ε and F. This shows that Definition 4.3.7 is in fact equivalent to [23, Definition 5.1], the classical way of defining the metric π-property as an approximation property where the approximating operators are norm one projections (we refer to [76] and [94] for more information on the π-property).

We have the following general result, which confirms that our intuition of finding a property of Banach spaces, which guarantees the existence of many 1-complemented subspaces, was in the right direction. This result will give us many positive examples of denseness in both norm-attaining tensor and nuclear operator cases (see Examples 4.3.12).

Theorem 4.3.8. *Let X be a Banach space satisfying the metric π-property.*

(a) If Y has the metric π-property, then $\overline{\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)}^{\|\cdot\|_\pi} = X\widehat{\otimes}_\pi Y$.

(b) If Y is uniformly convex, then $\overline{\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)}^{\|\cdot\|_\pi} = X\widehat{\otimes}_\pi Y$.

Proof. (a). Let $u \in S_{X\widehat{\otimes}_\pi Y}$ and $\varepsilon > 0$ be given. By [107, Proposition 2.8], there are bounded sequences $\{\lambda_n\}_{n=1}^\infty \subseteq \mathbb{R}^+$, $\{x_n\}_{n=1}^\infty \subseteq S_X$, and $\{y_n\}_{n=1}^\infty \subseteq S_Y$ with $u = \sum_{n=1}^\infty \lambda_n x_n \otimes y_n$ and

$$\sum_{n=1}^{\infty} \lambda_n < 1 + \varepsilon. \tag{4.3.2}$$

Find $k \in \mathbb{N}$ large enough so that $\|u-z\|_{\pi_{X\widehat{\otimes}_\pi Y}} < \frac{\varepsilon}{2}$ for $z := \sum_{n=1}^k \lambda_n x_n \otimes y_n$. Since X and Y have the metric π-property, we can find finite-dimensional subspaces X_0 of X and Y_0 of Y which are 1-complemented and such that, for every $n \in \{1,\ldots,k\}$, there are $x_n' \in X_0$ and $y_n' \in Y_0$ such that

$$\max\{\|x_n - x_n'\|, \|y_n - y_n'\|\} < \frac{\varepsilon}{4k\lambda_n}.$$

Define $z' = \sum_{n=1}^k \lambda_n x_n' \otimes y_n'$ and notice that $\|z'-z\|_{\pi_{X\widehat{\otimes}_\pi Y}} < \frac{\varepsilon}{2}$. Moreover, note that $z' \in X_0 \otimes Y_0$. We have that X_0 is 1-complemented in X and

Y_0 is 1-complemented in Y. Consequently, by [107, Proposition 2.4] we get that norm of $X\widehat{\otimes}_\pi Y$ agrees on $X_0 \otimes Y_0$ with the norm of $X_0\widehat{\otimes}_\pi Y_0$. In particular,

$$\|z'\|_{\pi_{X_0\widehat{\otimes}_\pi Y_0}} = \|z'\|_{\pi_{X\widehat{\otimes}_\pi Y}}. \tag{4.3.3}$$

Finally, since X_0, Y_0 are finite-dimensional spaces, we use Proposition 4.2.5 to show that z' attains its projective norm in $X_0\widehat{\otimes}_\pi Y_0$. Since (4.3.3) holds, z' attains its norm in $X\widehat{\otimes}_\pi Y$ and we are done.

(b). Let $u \in S_{X\widehat{\otimes}_\pi Y}$ and $\varepsilon > 0$ be given. There are bounded sequences $\{\lambda_n\}_{n=1}^\infty \subseteq \mathbb{R}^+$, $\{x_n\}_{n=1}^\infty \subseteq S_X$, and $\{y_n\}_{n=1}^\infty \subseteq S_Y$ with $u = \sum_{n=1}^\infty \lambda_n x_n \otimes y_n$ and (4.3.2) holds. We can find k large enough so that $\|u - z\|_{\pi_{X\widehat{\otimes}_\pi Y}} < \frac{\varepsilon}{3}$ for $z := \sum_{n=1}^k \lambda_n x_n \otimes y_n$. Since X satisfies the metric π-property, we can find a finite-dimensional subspace X_0 which is 1-complemented and such that for every $n \in \{1, \ldots, k\}$, there is $x_n' \in X_0$ such that $\|x_n - x_n'\| < \frac{\varepsilon}{6k\lambda_n}$. Define $z' = \sum_{n=1}^k \lambda_n x_n' \otimes y_n$. Notice that $\|z' - z\|_{\pi_{X\widehat{\otimes}_\pi Y}} < \frac{\varepsilon}{3}$ and that $z' \in X_0 \otimes Y$. Since X_0 is finite-dimensional and Y is uniformly convex, by Corollary 4.3.6, we can find $z'' \in X_0\widehat{\otimes}_\pi Y$ such that

$$\|z' - z''\|_{\pi_{X_0\widehat{\otimes}_\pi Y}} < \frac{\varepsilon}{3} \text{ with } z'' = \sum_{n=1}^\infty a_n \otimes b_n \text{ and } \|z''\|_{\pi_{X_0\widehat{\otimes}_\pi Y}} = \sum_{n=1}^\infty \|a_n\|\|b_n\|.$$

Since the norm of $X\widehat{\otimes}_\pi Y$ agrees on $X_0 \otimes Y$ with the norm of $X_0\widehat{\otimes}_\pi Y$, the result follows as in the previous item. ■

Let us notice that if a Banach space Z satisfies the metric π-property, then it has the metric approximation property, and then the analogous result for nuclear operators follows immediately from Theorem 4.3.8 and [107, Corollary 4.8].

Corollary 4.3.9. *Let X be Banach space such that X^* satisfies the metric π-property.*

(a) If Y has the metric π-property, then $\overline{\mathrm{NA}_{\mathcal{N}}(X,Y)}^{\|\cdot\|_{\mathcal{N}}} = \mathcal{N}(X,Y)$.

(b) If Y is uniformly convex, then $\overline{\mathrm{NA}_{\mathcal{N}}(X,Y)}^{\|\cdot\|_{\mathcal{N}}} = \mathcal{N}(X,Y)$.

To finish this section, let us see particular cases where Theorem 4.3.8 and Corollary 4.3.9 can be applied. The next examples show that we always have denseness in all classical Banach spaces. Note that item (a) below tells us that the metric π-property happens very often. Also, the stability results, (d), (e), (f), and (g), allow us to get more positive examples on denseness. We will first recall the following definition.

Definition 4.3.10. Let X be a Banach space. A sequence $\{X_n\}_{n=1}^{\infty}$ of finite-dimensional subspaces of X is called a *finite-dimensional decomposition* of X (FDD for short) if every $x \in X$ has a unique representation of the form $x = \sum_{n=1}^{\infty} x_n$ with $x_n \in X_n$ for every $n \in \mathbb{N}$.

Remark 4.3.11. A FDD on a Banach space X determines a sequence $\{P_n\}_{n=1}^{\infty}$ of projections (called the *partial sum projections* of the decomposition) such that if $x = \sum_{n=1}^{\infty} x_n \in X$, then $P_j(x) = \sum_{n=1}^{j} x_n$ for all $j \in \mathbb{N}$. These projections are commuting, have increasing range, and converge strongly to the identity operator on X. The supremum of the norms of those projections is finite and is called the *decomposition constant*.

Example 4.3.12. The following Banach spaces satisfy the metric π-property (we include the proofs for the sake of completeness).

(a) Banach spaces with a finite-dimensional decomposition with the decomposition constant 1 (consequently, every Banach space with Schauder basis can be renormed to have the metric π-property);

(b) $L_p(\mu)$-spaces for any $1 \leqslant p < \infty$ and any measure μ;

(c) Isometric preduals of L_1;

(d) $X \oplus_a Y$, whenever X, Y satisfy the metric π-property and $|\cdot|_a$ is an absolute norm;

(e) $X = \left[\bigoplus_{n\in\mathbb{N}} X_n\right]_{c_0}$ or $\left[\bigoplus_{n\in\mathbb{N}} X_n\right]_{\ell_p}$, for all $1 \leqslant p < \infty$, with X_n satisfying the metric π-property for all n;

(f) $X\widehat{\otimes}_\pi Y$, whenever X, Y satisfy the metric π-property;

(g) $X\widehat{\otimes}_\varepsilon Y$, whenever X, Y satisfy the metric π-property.

Proof. (a). Given a Banach space X, if there exists a sequence of finite-dimensional Banach spaces and 1-complemented subspaces $\{E_n\}_{n\in\mathbb{N}}$ such that $E_n \subseteq E_{n+1}$ holds for every n and such that $\bigcup_{n\in\mathbb{N}} E_n$ is dense in X, then X has the metric π-property. In particular, it applies whenever X is a Banach space with an FDD with the decomposition constant 1 (if $P_n : X \longrightarrow X$ are the associated norm-one projections, take $E_n := P_n(X)$).

(b). Let $1 \leqslant p < \infty$ be given. Let us write $X = L_p(\mu)$, for short. Consider $x_1, \ldots, x_n \in S_X, \varepsilon > 0$. For every $i \in \{1, \ldots, n\}$, we can find a simple function $x_i' \in S_X$ such that

$$\|x_i - x_i'\| < \varepsilon, \tag{4.3.4}$$

where $x_i' = \sum_{j=1}^m a_{ij}\chi_{A_j}$ for suitable $m \in \mathbb{N}$, $a_{ij} \in \mathbb{R}$ and pairwise disjoint $A_j \in \Sigma$. Now, in order to prove that X has the metric π-property, define $M := \operatorname{span}\{\chi_{A_j} : 1 \leqslant j \leqslant m\}$ and let us construct $P : X \longrightarrow X$ by the equation

$$T(f) := \sum_{j=1}^m \frac{1}{\mu(A_j)} \int_{A_j} f \, d\mu \chi_{A_j}.$$

It is clear from the disjointedness of $A_1, \ldots, A_m$ and the fact that $\|P(f)\| \leqslant \|f\|$ holds for every $f \in X$. Furthermore, it is clear from the definition that $P(f) = f$ holds for every $f \in M$, so P is a norm-one projection onto M. The result follows since $x_i' \in M$ and by the arbitrariness of $\varepsilon > 0$. This proves (b).

(c). Let X be an isometric predual of L_1. Let $\varepsilon > 0$ and $\{x_1, \ldots, x_n\} \subset S_X$ be given. Define $F_1 = \{0\}$ and $F_2 = \operatorname{span}\{x_1, \ldots, x_n\}$. By [92, Theorem 3.1] and [101, Theorem 1.3], we may find a subspace E of X such that E is isometric to ℓ_∞^m for some $m \in \mathbb{N}$ and $d(x, E) < \varepsilon$ for all $x \in F_2$. For each $1 \leqslant i \leqslant n$, pick $x_i' \in E$ so that $\|x_i - x_i'\| < \varepsilon$. By [100, Lemma 2.1], there exists a norm one projection P from X to E; hence E is indeed an 1-complemented finite-dimensional subspace of X.

(d). To prove that the metric π-property is stable by absolute sums, let us first notice that S_X, in its definition, can be replaced by B_X (indeed, let $\varepsilon > 0$ and $\{x_1, \ldots, x_n\} \subset B_X$ be given; without loss of generality, we may assume that $x_i \neq 0$ for all $1 \leqslant i \leqslant n$; from the metric π-property, we may find a 1-complemented finite-dimensional space M of X with $x_i' \in M$ such that $\|x_i/\|x_i\| - x_i'\| < \varepsilon$ for every $1 \leqslant i \leqslant n$; thus, $\|x_i - \|x_i\|x_i'\| < \varepsilon$ and $\{\|x_1\|x_1', \ldots, \|x_n\|x_n'\} \subset M$). Set $Z = X \oplus_a Y$. Let $\varepsilon > 0$ and $\{z_1, \ldots, z_n\} \subset S_Z$ be given. If we write $z_i = (x_i, y_i)$ for each $1 \leqslant i \leqslant n$, then $\max\{\|x_i\|, \|y_i\|\} \leqslant \|z_i\|_a = 1$ for every $1 \leqslant i \leqslant n$. As X has the metric π-property and $\{x_1, \ldots, x_n\} \subset B_X$, there exist a 1-complemented finite-dimensional subspace M of X and $\{x_1', \ldots, x_n'\} \subseteq M$ such that $\|x_i - x_i'\| < \varepsilon$. Similarly, there exist a 1-complemented finite-dimensional subspace N of Y and $\{y_1', \ldots, y_n'\} \subset N$ such that $\|y_i - y_i'\| < \varepsilon$. If we let $z_i' = (x_i', y_i')$ for each $1 \leqslant i \leqslant n$, then for every $1 \leqslant i \leqslant n$, we have

$$\|z_i - z_i'\|_a \leqslant \|x_i - x_i'\| + \|y_i - y_i'\| < 2\varepsilon.$$

Let P and Q be norm one projections from X onto M and Y onto N, respectively. Consider the map (P, Q) defined on $X \oplus_a Y$ as $(x, y) \mapsto (P(x), Q(y))$ for every $(x, y) \in X \oplus_a Y$. Note that (P, Q) is a projection with (closed) range $M \oplus_a N$. Moreover,

$$\|(P(x), Q(y))\|_a = |(\|P(x)\|, \|Q(y)\|)|_a \leqslant |(\|x\|, \|y\|)|_a = \|(x, y)\|_a$$

for every $(x, y) \in X \oplus_a Y$; hence $M \oplus_a N$ is a 1-complemented finite-dimensional subspace of Z with $\{z_1', \dots, z_n'\} \subset M \oplus_a N$ satisfying $\|z_i - z_i'\| < 2\varepsilon$ for each $1 \leqslant i \leqslant n$.

(e). This can be obtained by extending the proof of (d). Indeed, let $\{x_1, \dots, x_n\} \subseteq S_X$ be given. First, approximate x_i by x_i' of finite support. Now, say $x_i' = (x_{i1}, \dots, x_{ik}, 0, 0, \dots)$ with some common $k \in \mathbb{N}$. Find a 1-complemented subspace M_j in X_j containing $x_{1j}, \dots, x_{nj}$ from the assumption that X_j enjoys the metric π-property for each $1 \leqslant j \leqslant k$. Then, $M = \{(z_1, z_2, \dots, z_k, 0, 0, \dots) : z_i \in M_i, 1 \leqslant i \leqslant k\}$ is a finite-dimensional subspace of X which is 1-complemented by the projection $(P_1, P_2, \dots, P_k, 0, 0, \dots)$ (defined similarly as in the item (d)) and M contains the set $\{x_1', \dots, x_n'\}$.

(f). Let $\varepsilon > 0$ and $z_1, \dots, z_n \in S_{X \widehat{\otimes}_\pi Y}$ be given. For each $1 \leqslant i \leqslant n$, consider $\{x_j^{(i)}, y_j^{(i)}\} \subseteq B_X \times B_Y$ to be such that

$$z_i = \sum_{j=1}^{\infty} x_j^{(i)} \otimes y_j^{(i)} \quad \text{with} \quad \|z_i\|_\pi > \sum_{j=1}^{\infty} \|x_j^{(i)}\| \|y_j^{(i)}\| - \varepsilon.$$

For each $i = 1, \dots, n$, let $N_i \in \mathbb{N}$ be such that

$$\sum_{j=N_i+1}^{\infty} \|x_j^{(i)}\| \|y_j^{(i)}\| < \frac{\varepsilon}{2}.$$

Now, since X has the metric π-property, there exists a 1-complemented finite-dimensional subspace M with $\left\{ \widetilde{x_j}^{(i)} : 1 \leqslant j \leqslant N_i, 1 \leqslant i \leqslant n \right\} \subseteq M$ such that

$$\|\widetilde{x_j}^{(i)} - x_j^{(i)}\| < \min \left\{ \frac{\varepsilon}{4N_i} : 1 \leqslant i \leqslant n \right\}$$

and, analogously, there exists a 1-complemented finite-dimensional subspace N of Y with $\left\{\widetilde{y}_j^{(i)} : 1 \leqslant j \leqslant N_i, 1 \leqslant i \leqslant n\right\} \subseteq N$ such that

$$\|\widetilde{y}_j^{(i)} - y_j^{(i)}\| < \min\left\{\frac{\varepsilon}{4N_i} : 1 \leqslant i \leqslant n\right\}$$

for each $1 \leqslant j \leqslant N_i$ with $i = 1, \ldots, n$. By [107, Proposition 2.4], $M\widehat{\otimes}_\pi N$ is an 1-complemented space. Let $\widetilde{z}_i := \sum_{j=1}^{N_i} \widetilde{x}_j^{(i)} \otimes \widetilde{y}_j^{(i)}$. Then,

$$\left\| \widetilde{z}_i - \sum_{j=1}^{N_i} x_j^{(i)} \otimes y_j^{(i)} \right\|_\pi \leqslant 2N_i \min\left\{\frac{\varepsilon}{4N_i} : 1 \leqslant i \leqslant n\right\} \leqslant \frac{\varepsilon}{2}$$

for every $i = 1, \ldots, n$. Then, $X\widehat{\otimes}_\pi Y$ has the metric π-property, as desired.

(g). Let $z_1, \ldots, z_n \in S_{X\widehat{\otimes}_\varepsilon Y}$ and $\delta > 0$ be given. For each $i \in \{1, \ldots, n\}$, let $\widetilde{z}_i \in X \otimes Y$ be such that $\|z_i - \widetilde{z}_i\|_\varepsilon < \frac{\delta}{2}$. Let $\sum_{j=1}^{N_i} x_j^{(i)} \otimes y_j^{(i)}$ be a representation of $\widetilde{z}_i$ for each $i = 1, \ldots, n$. Since

$$\begin{aligned} &\{x_j^{(i)} : 1 \leqslant j \leqslant N_i, 1 \leqslant i \leqslant n\} \subseteq X \quad \text{and} \\ &\{y_j^{(i)} : 1 \leqslant j \leqslant N_i, 1 \leqslant i \leqslant n\} \subseteq Y, \end{aligned}$$

there are 1-complemented finite-dimensional subspaces $M \leqslant X$ and $N \leqslant Y$ with $\{\widetilde{x}_j^{(i)} : 1 \leqslant j \leqslant N_i, 1 \leqslant i \leqslant n\} \subseteq M$ and $\{\widetilde{y}_j^{(i)} : 1 \leqslant j \leqslant N_i, 1 \leqslant i \leqslant n\} \subseteq N$ such that

$$\begin{aligned} &\|x_j^{(i)} - \widetilde{x}_j^{(i)}\| < \min\left\{\frac{\varepsilon}{4N_i} : 1 \leqslant i \leqslant n\right\} \text{ and} \\ &\|y_j^{(i)} - \widetilde{y}_j^{(i)}\| < \min\left\{\frac{\varepsilon}{4N_i} : 1 \leqslant i \leqslant n\right\}. \end{aligned}$$

As $M\widehat{\otimes}_\varepsilon N$ is a 1-complemented subspace of $X\widehat{\otimes}_\varepsilon Y$ (see, for instance, [107, Proposition 3.2]),

$$\widetilde{v}_i = \sum_{j=1}^{N_i} \widetilde{x}_j^{(i)} \otimes \widetilde{y}_j^{(i)} \in M\widehat{\otimes}_\varepsilon N \quad \text{and} \quad \|\widetilde{z}_i - \widetilde{v}_i\|_\varepsilon \leqslant \|\widetilde{z}_i - \widetilde{v}_i\|_\pi \leqslant \frac{\delta}{2},$$

which implies that $\|z_i - \widetilde{v}_i\|_\varepsilon < \delta$, we have that $X\widehat{\otimes}_\varepsilon Y$ satisfies the metric π-property. ■

Remark 4.3.13. From the estimates of case (g) above it follows that $X\widehat{\otimes}_\alpha Y$ has the metric π-property whenever X and Y enjoy the metric π-property and α is a *uniform cross norm* (see [107, Section 6.1] for background and details).

Example 4.3.12.(g) allows us to extend Theorem 4.3.8 for larger projective tensor products.

Corollary 4.3.14. *Let $N \in \mathbb{N}$ be given. Let $X_1, \ldots, X_N$ be Banach spaces with the metric π-property, and Y be a Banach space. Then,*

$$\overline{\operatorname{NA}_\pi(X_1\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_N\widehat{\otimes}_\pi Y)}^{\|\cdot\|_\pi} = X_1\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_N\widehat{\otimes}_\pi Y.$$

4.4 Tensors not approximable by norm-attaining tensors

By the results from previous section, one may think that the denseness for norm-attaining tensors may always hold. In this section, we will see that this is not the case. We show that there are Banach spaces X and Y such that the set of all tensors in $X\widehat{\otimes}_\pi Y^*$ which attain their projective norms *is not* dense in $X\widehat{\otimes}_\pi Y^*$. In order to do that, let us notice that, by Theorem 4.2.1, it would be enough to show that

$\mathrm{NA}(X, Y^{**}) \cap B_{L(X,Y^{**})}$ is not norming for $X \widehat{\otimes}_\pi Y^*$ (and in fact that is what we do; see Remark 4.4.4). On the other hand, in view of the proof of [91, Proposition 2.3], note that if either X or Y^{**} satisfies the metric approximation property (respectively, bounded approximation property), then $\mathcal{F}(X, Y^{**})$ is norming (respectively, K-norming) for $X \widehat{\otimes}_\pi Y^*$, and this implies that $\mathcal{F}(X, Y^{**})$ is w^*-dense in $\mathcal{L}(X, Y^{**})$. This suggests us to look for our counterexample in the context of Banach spaces failing the approximation property and trying to guarantee that the set of operators which attain their norms is not bigger than the set of finite-rank operators. This is the reason why we will adapt [97, Theorem 1] taking into account all the previous considerations.

For this, we will use Read's space $\mathcal{R}$ (see [79, 80, 103] for all the details on this space). Read's space is a renorming of the Banach space c_0, $\mathcal{R} = (c_0, |||\cdot|||)$, whose bidual $\mathcal{R}^{**}$ is strictly convex (see [79, Theorem 4]). This implies that $\mathrm{NA}(X, \mathcal{R}^{**}) \subseteq \mathcal{F}(X, \mathcal{R}^{**})$ whenever X is a closed subspace of c_0 (see [97, Lemma 2]). It is worth mentioning that we are not using here the deep properties of $\mathcal{R}$ (that it contains no two-codimensional proximal subspaces) but only the fact that its bidual is strictly convex for the bidual norm and that it contains c_0 (this is in fact well-known; the existence of such norms can be justified, for instance, by using [80, Lemma 2.1] and taking R as a one-to-one operator from c_0 into ℓ_2).

Theorem 4.4.1. *Let $\mathcal{R}$ be Read's space. There exist subspaces X of c_0 and Y of $\mathcal{R}$ such that the set of tensors in $X \widehat{\otimes}_\pi Y^*$ which attain their projective norms is not dense in $X \widehat{\otimes}_\pi Y^*$.*

In order to prove Theorem 4.4.1, we would like to present several previous results which are interesting themselves.

Lemma 4.4.2. *Let X, Y be a Banach spaces such that Y^* is separable. If $\mathcal{F}(X, Y^{**})$ is viewed as a subspace of $(X \widehat{\otimes}_\pi Y^*)^* = \mathcal{L}(X, Y^{**})$, we* *have*

$$B_{\mathcal{F}(X,Y^{**})} \subset \overline{B_{\mathcal{F}(X,Y)}}^{w^*}.$$

Proof. Let $T \in \mathcal{F}(X, Y^{**})$ with $\|T\| < 1$. Choose a countable dense subset $\{y_n^*\}_{n=1}^\infty$ of Y^* and let $F_n = \operatorname{span}\{y_1^*, \ldots, y_n^*\}$ for each $n \in \mathbb{N}$. By the Principle of Local Reflexivity (see [53, Theorem 9.15] for instance), for each $n \in \mathbb{N}$, there exists an operator $\phi_n : T(X) \to Y$ such that

1. $\left(1 - \frac{1}{n}\right) \|T(x)\| \leqslant \|\phi_n(T(x))\| \leqslant \left(1 + \frac{1}{n}\right) \|T(x)\|$ for every $x \in X$,

2. $y^*(\phi_n(T(x))) = y^*(T(x))$ for every $y^* \in F_n$ and $x \in X$.

Choose $n_0 \in \mathbb{N}$ so that $\frac{1}{n} < \frac{1}{\|T\|} - 1$ whenever $n \geqslant n_0$. Let us define $K_n = \phi_n \circ T \in \mathcal{F}(X, Y)$ for each $n \geqslant n_0$. Then $\|K_n\| \leqslant \|\phi_n\|\|T\| < 1$ for each $n \geqslant n_0$. We claim that $K_n \xrightarrow{w^*} T$. First, observe that given $x \in X$ and $m \in \mathbb{N}$, we have

$$y_m^*(K_n(x)) = y_m^*(\phi_n(T(x))) = y_m^*(T(x)) \text{ for every } n \geqslant m. \tag{4.4.1}$$

Now, let $x \in X \backslash \{0\}$, $y^* \in Y^*$ and $\varepsilon > 0$ be given. Pick $n_0 \in \mathbb{N}$ so that $\|y_{n_0}^* - y^*\| < \frac{\varepsilon}{2\|x\|}$. By (4.4.1), we have for $n \geqslant n_0$,

$$\begin{aligned}
|y^*(K_n(x)) - y^*(T(x))| & \\
&\leqslant |y^*(K_n(x)) - y_{n_0}^*(K_n(x))| + |y_{n_0}^*(K_n(x)) - y_{n_0}^*(T(x))| \\
&\qquad + |y_{n_0}^*(T(x)) - y^*(T(x))| \\
&\leqslant \|y^* - y_{n_0}^*\|\|K_n\|\|x\| + \|y_{n_0}^* - y^*\|\|T\|\|x\| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

By a linearity argument we get that $K_n(z) \to T(z)$ for every $z \in X \otimes Y$. Finally, since the sequence K_n is bounded we get that $K_n \to T$ in the w^*-topology.

This implies that $\{T \in \mathcal{F}(X, Y^{**}) : \|T\| < 1\} \subset \overline{B_{\mathcal{F}(X,Y)}}^{w^*}$. As a w^*-closed set in $\mathcal{L}(X, Y^{**})$ is $\|\cdot\|$-closed, we conclude that $B_{\mathcal{F}(X,Y^{**})} \subset \overline{B_{\mathcal{F}(X,Y)}}^{w^*}$. ■

In what follows, we will be using the strong operator topology (SOT, for short) and the weak operator topology (WOT, for short). Recall that the *strong operator topology* in $\mathcal{L}(X, Y)$ is the topology defined by the basic neighborhoods

$$N(T; A, \varepsilon) = \{S \in \mathcal{L}(X, Y) : \|(T - S)(x)\| < \varepsilon, x \in A\},$$

where A is an arbitrary finite subset of X and $\varepsilon > 0$. Thus, in the SOT, a net (T_α) converges to T if and only if $(T_\alpha(x))$ converges to $T(x)$ for every $x \in X$. On the other hand, the *weak operator topology* is defined by the basic neighborhoods

$$N(T; A, A^*, \varepsilon) = \{S \in \mathcal{L}(X, Y), |y^*(T - S)(x)| < \varepsilon, y^* \in A^*, x \in A\},$$

where A and A^* are arbitrary finite sets in X and Y^*, respectively, and $\varepsilon > 0$. Thus, in the WOT, a net T_α converges to T if and only if $(y^*(T_\alpha(x)))$ converges to $y^*(T(x))$ for every $x \in X$ and $y^* \in Y^*$.

Let us notice that a convex set in $\mathcal{L}(X, Y)$ has the same closure in the WOT as it does in the SOT (see, for instance, [52, Corollary 5, page 477]). We will use this fact in the proof of Theorem 4.4.1 below.

Lemma 4.4.3. *Let X be a Banach space failing the approximation property. Then, the identity map on X does not belong to $R\overline{B_{\mathcal{F}(X,X)}}^{WOT}$ for any $R > 0$.*

Proof. Let X be a Banach space which fails the approximation property and let us denote the identity map on X by Id_X. Then, by definition, $\mathrm{Id}_X \notin \overline{\mathcal{F}(X, X)}^{\tau}$, where τ is the topology of uniform convergence on compact sets. For given $R > 0$, let us prove that $\mathrm{Id}_X \notin R\overline{B_{\mathcal{F}(X,X)}}^{SOT}$. In order to get a contradiction, let us assume $\mathrm{Id}_X \in R\overline{B_{\mathcal{F}(X,X)}}^{SOT}$. Then there exists a net $\{T_\alpha\}_{\alpha\in\Lambda} \subset RB_{\mathcal{F}(X,X)}$ such that $T_\alpha \xrightarrow{SOT} \mathrm{Id}_X$. Now, let K be a compact set in X and $\varepsilon > 0$ be given. Choose a $\left(\min\left\{\frac{\varepsilon}{3R}, \frac{\varepsilon}{3}\right\}\right)$-net $\{x_1, \ldots, x_k\}$ for K. Pick $\alpha_0 \in \Lambda$ such that for every $\alpha \geqslant \alpha_0$

$$\max_{1\leqslant i\leqslant k} \|T_\alpha(x_i) - \mathrm{Id}_X(x_i)\| = \max_{1\leqslant i\leqslant k} \|T_\alpha(x_i) - x_i\| < \frac{\varepsilon}{3}.$$

Given $x \in K$, take $i \in \{1, \ldots, k\}$ so that $\|x - x_i\| < \min\left\{\frac{\varepsilon}{3R}, \frac{\varepsilon}{3}\right\}$. Then,

$$\begin{aligned}
\|T_\alpha(x) - \mathrm{Id}_X(x)\| &\leqslant \|T_\alpha(x) - T_\alpha(x_i)\| + \|T_\alpha(x_i) - x_i\| + \|x_i - x\| \\
&\leqslant \|T_\alpha\|\|x - x_i\| + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\end{aligned}$$

for every $\alpha \geqslant \alpha_0$. This implies that $\mathrm{Id}_X \in \overline{\mathcal{F}(X,X)}^{\tau}$, a contradiction. So, $\mathrm{Id}_X \notin R\overline{B_{\mathcal{F}(X,X)}}^{SOT} = R\overline{B_{\mathcal{F}(X,X)}}^{WOT}$. ■

Now we are ready to prove Theorem 4.4.1.

Proof of Theorem 4.4.1. Let X be a closed subspace of c_0 which fails the approximation property (see, for instance, [95, Theorem 2.d.6]). Then, by Lemma 4.4.3, the identity map on X does not belong to $R\overline{B_{\mathcal{F}(X,X)}}^{WOT}$ for any $R > 0$. Let $Y = (X, |||\cdot|||)$, where $|||\cdot|||$ is the norm that defines Read's space. Let us denote by $\iota \in \mathcal{L}(X,Y)$ the formal identity map from X to Y. Then $T = \iota/\|\iota\|$ does not belong to $R\overline{B_{\mathcal{F}(X,Y)}}^{WOT}$ for any $R > 0$. It follows that T does not belong to $R\overline{B_{\mathcal{F}(X,Y)}}^{w^*}$ for any $R > 0$, where the previous weak-star topology refers to $\sigma(\mathcal{L}(X,Y^{**}), X\widehat{\otimes}_\pi Y^*)$. Indeed, if $T \in R\overline{B_{\mathcal{F}(X,Y)}}^{w^*}$ for some $R > 0$, given $x \in X$, $y^* \in Y^*$ and $\varepsilon > 0$, there exists $T_0 \in RB_{\mathcal{F}(X,Y)}$ such that

$$|y^*(T(x) - T_0(x))| = |(T - T_0)(x \otimes y^*)| < \varepsilon,$$

which implies that $T \in R\overline{B_{\mathcal{F}(X,Y)}}^{WOT}$, a contradiction. In particular, T does not belong to $\overline{B_{\mathcal{F}(X,Y)}}^{w^*}$. As Y^* is separable, by Lemma 4.4.2, T does not belong to $\overline{B_{\mathcal{F}(X,Y^{**})}}^{w^*}$. Thus, by the Hahn-Banach theorem we have that the unit ball $B_{\mathcal{F}(X,Y^{**})}$ is not norming for $X\widehat{\otimes}_\pi Y^*$. Take $z \in X\widehat{\otimes}_\pi Y^*$ with $\|z\|_\pi = 1$ and $\alpha > 0$ such that

$$\sup\{|G(z)| : G \in B_{\mathcal{F}(X,Y^{**})}\} < 1 - \alpha. \tag{4.4.2}$$

Claim: $\operatorname{dist}\big(z, \mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y^*)\big) > \dfrac{\alpha}{2}$.

If this is not the case, there exists $z' \in \mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y^*)$ such that $\|z-z'\|_\pi \leqslant \frac{\alpha}{2}$. This implies that $\|z'\|_\pi \geqslant 1 - \frac{\alpha}{2}$. Let $G \in \mathcal{L}(X, Y^{**})$ with $\|G\| = 1$ such that $|G(z')| = \|z'\|_\pi$. In particular, $G \in \mathrm{NA}(X, Y^{**})$ by Theorem 4.2.1. Notice that $Y^{**} = Y^{\perp\perp}$ is a closed subspace of $\mathcal{R}^{**}$, so Y^{**} is strictly convex. Thus, we have that $G \in \mathcal{F}(X, Y^{**})$ by [97, Lemma 2], which implies by (4.4.2) that $|G(z)| < 1 - \alpha$. Nevertheless,

$$|G(z)| \geqslant |G(z')| - \|z - z'\|_\pi \geqslant 1 - \frac{\alpha}{2} - \frac{\alpha}{2} = 1 - \alpha,$$

which is a contradiction. ■

Remark 4.4.4. Notice that from the above proof it follows that, given two Banach spaces X and Y, if $\mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y)$ is dense in $X\widehat{\otimes}_\pi Y$, then $\mathrm{NA}(X, Y^*) \cap B_{\mathcal{L}(X,Y^*)}$ is norming for $X\widehat{\otimes}_\pi Y$.

In fact, from the proof of Theorem 4.4.1 (and its lemmas) we extract more information. Recall that for every non-zero tensor $u \in X \otimes Y$, there is a smallest $N \in \mathbb{N} \cup \{\infty\}$ for which there is a representation for z containing N terms. The number N is known as the *rank* of u. Because of this, we will say that $u \in X \otimes Y$ is a *finite-rank tensor* if the rank of u is finite. Although it is not known whether every finite-rank operator can be approximated by norm-attaining operators, the case for tensors does not hold in general.

Proposition 4.4.5. *There are tensors of finite-rank which do not attain their projective norm.*

Proof. Consider X and Y^* as in Theorem 4.4.1. Then, there exist $\alpha > 0$ and $z \in X\widehat{\otimes}_\pi Y^*$ such that $\operatorname{dist}(z, \mathrm{NA}_\pi(X\widehat{\otimes}_\pi Y^*)) \geqslant \alpha$. Now, take u of finite-rank such that $\|z-u\|_\pi < \frac{\alpha}{2}$. Then, this element cannot attain its projective norm. ■

As we have commented at the beginning of this section, let us notice that from the proof of Theorem 4.4.1, there exist some Banach spaces X and Y such that $\mathrm{NA}(X, Y^{**}) \cap B_{\mathcal{L}(X,Y^{**})}$ is not w^*-dense in $B_{\mathcal{L}(X,Y^{**})}$. Actually, we have the following result.

Corollary 4.4.6. *There are Banach spaces X and Y such that*

$$\overline{\operatorname{conv}(\mathrm{NA}(X, Y^{**}) \cap B_{\mathcal{L}(X,Y^{**})})}^{w^*} \neq B_{\mathcal{L}(X,Y^{**})}.$$

Chapter 5

Linear spaces consisting of strongly norm-attaining Lipschitz mappings

5.1 Introduction and motivation

According to Rmoutil's result [104], there exists an infinite-dimensional Banach space X (namely c_0 in the equivalent norm constructed by Read [103]) such that the set $\mathrm{NA}(X, \mathbb{K}) \subset X^*$ of norm-attaining linear functionals does not contain two-dimensional linear subspaces. That is a negative answer to [64, Problem III] by Godefroy. Read's construction was generalized in [80], where such equivalent norms with "extremely nonlineable set of norm-attaining functionals" were constructed in all separable and some non-separable Banach spaces containing c_0.

In the first half of this chapter, we address an analogous question for metric spaces M and the set $\mathrm{SNA}(M)$ of strongly norm-attaining Lipschitz functions on M. Surprisingly, for Lipschitz functions the answer happens to be just the opposite, as we will see: for every infinite M the corresponding set $\mathrm{Lip}_0(M)$ always has linear subspaces of dimension at least 2 consisting of strongly norm-attaining functionals, and in fact, it contains much bigger linear spaces in general. After figuring out this new fact, we study some other natural questions about possible sizes for closed linear subspaces in $\mathrm{SNA}(M)$.

It was shown in [36, Theorem 3.2] that if M is an infinite metric space, then $\mathrm{Lip}_0(M)$ contains linear subspaces isomorphic to ℓ_∞, and later, an isometric version of this result was given in [37, Theorem 5]. However, the proofs cannot be adapted to the setting of strongly norm-attaining Lipschitz mappings in general and, as we will show in Theorem 5.3.3, for separable M, the non-separable space ℓ_∞ cannot be embedded in $\mathrm{SNA}(M)$, neither isometrically nor isomorphically.

5.1.1 Preliminaries

All vector spaces in this chapter will be assumed to be real without explicit mention. Let (M, d) be a *pointed metric space* (that is, a metric space consisting of at least 2 points and containing a distinguished point 0). We will usually consider only one metric on M which permits us to write just M for the metric space instead of (M, d). We use the standard notation $\mathrm{Lip}_0(M)$ for the space of all Lipschitz mappings $f : M \to \mathbb{R}$ such that $f(0) = 0$ endowed with the Lipschitz constant as the norm, that is

$$\|f\| = \sup\left\{\frac{|f(x) - f(y)|}{d(x,y)} : x, y \in M,\, x \neq y\right\}.$$

We refer to the book [114] for a solid background on Lipschitz mappings.

There is a natural way to define the norm-attainment in this context. According to [83], a Lipschitz function $f \in \mathrm{Lip}_0(M)$ is said to *strongly* *attain its norm* if there is a pair of points $x, y \in M$ with $x \neq y$ such that

$$\|f\| = \frac{|f(x) - f(y)|}{d(x, y)}.$$

We will denote the set of Lipschitz functions from $\mathrm{Lip}_0(M)$ that attain their norm strongly by $\mathrm{SNA}(M)$ (the notations $\mathrm{SA}(M)$ and $\mathrm{LipSNA}(M)$ have also been used in the literature).

The reason behind calling this natural norm-attainment *strong* is that this is a restrictive notion, and other weaker notions of norm-attainment, which are also natural and give interesting results, have also been introduced and studied since the initial works on the topic [66, 83] (see for instance [32, Section 1] for a very clean exposition of various kinds of norm-attainment for Lipschitz mappings and the relations between them).

The systematic study of norm-attaining Lipschitz mappings was started in [66] and [83]. Since then, a fruitful line of research arose and continues to be very active nowadays (just to list some relevant references on the topic, we refer to [15, 24, 27–30, 32, 60, 61, 65, 66, 78, 83, 84] and the references therein for a solid background on the topic). As we just mentioned, the notion of strong norm-attainment is a bit restrictive. This can be justified by the following facts:

- If a Lipschitz function f strongly attains its norm at some pair of points $x \neq y$, then f strongly attains its norm at any pair of different points in between them (see [83, Lemma 2.2] for the details).
- If M is a complete *length metric space* (that is, if for every $x \neq y \in M$, the distance $d(x, y)$ is equal to the infimum of the length of rectifiable curves joining them; note that every normed

space is a metric length space), then $\mathrm{SNA}(M)$ is never dense in $\mathrm{Lip}_0(M)$ (for the details, see [24, Theorem 2.2], which improves [83, Theorem 2.3], and check also [14, 62, 73] for more background and characterizations on complete length metric spaces).

However, despite the above results, positive results have also been achieved in this direction in the recent years for some metric spaces (see for instance [24, Section 3] and [29]). Evidently, if M is finite of cardinal $n \in \mathbb{N}$, then $\mathrm{SNA}(M) = \mathrm{Lip}_0(M)$, which means that it is already an $(n-1)$-dimensional linear space, so we are mainly focused in infinite metric spaces.

An important tool in the study of Lipschitz mappings is the concept of *Lipschitz-free spaces* (also referred to as *Arens-Eells spaces* and *Transportation cost spaces* in the literature). Given a metric space M, denote $\delta : M \to (\mathrm{Lip}_0(M))^*$ the canonical embedding given by $\delta(x) = \delta_x$, $x \in M$, where δ_x is the evaluation functional $f \mapsto f(x)$. Then the norm-closed linear span $\mathcal{F}(M)$ of $\delta(M)$ in $\mathrm{Lip}_0(M)$ is called the *Lipschitz-free space* over M. The space $\mathcal{F}(M)$ can be seen isometrically as a predual of $\mathrm{Lip}_0(M)$ (see [24, Section 1] and the survey [65] for a solid background on Lipschitz-free spaces). The identification $(\mathcal{F}(M))^* = \mathrm{Lip}_0(M)$ can be explained as follows: every Lipschitz mapping $f : M \to \mathbb{R}$ can be identified with the continuous linear mapping $\hat{f} : \mathcal{F}(M) \to \mathbb{R}$ given by $\hat{f}(\delta_p) \mapsto f(p)$ for $p \in M$ and extended to the whole $\mathcal{F}(M)$ by linearity and continuity. This identifies isometrically the spaces $\mathrm{Lip}_0(M)$ and $(\mathcal{F}(M))^*$. It is easy to check that $\mathrm{SNA}(M)$ can be identified with the set of those elements of $\mathcal{L}(\mathcal{F}(M), \mathbb{R})$ that attain their norm at a point of the form $\frac{\delta_x - \delta_y}{d(x,y)}$, for different $x, y \in M$. This identification has been used to get many results about strongly norm-attaining Lipschitz mappings (see for instance [24, Section 3], [28, Section 2], [29], [60, Section 7] and

[61, Section 4]). We refer to [37, 85, 102] for works where the possibility to embed ℓ_1 into Lipschitz-free spaces was studied.

Remark also, that the structure of $\mathrm{Lip}_0(M)$ and $\mathrm{SNA}(M)$ does not depend on the selection of the distinguished point 0: if M' is the same metric space but with another distinguished point $0'$ then the mapping $f \mapsto f - f(0')$ is a bijective linear isometry between $\mathrm{Lip}_0(M)$ and $\mathrm{Lip}_0(M')$, which maps $\mathrm{SNA}(M)$ to $\mathrm{SNA}(M')$ in fact, and it is $w^* - w^*$ continuous, which proves that the Lipschitz-free spaces do not depend on the choice of the origin either.

Another important tool is the well-known *McShane's extension theorem* [114, Theorem 1.33] that allows us to extend $f \in \mathrm{Lip}_0(M_1)$ to $\tilde{f} \in \mathrm{Lip}_0(M_2)$, with $M_1 \subset M_2$, in such a way that $\|f\| = \|\tilde{f}\|$.

Given a metric space M, in this chapter, the expression *linear subspaces of* $\mathrm{SNA}(M)$ should be understood as linear subspaces of $\mathrm{Lip}_0(M)$ consisting of strongly norm-attaining Lipschitz functions. Also, we use below the following slang. Let Y be a Banach space and M be a pointed metric space. We say that Y embeds in $\mathrm{SNA}(M)$ (or equivalently $\mathrm{SNA}(M)$ contains a copy of Y), if there is a linear isometric embedding $U : Y \to \mathrm{Lip}_0(M)$ such that $U(Y) \subset \mathrm{SNA}(M)$.

To finish this section, we will briefly discuss the structure of the rest of the chapter. The first half, contains two extra sections. In Section 5.2, we show that if a metric space M has more than $n \in \mathbb{N}$ points, then $\mathrm{SNA}(M)$ contains n-dimensional subspaces (see Theorem 5.2.7 and Corollary 5.2.9), solving in the positive our original question. In Section 5.3 we study other related questions, such as the possible sizes for linear subspaces of $\mathrm{SNA}(M)$ (see Proposition 5.3.1), or how "small" a metric space M can be if the Banach space Y is a subspace of $\mathrm{SNA}(M)$ (see Theorem 5.3.3). We also show that the existence of Banach spaces in $\mathrm{SNA}(M)$ has restrictions for σ-precompact metric spaces M (see

Proposition 5.3.7), and, on the other hand, we show that if a metric M contains $[0, 1]$ isometrically, then subspaces of $\mathrm{SNA}(M)$ can be infinite-dimensional (see Proposition 5.3.9).

In the very recent work [15], Avilés, Martínez-Cervantes, Rueda Zoca, and Tradacete, managed to show that, in fact, if M is any infinite complete metric space, then $\mathrm{SNA}(M)$ always contains c_0 isomorphically, answering [84, Questions 1 and 2], and they asked if this could be always done isometrically (see [15, Remark 3.6]). We will devote the second half of the chapter to tackle this question. In Section 5.4, some important tools will be presented. In Section 5.5, we will study and solve the question from [15, Remark 3.6]. In particular, we will show that the embedding of c_0 can be isometric if M is not uniformly discrete (see Subsection 5.5.2), but in the uniformly discrete case we find several counterexamples with very different behaviours (see Subsections 5.5.1 and 5.5.3). Finally, in Section 5.6, we will provide a result in the non-separable setting.

5.2 Finite-dimensional subspaces

In this section, we will study the existence of n-dimensional linear subspaces in $\mathrm{SNA}(M)$, where M is a pointed metric space and $n \in \mathbb{N}$. Our main result from the section states that if M contains at least 2^n points (in particular, if M is infinite), then $\mathrm{SNA}(M)$ contains an isometric copy of ℓ_1^n (see Theorem 5.2.7). This provides a shocking contrast when compared to Rmoutil's result from the classical theory of norm-attaining functionals (see [104]). In order to prove our main result in this direction, we need a bit of preparatory work.

First of all, recall that if a finite pointed metric space M has exactly $n > 1$ distinct points, for some $n \in \mathbb{N}$, then $\mathrm{Lip}_0(M) = \mathrm{SNA}(M)$, and it is an $(n-1)$-dimensional Banach space.

Remark 5.2.1. Note that, in general, we cannot claim that if a Banach space Y is a linear subspace of $\mathrm{SNA}(K)$ for some metric space K, then Y is also linearly isometric to a subspace of $\mathrm{SNA}(M)$ for metric spaces M containing K as a subspace. One may be tempted to use McShane's extension theorem in order to try to get such a result, but the extensions do not behave well like a linear space in general. However, the well-behaving norm $\|\cdot\|_1$ will allow us to get a result in this direction, as Lemma 5.2.2 below shows.

Lemma 5.2.2. *Let M be a pointed metric space such that for some subspace K of M,* $\mathrm{SNA}(K)$ *contains a linear subspace isometrically isomorphic to ℓ_1^n for some $n \in \mathbb{N}$ (respectively, ℓ_1). Then,* $\mathrm{SNA}(M)$ *also contains a linear subspace isometrically isomorphic to ℓ_1^n (respectively, ℓ_1).*

Proof. We will prove the finite-dimensional case, that is, ℓ_1^n, as the infinite-dimensional case (ℓ_1) can be proven with the same method. Let $E \subset \mathrm{Lip}_0(K)$ be a linear isometric copy of ℓ_1^n consisting of strongly norm-attaining functionals. Then, there are $f_1, \ldots, f_n \in S_E \subset S_{\mathrm{Lip}_0(K)}$ such that for all $a_1, \ldots, a_n \in \mathbb{R}$,

$$\left\| \sum_{k=1}^{n} a_k f_k \right\| = \sum_{k=1}^{n} |a_k|.$$

Let $g_1, \ldots, g_n \in S_{\mathrm{Lip}_0(M)}$ be norm-preserving extensions of $f_1, \ldots, f_n$ respectively. Then, by the triangle inequality, for all $a_1, \ldots, a_n \in \mathbb{R}$,

$$\left\| \sum_{k=1}^{n} a_k g_k \right\| \leqslant \sum_{k=1}^{n} |a_k|.$$

On the other hand, there is a pair of different points $t_1, t_2 \in K$ at which $\sum_{k=1}^n a_k f_k$ strongly attains its norm. This gives us

$$\left\| \sum_{k=1}^n a_k g_k \right\| \geqslant \frac{\left| \left(\sum_{k=1}^n a_k g_k\right)(t_1) - \left(\sum_{k=1}^n a_k g_k\right)(t_2) \right|}{d(t_1,t_2)}$$
$$= \frac{\left| \left(\sum_{k=1}^n a_k f_k\right)(t_1) - \left(\sum_{k=1}^n a_k f_k\right)(t_2) \right|}{d(t_1,t_2)} = \sum_{k=1}^n |a_k|,$$

so $\sum_{k=1}^n a_k g_k$ strongly attains its norm. ■

In particular, if we were able to embed ℓ_1^n spaces isometrically in $\mathrm{SNA}(M)$ for a finite pointed metric space M, we could use the previous lemma to obtain the same result for all metric spaces containing M. We will provide now a constructive proof of the fact that if a metric space M has 4 points, then $\mathrm{SNA}(M)$ contains ℓ_1^2 isometrically. Naturally, by Lemma 5.2.2, this implies that for bigger metric spaces, the result remains true.

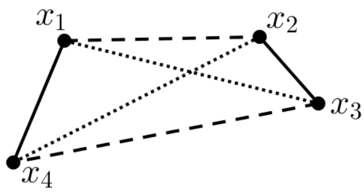
We will use the following notation. Given a function $f \in \mathrm{Lip}_0(M)$ and two points $\alpha, \beta \in M$ with $\alpha \neq \beta$, the notation $S(f,\alpha,\beta)$ will denote the incremental slope of f from α to β, that is,

$$S(f,\alpha,\beta) := \frac{f(\beta)-f(\alpha)}{d(\alpha,\beta)}.$$

Proposition 5.2.3. *Let $M = \{0,a,b,c\}$ be a pointed metric space consisting in exactly 4 points. Then $\mathrm{Lip}_0(M)$ has a linear subspace that is isometrically isomorphic to ℓ_1^2.*

Proof. We will first rename the points of M as x_1, x_2, x_3, x_4 satisfying that $x_1 = 0$ and that

$$d(x_1,x_4) + d(x_2,x_3) = \min\left\{d(\alpha,\beta) + d(\gamma,\delta) : M = \{\alpha,\beta,\gamma,\delta\}\right\}. \tag{5.2.1}$$



$$d(x_1, x_4) \leq d(x_1, x_2) + d(x_3, x_4)$$
$$d(x_1, x_4) \leq d(x_1, x_3) + d(x_2, x_4)$$

Figure 5.1. Visualization of the condition (5.2.1).

Define the following mappings:

$$f_1 : \begin{cases} f_1(x_1) = 0, \\ f_1(x_2) = d(x_1, x_4) - d(x_2, x_4), \\ f_1(x_3) = d(x_1, x_4) - d(x_2, x_4) + d(x_2, x_3), \\ f_1(x_4) = d(x_1, x_4), \end{cases}$$

$$f_2 : \begin{cases} f_2(x_1) = 0, \\ f_2(x_2) = d(x_1, x_2), \\ f_2(x_3) = d(x_1, x_2) - d(x_2, x_3), \\ f_2(x_4) = d(x_1, x_4). \end{cases}$$

We will verify that the linear space generated by f_1 and f_2 is isometrically isomorphic to ℓ_1^2. Let us study the slopes of all 6 pairs of points for each of these functions. For f_1 we have the following.

- $S(f_1, x_1, x_4) = 1$.
- $S(f_1, x_2, x_3) = 1$.
- $S(f_1, x_2, x_4) = 1$.
- Note that, by the triangle inequality, it holds that

$$-d(x_1, x_2) \leqslant d(x_1, x_4) - d(x_2, x_4) \leqslant d(x_1, x_2).$$

Therefore, $|S(f_1, x_1, x_2)| \leqslant 1$.

- $S(f_1, x_3, x_4) = \frac{d(x_2,x_4)-d(x_2,x_3)}{d(x_3,x_4)}$. Note now that, by the triangle inequality, it holds that

$$-d(x_3, x_4) \leqslant d(x_2, x_4) - d(x_2, x_3) \leqslant d(x_3, x_4).$$

Therefore, $|S(f_1, x_3, x_4)| \leqslant 1$.

- Finally, $S(f_1, x_1, x_3) = \frac{d(x_1,x_4)-d(x_2,x_4)+d(x_2,x_3)}{d(x_1,x_3)}$. Note once more that

$$-1 \leqslant S(f_1, x_1, x_3) \leqslant 1.$$

Indeed, the first inequality comes from applying the triangle inequality as follows

$$d(x_2, x_4) \leqslant d(x_2, x_3) + d(x_3, x_1) + d(x_1, x_4),$$

and the second inequality is derived directly from (5.2.1).

About f_2, we have the following.

- $S(f_2, x_1, x_4) = 1$.
- $S(f_2, x_2, x_3) = -1$.
- $S(f_2, x_1, x_2) = 1$.
- Note that, by the triangle inequality, it holds that

$$-d(x_2, x_4) \leqslant d(x_1, x_4) - d(x_1, x_2) \leqslant d(x_2, x_4).$$

Therefore, $|S(f_2, x_2, x_4)| \leqslant 1$.

- By the triangle inequality, it holds that

$$-d(x_1, x_3) \leqslant d(x_1, x_2) - d(x_2, x_3) \leqslant d(x_1, x_3).$$

  Therefore, $|S(f_2, x_1, x_3)| \leqslant 1$.

- Finally, $S(f_2, x_3, x_4) = \frac{d(x_1,x_4)-d(x_1,x_2)+d(x_2,x_3)}{d(x_3,x_4)}$. Note once more that

$$-1 \leqslant S(f_2, x_3, x_4) \leqslant 1.$$

  Indeed, the first inequality comes from applying the triangle inequality as follows

$$d(x_1, x_2) \leqslant d(x_1, x_4) + d(x_4, x_3) + d(x_3, x_2),$$

  and the second inequality is derived directly from (5.2.1).

Thus, we have obtained two mappings $f_1, f_2 \in \mathrm{SNA}(M)$, both with Lipschitz constant 1, and such that

$$\begin{aligned} S(f_1, x_1, x_4) &= 1, \quad S(f_1, x_2, x_3) = 1 \\ S(f_2, x_1, x_4) &= 1, \quad S(f_2, x_2, x_3) = -1. \end{aligned}$$

Let $a_1, b_1 \in \mathbb{R}$, and let $f = a_1 f_1 + a_2 f_2$. Note first that for all $i \neq j \in \{1, 2, 3, 4\}$,

$$S(f, x_i, x_j) = a_1 S(f_1, x_i, x_j) + a_2 S(f_2, x_i, x_j).$$

From here, it is clear that $\|f\| \leqslant |a_1| + |a_2|$. Also, if a_1 and a_2 have the same sign, one gets immediately that

$$|S(f, x_1, x_4)| = |a_1 + a_2| = |a_1| + |a_2|,$$

and if a_1 and a_2 have different sign, then

$$|S(f, x_2, x_3)| = |a_2 - a_3| = |a_2| + |a_3|.$$

This shows that the linear space generated by f_1 and f_2 is isometrically isomorphic to ℓ_1^2, and the proof is now finished. ■

Using Proposition 5.2.3 together with Lemma 5.2.2, we get that any metric space M with more than 2 points (in particular, any infinite metric space) satisfies that $\mathrm{SNA}(M)$ contains a 2-dimensional subspace, which shows a shocking contrast when compared to Rmoutil's result for functionals from [104]. However, we can actually go much further than this. In the recent works [85] and [102], the existence of ℓ_1^n and ℓ_1 subspaces of Lipschitz-free spaces was studied, providing an answer to [37, Question 2]. This has proven to be an important tool in our case, and we will use the cited below first half of [85, Theorem 14.5] in the proof of our main result. For the sake of completeness and easy reference, we include below the formal statement.

Lemma 5.2.4 (First half of [85, Theorem 14.5])**.** *For every $n \in \mathbb{N}$, if a pointed metric space M contains $2n$ elements, then $\mathcal{F}(M)$ contains a 1-complemented subspace isometric to ℓ_1^n.*

Recall that it is not true in general that if Y is a subspace of a Banach space X, then Y^* is isometric to a subspace of X^* (for instance, recall that ℓ_1 embeds isometrically in $C([0, 1])$, but ℓ_∞ does not embed isometrically in $C([0, 1])^*$); however, the scenario is different if Y is 1-complemented.

Lemma 5.2.5. *Let X be a Banach space that contains a 1-complemented subspace Y. Then Y^* embeds isometrically as a subspace of X^*.*

Proof. Let $P : X \to Y$ be a norm-one projection. Consider the mapping $U : Y^* \to X^*$ such that for all $y^* \in Y^*$, $U(y^*) := y^* \circ P$, that is, for all

$x \in X$, $U(y^*)(x) := y^*(P(x))$. Then U is an isometric embedding, as desired. Indeed, just note that for each $y^* \in Y^*$, we have

$$\|U(y^*)\| = \sup_{x \in B_X} \|U(y^*(x))\| = \sup_{x \in B_X} |y^*(P(x))| = \sup_{y \in B_Y} |y^*(y)| = \|y^*\|.$$

∎

Finally, recall the following well-known result, for which it is sufficient to consider the span of n vectors in $\ell_\infty^{2^{n-1}}$ with ± 1 coordinates, built analogously to the Rademacher functions on $[0,1]$ (that is, the basis would be formed by the vectors $(1,1,\ldots,1)$, $(1,-1,1,-1,\ldots,-1))$, $(1,1,-1,-1,1,\ldots,-1)$, and so on).

Lemma 5.2.6. *If $n \in \mathbb{N}$, then ℓ_1^n is isometric to a subspace of $\ell_\infty^{2^{n-1}}$.*

We now have all the necessary tools for the proof of the main result of the section.

Theorem 5.2.7. *Let $n > 1$ be a natural number, and let M be a pointed metric space with at least 2^n distinct points. Then, there exists a linear subspace of* $\mathrm{SNA}(M)$ *which is isometrically isomorphic to ℓ_1^n.*

Proof. First of all, consider a metric subspace K of M containing exactly 2^n distinct points. By Lemma 5.2.4, $\mathcal{F}(K)$ contains a 1-complemented subspace isometric to $\ell_1^{2^{n-1}}$. Recall that $\mathcal{F}(K)^*$ is isometric to $\mathrm{Lip}_0(K)$, and that $(\ell_1^{2^{n-1}})^*$ is isometric to $\ell_\infty^{2^{n-1}}$, so by Lemma 5.2.5, $\mathrm{Lip}_0(K) = \mathrm{SNA}(K)$, and it contains a subspace isometric to $\ell_\infty^{2^{n-1}}$. Applying Lemma 5.2.6 we deduce that $\mathrm{SNA}(K)$ contains a subspace isometric to ℓ_1^n as well. Finally, by Lemma 5.2.2, $\mathrm{SNA}(M)$ also contains a subspace isometric to ℓ_1^n. ∎

Corollary 5.2.8. *If M is an infinite pointed metric space, then for all $n \in \mathbb{N}$,* $\mathrm{SNA}(M)$ *contains an n-dimensional subspace isometric to ℓ_1^n.*

Corollary 5.2.9. *Let $n \in \mathbb{N}$. For a pointed metric space M, the following statements are equivalent:*

1. $\mathrm{SNA}(M)$ *contains n-dimensional linear subspaces.*
2. *M contains at least $n+1$ points.*

5.3 Infinite-dimensional subspaces

We start this section by showing that there exist metric spaces M for which $\mathrm{SNA}(M)$ contains "big" Banach subspaces. Actually, any Banach space Y can be a subspace of $\mathrm{SNA}(M)$ for a suitable metric space M.

Proposition 5.3.1. *If Y is a Banach space, then it is a subspace of* $\mathrm{SNA}(B_{Y^*})$.

Proof. Let Y be any Banach space. Consider the metric space B_{Y^*}. For each $y \in Y$, let $\delta_y : B_{Y^*} \to \mathbb{R}$ be the evaluation map $\delta_y(y^*) := y^*(y)$, for all $y^* \in B_{Y^*}$, which is clearly a linear mapping. For each $y \in Y$, there exists some $y^* \in B_{Y^*}$ such that $y^*(y) = \|y\|$. It is immediate to check that δ_y is in $\mathrm{Lip}_0(B_{Y^*})$ with Lipschitz constant $\|y\|$, and that it attains its norm strongly at the pair $(0, y^*)$. Therefore, Y is a subspace of $\mathrm{SNA}(B_{Y^*})$. ■

A natural question arises now: given a Banach space Y, how small can a metric space M be so that Y is a linear subspace of $\mathrm{SNA}(M)$? From the previous proposition, it is clear that if Y has separable dual, then M can be chosen to be separable.

What if Y^* is not separable? For instance, we have seen in Theorem 5.2.7 that if M is an infinite pointed metric space, then $\mathrm{SNA}(M)$ contains isometrically all the ℓ_1^n spaces as linear subspaces, so it is natural to wonder if it also contains, say, ℓ_1. However, this is not the case in general, as we are about to see. Theorem 5.3.3 below shows that separability of Y^* actually characterizes the possibility of M being separable. A set $B \subset B_{X^*}$ is a *James boundary of* X if for every $x \in X$, there is $g \in B$ such that $g(x) = \|x\|$ (see [54, Definition 3.118]). In order to prove it, we rely on the concept of James boundary and also on the following result by Gilles Godefroy.

Proposition 5.3.2 (Godefroy, [54, Corollary 3.125])**.** *Let X be a Banach space. If X has a separable James boundary, then X^* is separable.*

Theorem 5.3.3. *For a Banach space Y, the following assertions are equivalent.*

(1) *There is a separable pointed metric space M and a closed linear subspace $Z \subset \mathrm{Lip}_0(M)$ such that Z is isometric to Y and $Z \subset \mathrm{SNA}(M)$.*

(2) *There is a separable Banach space X and a closed linear subspace $Z_1 \subset X^*$ such that Z_1 is isometric to Y and $Z_1 \subset \mathrm{NA}(X, \mathbb{R})$.*

(3) *Y^* is separable.*

Proof. (1) implies (2): It is sufficient to consider $X = \mathcal{F}(M)$ and use the identification of $\mathrm{Lip}_0(M)$ with X^*. With this identification $Z \subset \mathrm{Lip}_0(M)$ identifies with a subspace of $Z_1 \subset X^*$ and all elements of Z_1 remain to be norm-attaining as elements of X^*.

(2) implies (3): Assume that such a separable Banach space X exists, denote as usual $J_X : X \to X^{**}$ the canonical embedding of X into its bidual and $R : X^{**} \to Z_1^*$ the natural restriction operator. The condition $Z_1 \subset \mathrm{NA}(X, \mathbb{R})$ means that for every $f \in Z_1$ there is $x \in B_X$ such that

$f(x) = \|f\|$, so in other words $((R \circ J_X)(x))(f) = \|f\|$. Consequently, $(R \circ J_X)(B_X)$ is a separable James boundary of Z_1, so Z_1^* is separable by Proposition 5.3.2). Hence, we have that Y^* is separable.

(3) implies (1): If Y^* is separable, take $M = B_{Y^*}$ and apply Proposition 5.3.1. ■

As a consequence of the above result, there exist infinite metric spaces M such that $\mathrm{SNA}(M)$ does not contain linear subspaces isometrically isomorphic to ℓ_1.

Remark 5.3.4. Note that a direct proof that (2) implies (1) in Theorem 5.3.3 can be achieved by considering $M = B_X$. In this case, the operator U that maps each $f \in X^*$ to its restriction on M is an isometric embedding with the property that if f is norm-attaining then $U(f)$ is strongly norm-attaining on M. So the subspace $Z := U(Z_1)$ is what we are looking for.

The previous result shows that for separable metric spaces M, the spaces we can find in $\mathrm{SNA}(M)$ must satisfy some restrictions. The next Theorem 5.3.7 shows in a similar way that if M is "small" then the restrictions on Banach subspaces in $\mathrm{SNA}(M)$ happen to be much stronger. In the proof we will use [58, Corollary 2.2]. Recall that a Banach space X is *polyhedral* if the unit ball of every finite-dimensional subspace of X is a polytope. A space that is isomorphic to a polyhedral space is said to be *isomorphically polyhedral*.

Proposition 5.3.5 ([58, Corollary 2.2])**.** *If X has a boundary that can be covered by a set of the form $\bigcup_{j=1}^{\infty} \overline{\mathrm{conv}}^{w^*}(K_j)$, where each K_j is countably infinite and w^*-compact, then X is isomorphically polyhedral.*

Corollary 5.3.6. *If X has a boundary that can be covered by a countable number of compact sets, then X is separable and isomorphically polyhedral.*

Proof. Let the boundary W of X be covered by $\bigcup_{j=1}^{\infty} W_j$ for compact sets W_j. Then, the boundary is separable, so by Godefroy's result (Proposition 5.3.2), X^* is separable, and then X is separable as well.

According to [96, Proposition 1.e.2], every compact subset W_j is included in a subset of the form $\overline{\text{conv}}\{x^*_{j,k}\}_{k=1}^{\infty} \subset X^*$ where $\|x^*_{j,k}\| \xrightarrow{k\to\infty} 0$. Thus, the boundary W has the property from Proposition 5.3.5, so X is isomorphically polyhedral. ■

Remark that the same result follows from an "internal" characterization from [57]. A metric space is said to be *σ-precompact* if it is a countable union of precompact sets. We can now prove the announced result.

Theorem 5.3.7. *Let M be a σ-precompact pointed metric space, then all Banach subspaces in* $\text{SNA}(M)$ *are separable and isomorphic to polyhedral spaces.*

Proof. Let $\{M_n\}_{n=1}^{\infty}$ be a sequence of precompact sets satisfying that $M \subset \bigcup_{j=1}^{\infty} M_n$. For each $n \in \mathbb{N}$, denote $\Delta_n = \{\delta_x : x \in M_n\} \subset \text{Lip}_0(M)^*$. By our assumption, each Δ_n is precompact in $\text{Lip}_0(M)^*$. Then $\overline{\text{aconv}}(\Delta_n - \Delta_m)$ is compact for every $n, m \in \mathbb{N}$ (where $\text{aconv}(A)$ denotes the *absolute convex hull* of A, and $\overline{\text{aconv}}(A)$ is its closure). The set

$$\text{Mol} = \left\{ \frac{\delta_t - \delta_\tau}{d(t,\tau)} : t \neq \tau \in M \right\} \subset \bigcup_{m,n,k=1}^{\infty} k \cdot \overline{\text{aconv}}(\Delta_n - \Delta_m)$$

is covered by a countable number of norm-compact sets.

Let $Y \subset \text{SNA}(M)$ be a Banach space. Denote $R : \text{Lip}_0(M)^* \to Y^*$ the natural restriction operator. Then by the continuity of R, $R(\text{Mol})$ is covered by a countable number of norm-compact sets as well. The set $R(\text{Mol}) \bigcap S_{Y^*}$ forms a boundary for Y (by the definition of strong norm-attainment), so the statement follows from Corollary 5.3.6. ■

Note that all compact spaces and all $\mathbb{R}^n$ spaces, with $n \in \mathbb{N}$, are σ-compact. In particular, every linear subspace in $\mathrm{SNA}([0,1])$ is separable and isomorphically polyhedral. It is worth noting, however, that such subspaces can be infinite-dimensional, as we will see in Example 5.3.8.

Since all Lipschitz functions are absolutely continuous, one can identify (see for instance [114, Example 1.6.5]) the space $\mathrm{Lip}_0([0,1])$ isometrically with the space $L_\infty([0,1])$, where the isometric isomorphism between them is just the differentiation operator (which exists almost everywhere):

$$\begin{aligned} U : \mathrm{Lip}_0([0,1)] &\to L_\infty([0,1]) \\ f &\mapsto U(f) = f'. \end{aligned}$$

It is clear from this and [83, Lemma 2.2] that $U(\mathrm{SNA}([0,1]))$ is the subset of $L_\infty([0,1])$ consisting of functions that attain their norm $\|\cdot\|_\infty$ throughout an interval with non-empty interior. We get the following result.

Example 5.3.8. If $M = [0,1]$, then $\mathrm{SNA}(M)$ contains linear subspaces isometrically isomorphic to c_0.

Proof. Consider the set A of functions $g : [0,1] \to \mathbb{R}$ such that the following holds for some $a = (a_1, a_2, \dots) \in c_0$:

$$\begin{cases} g(x) = a_k, & \text{if } x \in \left[1 - \frac{1}{k}, 1 - \frac{1}{k+1}\right), \ k = 1, 2, \dots \\ g(1) = 0 \end{cases}$$

Then $(A, \|\cdot\|_\infty)$ is a linear subspace of $U(\mathrm{SNA}([0,1]))$ which is isometrically isomorphic to c_0.

Finally, if we invert the mapping U, we actually get a linear subspace of SNA([0, 1]) which is isometrically isomorphic to c_0, as desired (see Figure 5.2 for a visualization of the process followed in this example). ■

Naturally, the previous example remains true if one changes $[0,1]$ with any other interval $[a,b] \subset \mathbb{R}$ with $a < b$.

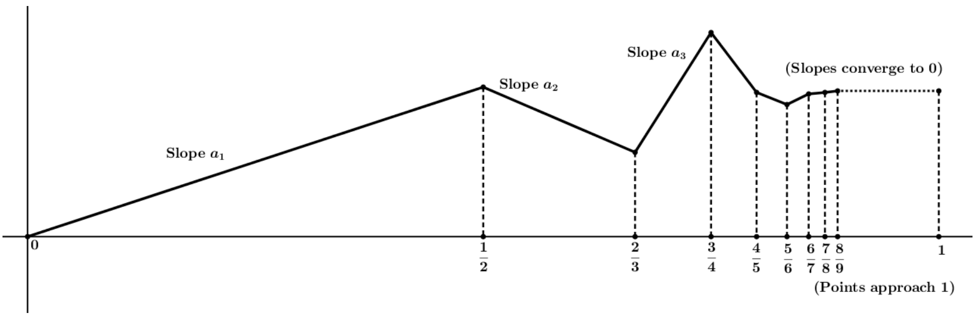


Figure 5.2. Visual construction of c_0 in SNA([0, 1]) as in Example 5.3.8.

Finally, the next result shows that one can extend the existence of c_0 in SNA([0, 1]) to SNA(M) for any pointed metric space M that contains $[0,1]$ isometrically (for instance, any normed space). Recall that a Lipschitz mapping $f : A \subset \mathbb{R}^n \to B \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) is a *Lipschitz retraction* if $f(x) = x$ for all $x \in A$.

Proposition 5.3.9. *If M is any pointed metric space containing* $[0,1]$ *isometrically, then* SNA(M) *contains linear subspaces isometrically isomorphic to c_0.*

Proof. Let M and Z be metric spaces such that $Z \subset M$. Assume that there exists some retraction $F : M \to Z$ with Lipschitz constant 1, that is:

$$\begin{cases} F(z) = z, & \text{for all } z \in Z, \\ d(F(a), F(b)) \leqslant d(a,b), & \text{for all } a, b \in M. \end{cases}$$

Let $T : \mathrm{Lip}_0(Z) \to \mathrm{Lip}_0(M)$ be such that for all $f \in \mathrm{Lip}_0(Z)$, $T(f) := f \circ F$. Thus, for all $x \in M$, we have $(T(f))(x) = f(F(x))$. It is clear that T is linear. Moreover, $T(\mathrm{Lip}_0(Z))$ is a subspace of $\mathrm{Lip}_0(M)$ (and $T(\mathrm{SNA}(Z)$ is a subset of $\mathrm{SNA}(M))$. Hence any linear subspace of $\mathrm{Lip}_0(Z)$ yields a subspace of $\mathrm{Lip}_0(M)$.

All that remains is to note that if X is any metric space containing $[0,1]$ isometrically, then the mapping F exists. Indeed, the identity operator Id on $[0,1]$ is a Lipschitz function with constant 1, and by McShane's extension theorem, it can be extended to the whole X preserving its Lipschitz constant. ■

Observe that Proposition 5.3.9 applies to all normed spaces. This should be once more compared with the classical theory of norm-attaining functionals, where there exist Banach spaces X such that $\mathrm{NA}(X,\mathbb{R})$ does not have 2-dimensional subspaces (see [104]).

5.4 The isometric embedding of c_0: Technical tools

As mentioned earlier, the existence of infinite-dimensional spaces in $\mathrm{Lip}_0(M)$ has already been studied (see for instance [36, 37, 71]), but, as Theorem 5.3.3 shows, the techniques used in those papers do not work for $\mathrm{SNA}(M)$. In this second half of the chapter, we will study the existence of infinite-dimensional subspaces of $\mathrm{SNA}(M)$ if M is infinite.

Recall that in Theorem 5.2.7 we saw that if M is infinite, then $\mathrm{SNA}(M)$ contains all the ℓ_1^n spaces isometrically for $n \in \mathbb{N}$. Note, however, that not every metric space M can satisfy that ℓ_1 is isometrically contained in $\mathrm{SNA}(M)$ (see Theorem 5.3.3). In fact, Theorems 5.3.3 and 5.3.7 tell us that if the metric space M is "small", then the subspaces of $\mathrm{SNA}(M)$ need to satisfy some restrictions.

On the other hand, recall that, by Proposition 5.3.9, any metric space M isometrically containing $[0,1]$ satisfies that $\mathrm{SNA}(M)$ contains c_0 isometrically (note that this includes all normed spaces). In fact, as we will see later, using some geometrical constructions, it is possible to show that every metric space M with an infinite amount of cluster points satisfies that c_0 is isometrically contained in $\mathrm{SNA}(M)$. But even without that condition, the spaces M that we were able to originally study in detail satisfy that $\mathrm{SNA}(M)$ contains c_0 at least isomorphically. This motivated us to ask the following two natural questions in [84].

Question 5.4.1 ([84, Question 1])**.** *Is it true that for every infinite complete pointed metric space M the corresponding $\mathrm{SNA}(M)$ contains infinite-dimensional closed (or at least non-closed) linear subspaces?*

Question 5.4.2 ([84, Question 2])**.** *Is it true that for every infinite complete pointed metric space M the corresponding $\mathrm{SNA}(M)$ contains an isomorphic copy of c_0?*

In the very recent work [15], Avilés, Martínez-Cervantes, Rueda Zoca, and Tradacete, by means of an elegant case distinction, and with the help of Ramsey's theorem, were able to solve in the positive both of these questions, as they proved the following.

Theorem 5.4.3 ([15, Main Theorem])**.** *Let M be an infinite complete pointed metric space. Then $\mathrm{SNA}(M)$ contains an isomorphic copy of c_0.*

As for the isometric case, they were able to show that metric spaces M satisfying a certain geometrical condition satisfy that $\mathrm{SNA}(M)$ contains c_0 isometrically (this includes, for instance, every metric space with an infinite amount of cluster points, and every discrete metric space which is not uniformly discrete).

Lemma 5.4.4 ([15, Lemma 3.1])**.** *Let M be a metric space. Assume that there is a sequence $B(x_n, R_n)$, $n \in \mathbb{N}$, of balls satisfying the following conditions:*

1. *$d(B(x_i, R_i), B(x_j, R_j)) > 0$ for every $n \neq m$,*
2. *$\frac{R_i+R_j}{d(B(x_i,R_i),B(x_j,R_j))} < \frac{1}{2}$ for every $i \neq j$, and*
3. *for every $n \in \mathbb{N}$, there is $y_n \in B(x_n, R_n) \backslash \{x_n\}$ such that $d(x_n, y_n) < d(y_n, M \backslash B(x_n, R_n))$.*

Then, for every $n \in \mathbb{N}$, there is a norm-one Lipschitz function f_n with $\frac{f_n(y_n)-f_n(x_n)}{d(x_n,y_n)} = 1$, so that $\{f_n\}$ is isometric to the c_0-basis and SNA(M) *contains* $\overline{\text{span}}\{f_n\}$.

At the end of their paper, in [15, Remark 3.6], the authors asked if this could be extended to all infinite metric spaces, that is: "if M is an infinite (complete) pointed metric space, then does SNA(M) contain c_0 isometrically?" In the remaining part of this chapter, we will provide a definitive answer to that question. We will show that there exist metric spaces for which such embedding of c_0 cannot be done isometrically. We will show, however, that the isometric embedding can be done for a wide class of metric spaces (in fact, for any metric space that is not uniformly discrete), and we will also provide some results about the non-separable scenario. In order to prove our main results, we need some preparatory work. We will first introduce, or recall, some notations and concepts which will be used throughout the rest of the chapter.

Recall once more that all vector spaces in this document are real, and so, the notation $c_0(\Gamma)$ should always be understood as the space $c_0(\Gamma, \mathbb{R})$. Let M be a metric space. The *separation radius* of a point $x \in M$ is defined by

$$R(x) := \inf \Big\{ d(x, y) \colon y \in M \backslash \{x\} \Big\},$$

and it will be central in some of the upcoming results. We will say that a point x from a metric space M *attains its separation radius* whenever there is $y \in M$ such that $R(x) = d(x, y)$.

The symbol M' stands for the set of all cluster points of M. Recall that a metric space M is said to be *discrete* if $M' = \varnothing$, *uniformly discrete* if $\inf\{R(x) : x \in M\} > 0$, and *proper* if every closed and bounded subset of M is compact (note that every proper space is σ-compact, although the converse is not true in general).

Let X be a separable Banach space with a Schauder basis denoted by $\{x_n\}_{n=1}^{\infty}$. We say that a sequence $\{y_n\}_{n=1}^{\infty}$ in a Banach space Y is *(isometrically) equivalent* to the basis $\{x_n\}_{n=1}^{\infty}$ if there exists a linear (isometric) isomorphism $T : \overline{\operatorname{span}}\{y_n : n \in \mathbb{N}\} \to X$ such that $T(y_n) = x_n$ for all $n \in \mathbb{N}$. The following straightforward facts will be used throughout the text without any explicit reference.

(i) A sequence $\{x_n\}_{n=1}^{\infty}$ is isometrically equivalent to the canonical basis of c_0 if and only if the equality $\left\| \sum_{n=1}^{\infty} \lambda_n x_n \right\| = \max_n |\lambda_n|$ holds for every sequence $\{\lambda_n\}_{n=1}^{\infty} \in c_0$.

(ii) If a sequence $\{x_n\}_{n=1}^{\infty}$ is isometrically equivalent to the canonical basis of c_0, then so is the sequence $\{\varepsilon_n x_n\}_{n=1}^{\infty}$, where $\varepsilon_n \in \{-1, 1\}$ for every $n \in \mathbb{N}$.

(iii) Any subsequence of a sequence which is isometrically equivalent to the canonical basis of c_0 is once again isometrically equivalent to the same basis.

Given any set A and a natural number $k \in \mathbb{N}$, we denote by $A^{[k]}$ the set of all subsets of A with exactly k elements. We will use Ramsey's Theorem intensively throughout the text, which ensures that given any infinite set A and any finite partition of the set $A^{[k]}$, $\{B_1, \ldots, B_n\}$ for some $n \in \mathbb{N}$, there exists an infinite subset S of A and a number $i \in \{1, \ldots n\}$ such that $S^{[k]}$ is contained in B_i (see, for instance, [54, Proposition 6.4]).

We will end this section by stating and proving some auxiliary results that will be crucial for the rest of the chapter. The following are three essential, yet straightforward, statements that hold in any complete metric space. We provide their proofs for the sake of completeness.

Lemma 5.4.5. *Let M be a complete metric space. Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of functions in $\mathrm{Lip}_0(M)$ which is isometrically equivalent to the canonical basis of c_0. Then, for every $n \in \mathbb{N}$, if the function f_n strongly attains its Lipschitz norm at a pair of points $x_n, y_n \in M$, then $f_m(x_n) = f_m(y_n)$ for every $m \in \mathbb{N}\backslash\{n\}$.*

Proof. Let $n \in \mathbb{N}$ be fixed. Suppose that the function f_n strongly attains its Lipschitz norm at a pair of points $x_n, y_n \in M$. Without loss of generality, we may (and we do) assume that $|f_n(x_n) - f_n(y_n)| = f_n(x_n) - f_n(y_n) = d(x_n, y_n)$. We will proceed by contradiction. Let us suppose to the contrary that there exist natural numbers $m \neq n$ such that $f_m(x_n) \neq f_m(y_n)$. We may again suppose without loss of generality that $f_m(x_n) > f_m(y_n)$ (otherwise we may consider the sequence $\{g_k\}_{k=1}^{\infty}$ defined as $g_m = -f_m$ and $g_k = f_k$ for $k \neq m$, which is still equivalent to the c_0 basis). Set $f := f_n + f_m$. Then, we have that

$$
\begin{aligned}
|f(x_n) - f(y_n)| &\geqslant (f_n + f_m)(x_n) - (f_n + f_m)(y_n) \\
&= f_n(x_n) - f_n(y_n) + f_m(x_n) - f_m(y_n) \\
&> d(x_n, y_n),
\end{aligned}
$$

which yields a contradiction with the fact that f is 1-Lipschitz. ■

Lemma 5.4.6. *Let M be a complete metric space. Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of functions in $\mathrm{Lip}_0(M)$ which is equivalent to the canonical basis of c_0. Then, for all $p \in M$, $\lim_{n\to\infty} |f_n(p)| = 0$.*

Proof. Let $T\colon c_0 \to \overline{\operatorname{span}}\{f_n\colon n \in \mathbb{N}\}$ be a linear isomorphism with $T(e_n) = f_n$ for all $n \in \mathbb{N}$ and set $C = \|T\|$. Suppose that for some $p \in M$, the sequence $\{f_n(p)\}_{n=1}^{\infty}$ does not converge to 0. Then, there exists $N \in \mathbb{N}$ such that $\sum_{n=1}^{N} |f_n(p)| > C \cdot d(p, 0)$. However, this implies that there exist $\{\varepsilon_n\}_{n=1}^{N} \subset \{-1, 1\}^N$ such that the function $\sum_{k=1}^{N} \varepsilon_n f_n$ is not C-Lipschitz, contradicting the fact that the operator norm of T is C. ■

The following remark is an immediate consequence of the triangle inequality, and it will be used several times in the chapter together with the following clear fact (that will be used implicitly): if $A, B, C, D > 0$ and $A + B \leqslant C + D$, then one of the elements in $\{A, B\}$ is smaller or equal than one of the elements in $\{C, D\}$.

Remark 5.4.7. Let $f \in \operatorname{Lip}_0(M)$ be given. Suppose that $x, y \in M$ with $x \neq y$ are such that $|f(x) - f(y)| = d(x, y)$. Then, we have that

$$|f(x) - C| + |f(y) - C| \geqslant d(x, y)$$

for every $C \in \mathbb{R}$.

Finally, for the upcoming positive results of the chapter, we need the following generalization of [15, Lemma 3.1].

Lemma 5.4.8. *Let Γ be a nonempty index set. Let M be a pointed metric space such that there exist two sets $\{x_\gamma\}_{\gamma\in\Gamma}, \{y_\gamma\}_{\gamma\in\Gamma} \subset M$ with $x_\gamma \neq y_\gamma$, $x_\alpha \neq x_\beta$ for $\gamma, \alpha, \beta \in \Gamma$, $\alpha \neq \beta$. If $d(x_\alpha, x_\beta) \geqslant d(x_\alpha, y_\alpha) + d(x_\beta, y_\beta)$ for every $\alpha \neq \beta \in \Gamma$, then there is a linear subspace of* $\operatorname{SNA}(M)$ *isometric to $c_0(\Gamma)$.*

Proof. For each $\gamma \in \Gamma$, define $f_\gamma : M \to \mathbb{R}$ by

$$f_\gamma(x) := \max\{0, d(x_\gamma, y_\gamma) - d(x, x_\gamma)\} \quad (x \in M).$$

Let $\lambda := \{\lambda_\gamma \colon \gamma \in \Gamma\} \in c_0(\Gamma)$ and let $\gamma_0 \in \Gamma$ be such that $|\lambda_{\gamma_0}| = \|\lambda\|_\infty$. Finally, set $f : M \to \mathbb{R}$ to be defined as

$$f := \sum_{\gamma \in \Gamma} \lambda_\gamma (f_\gamma - f_\gamma(0)).$$

We will be done when we check that f is an element of $\mathrm{Lip}_0(M)$ with Lipschitz norm $\|\lambda\|_\infty$ strongly attaining its norm at the pair $(x_{\gamma_0}, y_{\gamma_0})$.

It is easy to check that, for all $\gamma \in \Gamma$, f_γ strongly attains its norm at the pair (x_γ, y_γ) with $\|f_\gamma\| = 1$. Also, let us notice that the support of f lies in $\bigcup_{\gamma \in \Gamma} B(x_\gamma, d(x_\gamma, y_\gamma))$. Note that we can assume without loss of generality that $f_\gamma(0) = 0$ for all $\gamma \in \Gamma$.

Let us now prove that $\|f\| = |\lambda_{\gamma_0}|$. Let $x \neq y$ be two points in M. We will distinguish several cases. For simplicity, for each $\gamma \in \Gamma$, denote $\delta_\gamma := d(x_\gamma, y_\gamma)$.

(a) If both x and y lie outside of $\bigcup_{\gamma \in \Gamma} B(x_\gamma, d(x_\gamma, y_\gamma))$, then clearly $|f(x) - f(y)| = 0$.

(b) Assume that $x \notin \bigcup_{\gamma \in \Gamma} B(x_\gamma, \delta_\gamma)$ and that there exists some $\alpha \in \Gamma$ such that $y \in B(x_\alpha, d(x_\alpha, y_\alpha))$. Since $d(x, y) \geqslant d(x_\alpha, x) - d(x_\alpha, y) \geqslant d(x_\alpha, y_\alpha) - d(x_\alpha, y) \geqslant 0$, we have

$$\frac{|f(x) - f(y)|}{d(x, y)} = \frac{|\lambda_\alpha|(d(x_\alpha, y_\alpha) - d(x_\alpha, y))}{d(x, y)} \leqslant |\lambda_{\gamma_0}|.$$

(c) Assume now that there is some $\gamma \in \Gamma$ satisfying that $x, y \in B(x_\gamma, d(x_\gamma, y_\gamma))$. Then, since $d(x_\gamma, y) \leqslant d(x_\gamma, x) + d(x, y)$ and $d(x_\gamma, x) \leqslant d(x_\gamma, y) + d(x, y)$, we have

$$\frac{|f(x) - f(y)|}{d(x, y)} = |\lambda_\gamma| \frac{|(\delta_\gamma - d(x_\gamma, x)) - (\delta_\gamma - d(x_\gamma, y))|}{d(x, y)} \leqslant |\lambda_{\gamma_0}|.$$

(d) Finally, if there are different $\alpha, \beta \in \Gamma$ such that $x \in B(x_\alpha, d(x_\alpha, y_\alpha))$ and $y \in B(x_\beta, d(x_\beta, y_\beta))$, assuming without loss of generality that $|\lambda_\alpha| \geqslant |\lambda_\beta| > 0$, we have

$$\frac{|f(x) - f(y)|}{d(x,y)} = \frac{|\lambda_\alpha(\delta_\alpha - d(x, x_\alpha)) - \lambda_\beta(\delta_\beta - d(y, x_\beta))|}{d(x,y)} = (*).$$

We will distinguish 2 cases now.

Case 1: λ_α and λ_β have the same sign.

We can assume for the following computation that both are positive (if not, we can multiply by $\text{sign}(\lambda_\alpha)$ when needed). For $\gamma \in \{\alpha, \beta\}$, define $g_\gamma : M \to \mathbb{R}$ by $g_\gamma(z) := \lambda_\gamma(\delta_\gamma - d(x_\gamma, z))$, for all $z \in M$. Note that we can also assume that $f(x) \neq f(y)$, since if they were equal we would just have $(*) = 0$.

If $f(x) > f(y)$, we have

$$(*) = \frac{f(x) - f(y)}{d(x,y)} \leqslant \frac{f(x) - g_\alpha(y)}{d(x,y)} = (**),$$

since $f(y) \geqslant 0 \geqslant g_\alpha(y)$. By definition of f and g_α we would then have

$$(**) = \lambda_\alpha \frac{\delta_\alpha - d(x, x_\alpha) - \delta_\alpha + d(x_\alpha, y)}{d(x,y)} \leqslant \lambda_\alpha,$$

where the last inequality comes from using the triangle inequality (indeed, just note that $d(x_\alpha, y) \leqslant d(x_\alpha, x) + d(x, y)$).

On the other hand, if $f(y) > f(x)$, we can do a symmetric argument:

$$(*) = \frac{f(y) - f(x)}{d(x,y)} \leqslant \frac{f(y) - g_\beta(x)}{d(x,y)} = (**),$$

since $f(x) \geqslant 0 \geqslant g_\beta(x)$. By definition of f and g_β we would then have

$$(**) = \lambda_\beta \frac{\delta_\beta - d(y, x_\beta) - \delta_\beta + d(x_\beta, x)}{d(x, y)} \leqslant \lambda_\beta,$$

where the last inequality comes from using the triangle inequality once more (since $d(x_\beta, x) \leqslant d(x_\beta, y) + d(x, y)$).

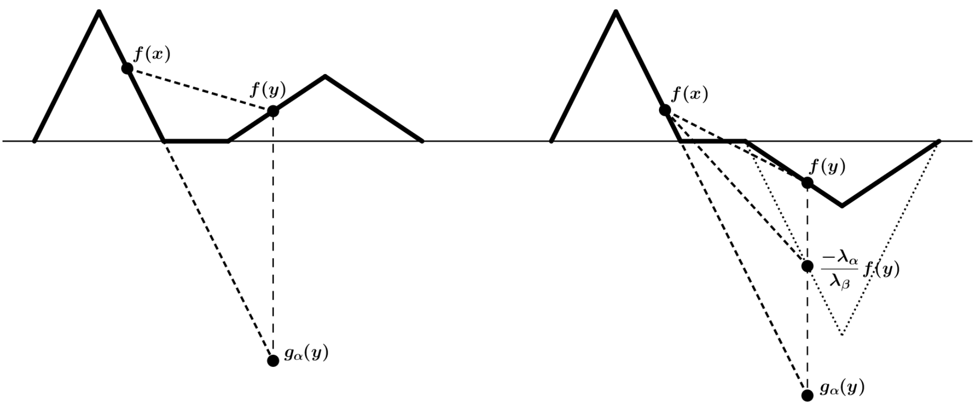


Figure 5.3. Visualization of proof for Case 1 (left) and Case 2 (right). The main inequalities can be seen as a comparison of slopes.

Case 2: λ_α and λ_β have different signs.

We can assume that λ_α is positive and λ_β is negative (else, we could repeat the argument multiplying by (-1) when needed). In this case, we have

$$(*) = \frac{f(x) - f(y)}{d(x, y)} = (**).$$

Note now that $d(x_\alpha, y) + d(y, x_\beta) \geqslant d(x_\alpha, x_\beta) \geqslant \delta_\alpha + \delta_\beta$, so in particular we have that

$$\lambda_\alpha(\delta_\alpha + \delta_\beta) \leqslant \lambda_\alpha(d(x_\alpha, y) + d(y, x_\beta)),$$

which, regrouping terms, leads to

$$\lambda_\alpha(\delta_\alpha - d(x_\alpha, y)) \leqslant (-\lambda_\alpha)(\delta_\beta - d(x_\beta, y)).$$

In other words, if g_α is defined as in Case 1,

$$g_\alpha(y) \leqslant \frac{-\lambda_\alpha}{\lambda_\beta} f(y) \leqslant f(y),$$

since $f(y) \leqslant 0$. Therefore now we can repeat the same argument as in the first subcase of Case 1:

$$(**) \leqslant \frac{f(x) - g_\alpha(y)}{d(x,y)} = \lambda_\alpha \frac{\delta_\alpha - d(x, x_\alpha) - \delta_\alpha + d(x_\alpha, y)}{d(x,y)} \leqslant \lambda_\alpha,$$

by the triangle inequality.

Therefore, in both cases we clearly have

$$\frac{|f(x) - f(y)|}{d(x,y)} \leqslant |\lambda_{\gamma_0}|.$$

This proves that $\|f\| \leqslant |\lambda_{\gamma_0}|$. Finally, it is clear that f strongly attains its Lipschitz norm at the pair $(x_{\gamma_0}, y_{\gamma_0})$, and the proof is over. ■

Remark 5.4.9. Note that the previous lemma is, indeed, a slight generalization of [15, Lemma 3.1] in the following sense. On the one hand, the existence of c_0 is generalized to the existence of $c_0(\Gamma)$ (which can also be done in [15, Lemma 3.1]). On the other hand, it is clear that any metric space satisfying the conditions from [15, Lemma 3.1], trivially satisfies the conditions from Lemma 5.4.8, but the converse is not true in general even if Γ is countable. For instance, consider the metric space M formed by the following points of c_0 endowed with the c_0 metric:

$$\begin{aligned}
x_0 &= (0,0,0,0,\dots) \\
x_1 &= (1,0,0,0,\dots) \\
x_2 &= \left(-\frac{1}{2},\frac{1}{2},0,0,\dots\right) \\
x_3 &= \left(-\frac{1}{3},-\frac{1}{3},\frac{1}{3},0,0,\dots\right) \\
&(\dots) \\
x_n &= \left(\overbrace{-\frac{1}{n},\dots,-\frac{1}{n}}^{(n-1)},\frac{1}{n}0,0,\dots\right), \quad n \in \mathbb{N}.
\end{aligned}$$

Then, this space cannot satisfy the conditions from [15, Lemma 3.1], since for all $r > 0$, $B(x_0, r)$ is a cofinite set of M, and for all $r \neq s \in \mathbb{N}$, $\rho(x_r, x_s) = \rho(x_r, x_0) + \rho(x_0, x_s)$. However, M trivially satisfies the conditions from Lemma 5.4.8 if we consider $\{x_n\}_{n=1}^{\infty}$ as the center points and $y_n = x_0$ for all $n \in \mathbb{N}$.

5.5 The isometric embedding of c_0: the results

In this section, we will prove the main results about the isometric embedding of c_0 in $\mathrm{SNA}(M)$. We divide the contents in 3 subsections: the construction of a first counterexample, a positive result for a wide class of metric spaces, and the construction of a second counterexample with a radically different behaviour as a metric space than the first one.

5.5.1 A bounded counterexample

In this subsection, we construct an infinite complete metric space M such that the set $\mathrm{SNA}(M)$ of strongly norm-attaining Lipschitz functions does not contain an isometric copy of c_0, answering the question from [15, Remark 3.6] in the negative. It is worth mentioning that no point of this constructed metric space attains its separation radius.

Theorem 5.5.1. *There exists an infinite bounded uniformly discrete complete metric space M such that c_0 is not isometrically contained in* $\mathrm{SNA}(M)$ *and for which no point in M attains its separation radius.*

Proof. Let $M = \{p_n\}_{n=1}^{\infty}$ be any countable set endowed with the metric d given by $d(p_n, p_m) := 1 + \frac{1}{\max\{m,n\}}$, for $n \neq m$. Note that the diameter of M is $3/2$.

For the sake of contradiction, let us suppose that there exists a sequence $\{f_n\}_{n=1}^{\infty}$ of strongly norm-attaining functions which is isometrically equivalent to the canonical basis of c_0. For every $n \in \mathbb{N}$, let $x_n, y_n \in M$ be such that $x_n \neq y_n$ and $\left|f_n(x_n) - f_n(y_n)\right| = d(x_n, y_n)$. Our goal is to find two natural numbers $n_0 \neq m_0$ and $\delta \in \{-1, 1\}$ such that the Lipschitz function $f_{n_0} + \delta f_{m_0}$ has Lipschitz norm strictly greater than 1. This will lead to a contradiction.

Let us consider the sets

$$
\begin{aligned}
A &:= \left\{\{n,m\} \in \mathbb{N}^{[2]} : \{x_n, y_n\} \cap \{x_m, y_m\} = \varnothing\right\}, \\
B_1 &:= \left\{\{n,m\} \in \mathbb{N}^{[2]} : x_n = x_m\right\}, \\
B_2 &:= \left\{\{n,m\} \in \mathbb{N}^{[2]} : y_n = y_m\right\}, \text{ and} \\
B_3 &:= \left\{\{n,m\} \in \mathbb{N}^{[2]} : x_n = y_m \text{ or } x_m = y_n\right\}.
\end{aligned}
$$

By Ramsey's theorem, there exists $C \in \{A, B_1, B_2, B_3\}$ and an infinite set $S \subset \mathbb{N}$ such that $S^{[2]} \subset C$.

Case 1: $C = A$.

We may assume by passing to a subsequence that $\{x_n, y_n\} \cap \{x_m, y_m\} = \varnothing$ for every $n, m \in \mathbb{N}$ with $n \neq m$. For each $n \in \mathbb{N}$, let us set

$$\varepsilon_n := \frac{1}{2k(n)}, \quad \text{where} \quad k(n) := \max\{k \in \mathbb{N} : p_k = x_n \quad \text{or} \quad p_k = y_n\}.$$

Let us fix $n_0 \in \mathbb{N}$. Since $\{x_n, y_n\} \cap \{x_m, y_m\} = \varnothing$ for every $n, m \in \mathbb{N}$ with $n \neq m$, by Lemma 5.4.6 and the definition of the metric d, there exists $m_0 \in \mathbb{N} \backslash \{n_0\}$ such that

(*i*) $\max\{|f_{m_0}(x_{n_0})|, |f_{m_0}(y_{n_0})|\} \leqslant \dfrac{\varepsilon_{n_0}}{3}$ and

(*ii*) $\max\{d(x_{n_0}, x_{m_0}), d(x_{n_0}, y_{m_0}), d(y_{n_0}, x_{m_0}), d(y_{n_0}, y_{m_0})\} \leqslant 1 + \dfrac{\varepsilon_{n_0}}{3}$.

Now, by Lemma 5.4.5, there is a constant $C_{m_0} \in \mathbb{R}$ such that $f_{n_0}(x_{m_0}) = f_{n_0}(y_{m_0}) = C_{m_0}$. By relabeling the pairs (x_{n_0}, y_{n_0}) and (x_{m_0}, y_{m_0}) if necessary, we may assume that

$$|f_{n_0}(x_{n_0}) - C_{m_0}| \geqslant |f_{n_0}(y_{n_0}) - C_{m_0}| \qquad \text{and} \qquad |f_{m_0}(x_{m_0})| \geqslant |f_{m_0}(y_{m_0})|.$$

With this assumption, Remark 5.4.7 yields that

$$|f_{m_0}(x_{m_0})| \geqslant \frac{1}{2} + \varepsilon_{m_0} \qquad \text{and} \qquad |f_{n_0}(x_{n_0}) - C_{m_0}| \geqslant \frac{1}{2} + \varepsilon_{n_0}. \quad (5.5.1)$$

In particular, $f_{m_0}(x_{m_0}) \neq 0$. Set now $\delta := \frac{|f_{m_0}(x_{m_0})|}{f_{m_0}(x_{m_0})} \in \{-1, 1\}$. To finish the proof of this case, we distinguish two possibilities according to the sign of $|f_{n_0}(x_{n_0}) - C_{m_0}|$:

If $f_{n_0}(x_{n_0}) < C_{m_0}$, consider the function $f = f_{n_0} + \delta f_{m_0}$, which is 1-Lipschitz by assumption. However, using properties (i) and (ii) and equation (5.5.1) we obtain that

$$\begin{aligned} |f(x_{n_0}) - f(x_{m_0})| &\geqslant -f_{n_0}(x_{n_0}) - \delta f_{m_0}(x_{n_0}) + f_{n_0}(x_{m_0}) + \delta f_{m_0}(x_{m_0}) \\ &> \frac{1}{2} + \varepsilon_{n_0} + \frac{1}{2} - |f_{m_0}(x_{n_0})| > d(x_{n_0}, x_{m_0}), \end{aligned}$$

a contradiction.

On the other hand, if $f_{n_0}(x_{n_0}) \geqslant C_{m_0}$, an analogous procedure shows that the function $g = f_{n_0} - \delta f_{m_0}$ has a Lipschitz norm greater than 1 witnessed by the same pair (x_{n_0}, x_{m_0}). This again yields a contradiction.

Case 2: $C \in \{B_1, B_2, B_3\}$.

We will prove it for $C = B_1$, since the two remaining possibilites can be reduced to this one. Indeed, it is straightforward to check that, if $C = B_2$, by relabelling the pairs (x_n, y_n), for $n \in \mathbb{N}$, we may assume that $C = B_1$. Else, if Ramsey applied to B_3, fix $n_0 \in S$. For each $m \in S \backslash \{n_0\}$, we have that either $x_{n_0} = y_m$ or $x_m = y_{n_0}$, so one of the sets $S_1 = \{m \in S \backslash \{n_0\} \colon x_m = y_{n_0}\}$ or $S_2 = \{m \in S \backslash \{n_0\} \colon y_m = x_{n_0}\}$ is infinite, and we have now reduced this situation to the case where Ramsey applies to B_1 or B_2, respectively.

Hence, by taking subsequences if necessary, we assume that there exists $k^* \in \mathbb{N}$ such that $x_n = p_{k*}$ for every $n \in \mathbb{N}$. By Lemma 5.4.5, we have that $y_n \neq y_m$ for every $n, m \in \mathbb{N}$ with $n \neq m$. Using Lemma 5.4.6, we may find $n_0, m_0 \in \mathbb{N}$ with $n_0 \neq m_0$ such that

$$|f_{n_0}(x)| \leqslant \frac{1}{10} \qquad \text{and} \qquad |f_{m_0}(x)| \leqslant \frac{1}{10}. \tag{5.5.2}$$

Hence, applying now Remark 5.4.7, we have that

$$|f_{n_0}(y_{n_0})| \geqslant \frac{9}{10} \qquad \text{and} \qquad |f_{m_0}(y_{m_0})| \geqslant \frac{9}{10}. \tag{5.5.3}$$

Changing signs of f_{n_0} and f_{m_0} if needed, we may assume that $f_{n_0}(y_{n_0}) > 0$ and $f_{m_0}(y_{m_0}) > 0$. Finally, consider the function $f := f_{n_0} - f_{m_0}$, which is 1-Lipschitz by the assumption on the sequence $\{f_n\}_{n=1}^{\infty}$. However, applying (5.5.2) and (5.5.3), and recalling that the diameter of M is 3/2 we obtain that

$$\begin{aligned} |f(y_{n_0}) - f(y_{m_0})| &\geqslant f_{n_0}(y_{n_0}) - f_{m_0}(y_{m_0}) + f_{m_0}(y_{n_0}) - f_{n_0}(y_{m_0}) \\ &\geqslant \frac{9}{5} - (|f_{m_0}(y_{n_0})| + |f_{n_0}(y_{m_0})|) > d(y_{n_0}, y_{m_0}). \end{aligned}$$

This is a contradiction and the proof is over. ■

It is worth mentioning that there exist countable bounded uniformly discrete complete metric spaces M with the condition that no point x in M attains its separation radius, but such that c_0 embeds isometrically in $\mathrm{SNA}(M)$. Indeed, it suffices to consider a countable collection $\{M_n\}_{n=1}^{\infty}$ of copies of the previous space in such a way that $d(M_n, M_m) = 3$ for all different $n, m \in \mathbb{N}$, and observe that, in this context, Lemma 5.4.8 trivially applies. This means that the aforementioned property is not sufficient for c_0 not to be contained in $\mathrm{SNA}(M)$ isometrically.

Likewise, one could be tempted to assume that the condition of not attaining the separation radii is at least necessary in negative results as Theorem 5.5.1. However, this is far from being true as well. In fact, later in this section we will exhibit a proper but not bounded uniformly discrete complete metric space M such that c_0 cannot be embedded in $\mathrm{SNA}(M)$ isometrically (see Theorem 5.5.4) (in particular, every point of M attains its separation radius, since closed bounded sets in M are compact). On the other hand, we will see in the next subsection that the property of being uniformly discrete is indeed necessary in order to get such negative results (see Theorem 5.5.2).

5.5.2 Non uniformly discrete metric spaces

In the previous subsection, we found a metric space M for which $\mathrm{SNA}(M)$ does not contain c_0 isometrically. Note that the constructed space was uniformly discrete. As announced, in this subsection we will show that this is actually necessary for all counterexamples. The main positive result is the following.

Theorem 5.5.2. *Let M be an infinite non uniformly discrete metric space. Then, the set* $\mathrm{SNA}(M)$ *contains an isometric copy of* c_0.

Proof. By [15, Theorems 3.2 and 3.4] it suffices to assume that M' is non-empty and finite. We also assume without loss of generality that $0 \in M'$. Now we can find a sequence $\{x_n\}_{n=1}^{\infty}$ in M converging to 0 such that $R(x_n) > 0$ for all $n \in N$. It is clear that the sequence $\{R(x_n)\}_{n=1}^{\infty}$ converges to 0.

We are going to define a sequence $\{f_k\}_{k=1}^{\infty}$ of 1-Lipschitz functions in $\mathrm{SNA}(M)$ which will be isometrically equivalent to the canonical basis of c_0 and such that the subspace $\overline{\mathrm{span}}\{f_k : k \in \mathbb{N}\}$ (which is isometric to c_0) is contained in $\mathrm{SNA}(M)$.

We define the following sets:

$$A := \big\{\{n, m\} \in \mathbb{N}^{[2]} : d(x_n, x_m) \geqslant R(x_n) + R(x_m)\big\},$$
$$B := \big\{\{n, m\} \in \mathbb{N}^{[2]} : d(x_n, x_m) < R(x_n) + R(x_m)\big\},$$

which form a partition of $\mathbb{N}^{[2]}$. By Ramsey's theorem, there is $C \in \{A, B\}$ and an infinite subset $S \subset \mathbb{N}$ such that $S^{[2]} \subset C$. These two possibilities give us two separate cases.

Case 1: $C = A$.

Consider the subset $\{x_n\}_{n\in S}$, which satisfies that $d(x_n, x_m) \geqslant R(x_n) + R(x_m)$ for all $n \neq m \in S$. Assume first that there is an infinite subset of S, which we denote by S again, such that $R(x_n)$ is attained for every $n \in S$. Consider now for each $n \in S$ an element $y_n \in M$ such that $d(x_n, y_n) = R(x_n)$. It is straightforward to see that the sequences $\{x_n\}_{n\in S}, \{y_n\}_{n\in S}$ satisfy the assumptions of Lemma 5.4.8 and we are done.

Otherwise, we may assume that $R(x_n)$ is not attained for any $n \in S$. Let us then choose inductively a sequence $\{a_k\}_{k=1}^{\infty}$ among the elements of the sequence $\{x_n\}_{n\in S}$ satisfying that for every $k \in \mathbb{N}$,

$$d(a_k, 0) \leqslant \frac{d(a_j, 0) - R(a_j)}{4} \quad \forall j < k. \tag{5.5.4}$$

For the sake of clarity, let us denote $\Delta_k = \frac{d(a_k,0)-R(a_k)}{4}$ for each $k \in \mathbb{N}$. It is clear from (5.5.4) that $\{\Delta_k\}_{k=1}^{\infty}$ is a decreasing sequence. Now, from the fact that $R(a_k)$ is not attained, we deduce that for each $k \in \mathbb{N}$, there is $b_k \in B(a_k, R(a_k) + \Delta_k)$. Finally, let us prove that the sequences $\{a_k\}, \{b_k\}$ are under the assumptions of Lemma 5.4.8. Pick now $n, m \in \mathbb{N}$ with $n < m$. Clearly, the following expressions hold

$$\begin{aligned} d(a_n, 0) =& R(a_n) + 4\Delta_n, \quad d(a_m, 0) \leqslant \Delta_n, \quad d(a_n, b_n) \leqslant R(a_n) + \Delta_n \\ & d(a_m, b_m) \leqslant R(a_m) + \Delta_m \leqslant d(a_m, 0) + \Delta_m \leqslant 2\Delta_n. \end{aligned} \tag{5.5.5}$$

Hence, by (5.5.5) we have that

$$d(a_n, a_m) \geqslant d(a_n, 0) - d(a_m, 0) \geqslant R(a_n) + 3\Delta_n \geqslant d(a_n, b_n) + d(a_m, b_m).$$

This finishes the first case.

Case 2: $C = B$.

Since the set S is infinite and the sequence $\{x_n\}_{n=1}^{\infty}$ is convergent, we can inductively define a pair of sequences $\{a_k\}_{k=1}^{\infty} \subset \{x_n\}_{n\in S}$ and $\{b_k\}_{k=1}^{\infty} \subset \{x_n\}_{n\in S}$ satisfying the following properties:

(*i*) $R(a_k) < \varepsilon_j/2$, for $j, k \in \mathbb{N}$ with $j < k$, where $\varepsilon_j = R(a_j) + R(b_j) - d(a_j, b_j) > 0$.

(*ii*) $R(b_k) < R(a_k)/2$ for every $k \in \mathbb{N}$.

Fixed $k \in \mathbb{N}$, we define $f_k : M \to \mathbb{R}$ by

$$f_k(p) = \begin{cases} R(a_k) - \dfrac{\varepsilon_k}{2} & \text{if } p = a_k, \\ -R(b_k) + \dfrac{\varepsilon_k}{2} & \text{if } p = b_k, \\ 0 & \text{otherwise.} \end{cases}$$

Property (*i*) and the definition of ε_k ensure that $f_k(a_k) \geqslant 0$ and $f_k(b_k) \leqslant 0$ for every $k \in \mathbb{N}$. With this, we obtain that

$$|f_k(a_k)| = R(a_k) - \frac{\varepsilon_k}{2}, \quad \text{and} \quad |f_k(b_k)| = R(b_k) - \frac{\varepsilon_k}{2}, \quad \text{for all } k \in \mathbb{N}. \tag{5.5.6}$$

Let $\{\lambda_k\}_{k=1}^{\infty} \in c_0$. Again we will show that $f := \sum_{k=1}^{\infty} \lambda_k f_k \in \mathrm{SNA}(M)$ and also that $\|f\| = \max_{k\in\mathbb{N}}\{|\lambda_k|\}$. Choose $k_0 \in \mathbb{N}$ such that $|\lambda_{k_0}| = \max_{k\in\mathbb{N}}\{|\lambda_k\}$.

We again start by proving that f is $|\lambda_{k_0}|$-Lipschitz. Take $p, q \in M$ with $p \neq q$. We show that $|f(p) - f(q)| \leqslant |\lambda_{k_0}| d(p, q)$. If both p and q form a pair $\{a_k, b_k\}$ for some $k \in \mathbb{N}$, the previous inequality is clear. We need to study now the two remaining possibilities:

(a) Suppose that there exist $k_1, k_2 \in \mathbb{N}$ with $k_1 < k_2$ such that $p \in \{a_{k_1}, b_{k_1}\}$ and $q \in \{a_{k_2}, b_{k_2}\}$. Then, by (5.5.6) we have in particular that $|f(p)| = |\lambda_{k_1}| \left(R(p) - \frac{\varepsilon_{k_1}}{2}\right)$ and $|f(q)| < |\lambda_{k_2}| R(a_{k_2})$. Hence, we obtain that

$$\begin{aligned}|f(p) - f(q)| < |\lambda_{k_0}| \cdot \left(R(p) - \frac{\varepsilon_{k_1}}{2} + R(a_{k_2})\right)\\ \leqslant |\lambda_{k_0}| \cdot R(p) \leqslant |\lambda_{k_0}| \cdot d(p, q).\end{aligned}$$

(b) If $p \in M \backslash \{x_k\}_{k=1}^{\infty}$, then $f(p) = 0$ and, using (5.5.6) again, we have that

$$|f(p) - f(q)| = |f(q)| < |\lambda_{k_0}| \cdot R(q) \leqslant |\lambda_{k_0}| \cdot d(p, q).$$

We have proven then that the Lipschitz norm of f is smaller or equal than $|\lambda_{k_0}|$. Finally, considering the pair of points a_{k_0} and b_{k_0}, we quickly observe that $\|f\| = |\lambda_{k_0}|$ and that f strongly attains its Lipschitz norm at this pair of points. This finishes the proof. ■

In [15, Theorem 3.3], c_0 is isomorphically embedded into $\mathrm{SNA}(M)$ for countable compact metric spaces M in a non constructive way. Indeed, the authors show that the little Lipschitz space is an infinite-dimensional subspace of c_0 contained in $\mathrm{SNA}(M)$. The following corollary, which is an immediate consequence of Theorem 5.5.2, improves that part of [15, Theorem 3.3] with a different approach.

Corollary 5.5.3. *Let M be an infinite compact metric space. Then, the subset* $\mathrm{SNA}(M)$ *contains an isometric copy of c_0.*

On the other hand, unlike in [15, Theorem 3.3], it is not possible to extend Corollary 5.5.3 to proper spaces, as we will see in the next subsection.

5.5.3 A proper counterexample

In Subsection 5.5.1, we provided a metric space M where no point attains its separation radius and such that $\mathrm{SNA}(M)$ does not contain c_0 isometrically. Then, in Subsection 5.5.2, we saw that if M is not uniformly discrete (in particular, if M is compact), then $\mathrm{SNA}(M)$ always contain c_0 isometrically. In this subsection, we will provide an example of proper metric space M (and so, every point attains its separation radius) such that $\mathrm{SNA}(M)$ does not contain c_0 isometrically.

Theorem 5.5.4. *There exists an infinite proper uniformly discrete complete metric space M such that c_0 is not isometrically contained in* $\mathrm{SNA}(M)$ *and for which every point in M attains its separation radius.*

Proof. Let $M = \{p_k\}_{k=0}^{\infty}$, with distinguished point $p_0 = 0$, be a countable set endowed with the metric $d\colon M \times M \to \mathbb{R}$ given by:

$$d(p_k, p_j) = \begin{cases} k + j - \varepsilon_{\max\{k,j\}} & \text{if } k \neq j \in \mathbb{N}\backslash\{0\}, \\ k & \text{if } j = 0, \\ j & \text{if } k = 0, \\ 0 & \text{if } j = k, \end{cases}$$

where $\{\varepsilon_k\}_{k=1}^{\infty}$ is a sequence of positive numbers such that $\varepsilon_{k+1} > \varepsilon_k$ and $\varepsilon_k < 1/2$ for all $k \in \mathbb{N}$. For convenience, write $\delta_k = \varepsilon_{k+1} - \varepsilon_k > 0$ for all $k \in \mathbb{N}$. It is clear that M is proper since every bounded set is finite.

As in the proof of Theorem 5.5.1, we start by assuming that there exists a sequence $\{f_n\}_{n=1}^{\infty}$ of functions in $\mathrm{SNA}(M)$ isometrically equivalent to the canonical basis of c_0, and we are going to find two natural numbers $n_0 \neq m_0$ such that $f_{n_0} - f_{m_0}$ is not 1-Lipschitz, which will yield a contradiction. For each $n \in \mathbb{N}$, since f_n is strongly norm-attaining, we may consider two points $x_n \neq y_n$ such that $|f(x_n) - f(y_n)| = d(x_n, y_n)$.

We write $k(n)$ and $j(n)$ to denote the natural numbers such that $x_n = p_{k(n)}$ and $y_n = p_{j(n)}$ for every $n \in \mathbb{N}$. By relabelling the pair (x_n, y_n), we may assume that $k(n) < j(n)$ for all $n \in \mathbb{N}$.

We now define the sets A, B_1, B_2, and B_3 as in the proof of Theorem 5.5.1. By Ramsey's theorem, there exists $C \in \{A, B_1, B_2, B_3\}$ and an infinite set $S \subset \mathbb{N}$ such that $S^{[2]} \subset C$. Note, however, that the case $C = B_3$ can be reduced to $C \subset \{B_1, B_2\}$ as in Theorem 5.5.1, and the case $C = B_2$ cannot happen, since in that case we would forcefully get functions f_n and f_m from the basis that would strongly attain their norms at the same pair of points, contradicting Lemma 5.4.5. Hence, the conclusion of Ramsey's theorem can only apply to sets A and B_1 in this scenario, and thus, by passing to a subsequence if needed, this allows us to reduce the possibilities to only two cases.

Case 1: For every $n \neq m$ we have $\{x_n, y_n\} \cap \{x_m, y_m\} = \varnothing$.

In this case, choose an arbitrary $n_0 \in \mathbb{N}$ such that $k(n_0), j(n_0) \neq 0$. By Lemma 5.4.6, and using that $\{x_n, y_n\} \cap \{x_m, y_m\} = \varnothing$ for all $n \neq m \in \mathbb{N}$, we can find $m_0 \in \mathbb{N}$ with $k(m_0) > j(n_0)$ such that

$$|f_{m_0}(x_{n_0})| < \frac{1}{2}\delta_{j(n_0)} \qquad \text{and} \qquad |f_{m_0}(y_{n_0})| < \frac{1}{2}\delta_{j(n_0)}. \tag{5.5.7}$$

Using Lemma 5.4.5 we can define $C_{m_0} \in \mathbb{R}$ such that $C_{m_0} = f_{n_0}(x_{m_0}) = f_{n_0}(y_{m_0})$. With Remark 5.4.7 we obtain that either

(a_0) $|f_{n_0}(x_{n_0}) - C_{m_0}| \geqslant k(n_0) - \frac{1}{2}\varepsilon_{j(n_0)}$, or

(a_1) $|f_{n_0}(y_{n_0}) - C_{m_0}| \geqslant j(n_0) - \frac{1}{2}\varepsilon_{j(n_0)}$.

Similarly and by the same lemma, we have that either

(b_0) $|f_{m_0}(x_{m_0})| \geqslant k(m_0) - \left(\frac{1}{2}\varepsilon_{j(n_0)} + \frac{1}{2}\delta_{j(n_0)}\right)$, or

(b_1) $|f_{m_0}(y_{m_0})| \geqslant j(m_0) - \left(\varepsilon_{j(m_0)} - \frac{1}{2}\varepsilon_{j(n_0)} - \frac{1}{2}\delta_{j(n_0)}\right)$.

In total, there are now 4 different possibilities that must be checked for contradiction. We will only expand on the two possibilities where (a_0) holds, since the two remaining possibilities (when (a_1) holds) are proven similarly. Hence, suppose first that (a_0) and (b_0) hold. By changing the signs of f_{n_0} and f_{m_0} if necessary, we may suppose that $f_{n_0}(x_{n_0}) - C_{m_0} \geqslant k(n_0) - \frac{1}{2}\varepsilon_{j(n_0)}$ and $f_{m_0}(x_{m_0}) \geqslant k(m_0) - \left(\frac{1}{2}\varepsilon_{j(n_0)} + \frac{1}{2}\delta_{j(n_0)}\right)$. Consider the function $f = f_{n_0} - f_{m_0}$, which is 1-Lipschitz since we are assuming that $\{f_n\}_{n=1}^{\infty}$ is isometrically equivalent to the canonical basis of c_0. However, using (5.5.7), we have that

$$\begin{aligned}
|f(x_{n_0}) - f(x_{m_0})| &\geqslant f_{n_0}(x_{n_0}) - f_{m_0}(x_{n_0}) - C_{m_0} + f_{m_0}(x_{m_0}) \\
&> k(n_0) - \frac{1}{2}\varepsilon_{j(n_0)} - \frac{1}{2}\delta_{j(n_0)} + k(m_0) - \frac{1}{2}\varepsilon_{j(n_0)} - \frac{1}{2}\delta_{j(n_0)} \\
&\geqslant k(n_0) + k(m_0) - \varepsilon_{k(m_0)} = d(x_{n_0}, x_{m_0}),
\end{aligned}$$

which yields a contradiction. Suppose now that (a_0) and (b_1) hold. Again we may suppose that $f_{n_0}(x_{n_0}) - C_{m_0} \geqslant k(n_0) - \frac{1}{2}\varepsilon_{j(n_0)}$ and $f_{m_0}(y_{m_0}) \geqslant j(m_0) - \big(\varepsilon_{j(m_0)} - \frac{1}{2}\varepsilon_{j(n_0)} - \frac{1}{2}\delta_{j(n_0)}\big)$. Using (5.5.7) again, the 1-Lipschitz function $f = f_{n_0} - f_{m_0}$ now tested at the pair (x_{n_0}, y_{m_0}) yields

$$\begin{aligned}|f(x_{n_0}) - f(y_{m_0})|&\\ &> k(n_0) - \frac{1}{2}\varepsilon_{j(n_0)} - \frac{1}{2}\delta_{j(n_0)} + j(m_0) - \varepsilon_{j(m_0)} + \frac{1}{2}\varepsilon_{j(n_0)} + \frac{1}{2}\delta_{j(n_0)}\\ &= k(n_0) + j(m_0) - \varepsilon_{j(m_0)} = d(x_{n_0}, y_{m_0}),\end{aligned}$$

which is again a contradiction. This finishes the proof for Case 1.

Case 2: $x_n = x_m$ for all $n, m \in \mathbb{N}$.

Write k^* to denote the natural number (including 0) such that $p_{k^*} = x_n$ for all $n \in \mathbb{N}$. Suppose first that $k^* = 0$. Then, choose any two different numbers $n_0 \neq m_0 \in \mathbb{N}$. Since both f_{n_0} and f_{m_0} strongly attain their norm at the pair $(0, y_{n_0})$ and $(0, y_{m_0})$ respectively, and both f_{n_0} and f_{m_0} vanish at 0, we have that $|f_{n_0}(y_{n_0})| = j(n_0)$ and $|f_{m_0}(y_{m_0})| = j(m_0)$. With Lemma 5.4.5 we obtain that $f_{n_0}(y_{m_0}) = f_{m_0}(y_{n_0}) = 0$. By changing the signs of both functions if needed, we may suppose that $f_{n_0}(y_{n_0}) = j(n_0)$ and $f_{m_0}(y_{m_0}) = j(m_0)$, producing a contradiction directly by considering the mapping $f = f_{n_0} - f_{m_0}$, which is not 1-Lipschitz as witnessed by the pair (y_{n_0}, y_{m_0}). Indeed,

$$|f(y_{n_0}) - f(y_{m_0})| = j(n_0) + j(m_0) > d(y_{n_0}, y_{m_0}).$$

Suppose now that $k^* \neq 0$. Using Lemma 5.4.6, choose two different natural numbers $n_0 \neq m_0 \in \mathbb{N}$ with $j(m_0) > j(n_0) > k^*$ such that

$$|f_{n_0}(p_{k^*})| < \frac{1}{4} \qquad \text{and} \qquad |f_{m_0}(p_{k^*})| < \frac{1}{4}.$$

On the one hand, this means that $|f_{n_0}(y_{n_0})| > k^* + j(n_0) - \varepsilon_{j(n_0)} - \frac{1}{4}$ and $|f_{m_0}(y_{m_0})| > k^* + j(m_0) - \varepsilon_{j(m_0)} - \frac{1}{4}$, while, on the other hand, it implies by Lemma 5.4.5 that

$$|f_{n_0}(y_{m_0})| < \frac{1}{4} \qquad \text{and} \qquad |f_{m_0}(y_{n_0})| < \frac{1}{4}.$$

Finally, we may again suppose without loss of generality that f_{n_0} and f_{m_0} are both positive at the points y_{n_0} and y_{m_0} respectively, and consider the function $f = f_{n_0} - f_{m_0}$, which is assumed to be 1-Lipschitz. However, we have that

$$\begin{aligned} |f(y_{n_0}) - f(y_{m_0})| &\geqslant f_{n_0}(y_{n_0}) - f_{m_0}(y_{n_0}) - f_{n_0}(y_{m_0}) + f_{m_0}(y_{m_0}) \\ &\geqslant j(n_0) + j(m_0) + 2k^* - 1 - \varepsilon_{j(n_0)} - \varepsilon_{j(m_0)} \\ &> j(n_0) + j(m_0) > d(y_{n_0}, y_{m_0}), \end{aligned}$$

a contradiction. This finishes the proof of Case 2 and so the theorem is finally proven. ■

Remark 5.5.5. Note that there exist infinite proper uniformly discrete complete metric spaces M where every point attains its separation radius and such that $\mathrm{SNA}(M)$ contains c_0 isometrically. For instance, if M is the space of natural numbers $\mathbb{N}$ with the usual metric, then Lemma 5.4.8 trivially applies to M.

5.6 The non-separable case

To end this chapter, in this section we tackle the problem of embedding $c_0(\Gamma)$ in $\mathrm{SNA}(M)$ isometrically, where Γ is an arbitrary set of large cardinality. We need to first introduce some basic concepts and results of set theory that will be heavily used in this section. We will use the notation from the book [90].

An ordinal α is a *cardinal* if for every smaller ordinal $\beta < \alpha$, β is not equivalent to α. We denote by $\operatorname{dens}(M)$ the *density character* of a metric space M, defined as the smallest cardinal Γ such that there is a dense subset of M of cardinality $\operatorname{card}(\Gamma)$. The *cofinality* $\operatorname{cof}(\alpha)$ of an ordinal α is the smallest ordinal β such that $\alpha = \sup_{\gamma<\beta} \alpha_\gamma$, where $\{\alpha_\gamma\}_{\gamma<\beta}$ is an ordinal sequence of length β with $\alpha_\gamma < \alpha$ for all $\gamma < \beta$. An ordinal Γ is *regular* if $\operatorname{cof}(\Gamma) = \Gamma$, and note that regular ordinals are always cardinals (see [90, Lemma 10.35]). For an ordinal α, we denote by α^+ the least cardinal strictly bigger than α, which is always a regular cardinal (see [90, Lemma 10.37]). We again refer to the book [90] for a comprehensive background on this topic. Finally, recall that a subset of a metric space $S \subset M$ is called *r-separated* for some $r > 0$ if $d(x, y) \geqslant r$ for all $x \neq y \in S$.

The next result is essentially based on the proof of [71, Proposition 3].

Proposition 5.6.1. *Let M be a metric space with $\operatorname{dens}(M) = \Gamma$, for some uncountably infinite cardinal Γ. Then, there exists a discrete set $L \subset M$ with $\operatorname{card}(L) = \Gamma$. Moreover, if $\operatorname{cof}(\Gamma)$ is uncountable, then L can be chosen to be uniformly discrete.*

Proof. For every $k \in \mathbb{N}$, let M_k be some maximal $\frac{1}{2^k}$-separated subset of M. Denote $\Gamma_k := \operatorname{card}(M_k)$ for all $k \in \mathbb{N}$. If $\operatorname{cof}(\Gamma)$ is uncountable, then, since $\overline{\bigcup_{k=1}^{\infty} M_k} = M$, we have that there is $k_0 \in \mathbb{N}$ such that $\Gamma_{k_0} = \Gamma$ and so we take $L := M_{k_0}$.

Now, let us assume that $\operatorname{cof}(\Gamma)$ is countable. If there exists $k_0 \in \mathbb{N}$ such that $\Gamma_{k_0} = \Gamma$, we are done, since we can take once again $L := M_{k_0}$. On the other hand, if this is not the case, we have that $\Gamma_k < \Gamma$ for every $k \in \mathbb{N}$ and $\operatorname{cof}(\Gamma)$ is countable. Since Γ is not regular, we know that

$\Gamma_k^+ < \Gamma$, for every $k \in \mathbb{N}$. Using this, and the fact that $\sup_{k\in\mathbb{N}} \Gamma_k = \Gamma$, it is straightforward to inductively construct a subsequence $\{\Gamma_{k_n}\}_{n=1}^\infty$ of $\{\Gamma_k\}_{k=1}^\infty$ with Γ_{k_1} infinite and such that $\Gamma_{k_n}^+ < \Gamma_{k_{n+1}}$ for all $n \in \mathbb{N}$.

Now, for each $n \in \mathbb{N}$, let us consider a sequence of sets $\{\widetilde{M}_n\}_{n=1}^\infty$ such that $\widetilde{M}_n$ is a subset of $M_{k_{n+1}}$ with $\mathrm{card}(\widetilde{M}_n) = \Gamma_{k_n}^+$ for all $n \in \mathbb{N}$. Let us write $\widetilde{M}_n = \{x_\alpha^n : \alpha \in \Gamma_{k_n}^+\}$.

For each $n \in \mathbb{N}$, each $j \leqslant n$, and each $\alpha \in \Gamma_{k_j}^+$, we define

$$A_{j,\alpha}^n := \widetilde{M}_{n+1} \cap B\left(x_\alpha^j, \frac{1}{2^{k_{j+1}+1}}\right).$$

We will inductively construct, for every $n \in \mathbb{N}$, a set $L_n \subset \widetilde{M}_n$ with $\mathrm{card}(L_n) = \Gamma_{k_n}^+$ and a finite subset $N_n \subset M$ such that whenever $j < n$,

$$d(L_n, L_j \backslash N_n) \geqslant \frac{1}{2^{k_{j+1}+2}}. \tag{5.6.1}$$

Set $L_1 := \widetilde{M}_1$ and $N_1 := \varnothing$. Now, assuming that for some $n \in \mathbb{N}$ we have constructed L_j and N_j for all $j \leqslant n$, we can do the inductive step towards $n+1$.

(a) Suppose that $\mathrm{card}(A_{j,\alpha}^n) < \Gamma_{k_{n+1}}^+$ for every $j \leqslant n$ and $\alpha \in \Gamma_{k_n}^+$. Since $\Gamma_{k_n}^+ < \Gamma_{k_{n+1}}^+$ and $\Gamma_{k_{n+1}}^+$ is regular, we have that

$$\mathrm{card}\left(\bigcup_{j\leqslant n,\, \alpha\in\Gamma_{k_j}^+} A_{j,\alpha}^n\right) < \Gamma_{k_{n+1}}^+ = \mathrm{card}(\widetilde{M}_{n+1}).$$

Therefore, the set

$$L_{n+1} := \widetilde{M}_{n+1} \backslash \bigcup_{j\leqslant n,\, \alpha\in\Gamma_{k_j}^+} A_{j,\alpha}^n$$

satisfies $\operatorname{card}(L_{n+1}) = \Gamma^+_{k_{n+1}}$ and (5.6.1) holds by setting $N_{n+1} = \varnothing$. Indeed, for any $j \in \{1, \dots, n\}$, every point in L_j is of the form x^j_α for some $\alpha \in \Gamma^+_{k_j}$. Hence, if there exists a point $p \in L_{n+1}$ such that $d(p, x^j_\alpha) < \frac{1}{2^{k_{j+1}+2}}$, then p belongs to the set $A^n_{j,\alpha}$, which leads to a contradiction with the definition of L_{n+1}.

(b) Suppose now that $\operatorname{card}(A^n_{j_0,\alpha_0}) = \Gamma^+_{k_{n+1}}$ for some $j_0 \leqslant n$ and some $\alpha_0 \in \Gamma^+_{k_{j_0}}$. Without loss of generality we consider $j_0 \in \{1, \dots, n\}$ to be such that $\operatorname{card}(A^n_{j,\alpha}) < \Gamma^+_{k_{n+1}}$ for all $j_0 < j \leqslant n$ and all $\alpha \in \Gamma^+_{k_j}$. Define

$$L_{n+1} := A^n_{j_0,\alpha_0} \setminus \bigcup_{j_0 < j \leqslant n,\, \alpha \in \Gamma^+_{k_j}} A^n_{j,\alpha}.$$

Arguing as in case (a), we obtain that $\operatorname{card}(L_{n+1}) = \Gamma^+_{k_{n+1}}$. Finally, define $N_{n+1} := \big\{x \in M \colon \exists i \in \{1, \dots, j_0\}$ such that $x \in L_i$ and $d(x, L_{n+1}) < \frac{1}{2^{k_{i+1}+2}}\big\}$, which is finite since for each $i \in \{1, \dots, j_0\}$, there can only be at most a single point x_i in L_i such that $d(x_i, L_{n+1}) < \frac{1}{2^{k_{i+1}+2}}$. Indeed, if $i = j_0$, the only point in L_{j_0} that can satisfy that property is $x^{j_0}_{\alpha_0}$, since for every $\beta \in \Gamma^+_{k_{j_0}} \setminus \{\alpha_0\}$, $d(x^{j_0}_\beta, A^n_{j_0,\alpha_0}) \geqslant \frac{1}{2^{k_{j_0+1}+1}}$. On the other hand, if $i < j_0$, if there were two points $x_i \neq y_i \in L_i$ with that property, we would have that

$$d(x_i, y_i) \leqslant d(x_i, A^n_{j_0,\alpha_0}) + d(y_i, A^n_{j_0,\alpha_0}) + \operatorname{diam}(A^n_{j_0,\alpha_0}) < \frac{1}{2^{k_{i+1}}},$$

a contradiction with the fact that L_i is $\frac{1}{2^{k_{i+1}}}$-separated.

Let us check that the sets L_{n+1} and N_{n+1} satisfy equation (5.6.1) for each $j \in \{1, \dots, n\}$. Fix $j \in \{1, \dots, n\}$. If $j \leqslant j_0$ then the inequality follows directly by definition of N_{n+1}. Otherwise, if $j > j_0$, then the inequality holds following the same argument as in case (a).

Having discussed both possibilities, the induction is finished. To finish the proof, set $L := \left(\bigcup_{n=1}^{\infty} L_n\right) \setminus \left(\bigcup_{n=1}^{\infty} N_n\right)$. It is clear that $\mathrm{card}(L) = \Gamma$, and using equation (5.6.1), it is straightforward to prove that all convergent sequences in L are eventually constant, and thus, L is discrete. ■

As an application of Lemma 5.4.8 and Proposition 5.6.1, we have the following isometric result.

Theorem 5.6.2. *Let M be a pointed metric space such that* $\mathrm{dens}(M') = \Gamma$ *for some infinite cardinal* Γ*. Then there is a linear subspace of* $\mathrm{SNA}(M)$ *that is isometrically isomorphic to* $c_0(\Gamma)$*.*

Proof. The case where Γ is countable is already covered in [15, Theorem 3.2], and is also a direct consequence of Lemma 5.4.8. Assume now that Γ is uncountable. If we apply Proposition 5.6.1 to the set M', we find a discrete set $L \subset M'$ with $\mathrm{card}(L) = \mathrm{dens}(L) = \Gamma$ and such that all points of L are cluster points of M. If I is an index set of cardinality Γ, if we write now $L = \{x_i\}_{i\in I}$, since L is discrete, for all $j \in I$,

$$r_j := \inf\{d(x_j, x_k)\colon\ k \in I\backslash\{j\}\} > 0.$$

Using now that $L \subset M'$, for each $i \in I$, let $y_i \in M$ be such that $d(x_i, y_i) < \frac{r_i}{2}$. Now, for each $j, k \in I$ such that $j \neq k$, we have

$$d(x_j, y_j) + d(x_k, y_k) < \frac{r_j + r_k}{2} \leqslant \max\{r_j, r_k\} \leqslant d(x_j, x_k),$$

by the definition of r_j and r_k. Finally, applying Lemma 5.4.8 with the sets $\{x_i\}_{i\in I}$ and $\{y_i\}_{i\in I}$, we get that $\mathrm{SNA}(M)$ contains $c_0(\Gamma)$ (or just $c_0(I)$) isometrically. ■

Chapter 6

Conclusions and open questions

In this chapter we will summarize the conclusions we got in this work, and we will also discuss some remarks and open questions that arose from our study.

6.1 Chapter 2

The contents of Chapter 2 were extracted from the published paper [42], where we studied classes of operators (denoted respectively as $\mathcal{A}_{\|\cdot\|}$ and $\mathcal{A}_{\mathrm{nu}}$) that satisfy that whenever they almost attain their norm (respectively, their numerical radius) at a point (respectively, at a state), they do attain it at a nearby point (respectively, a nearby state). These classes of operators were inspired by the $\mathbf{L}_{o,o}$ property and its applications, and were defined in a similar manner (see Definition 2.1.1).

In Theorem 2.2.1 we characterized what operators are in $\mathcal{A}_{\|\cdot\|}$ and $\mathcal{A}_{\rm nu}$ when the domain space is finite-dimensional, and also what functionals on c_0 are in $\mathcal{A}_{\|\cdot\|}$. This made us wonder whether we could get some more results for functionals in some other sequence spaces. Both positive and negative results were achieved about this in Proposition 2.2.2. For the more general scenario of operators bewtween two Banach spaces, we saw that there are operators that belong to $\mathcal{A}_{\|\cdot\|} \cap \mathcal{A}_{\rm nu}$, to $\mathcal{A}_{\|\cdot\|} \backslash \mathcal{A}_{\rm nu}$, to $\mathcal{A}_{\rm nu} \backslash \mathcal{A}_{\|\cdot\|}$, and operators that do not belong to $\mathcal{A}_{\|\cdot\|} \cup \mathcal{A}_{\rm nu}$ despite attaining their norm and numerical radius (see Example 2.2.5).

For compact operators we got a positive result if the involved spaces satisfy certain conditions (see Theorem 2.2.6 and Corollary 2.2.7), and we provided some examples that show how sharp the conditions in the statement are (see the operators from (2.2.3) and (2.2.5), and see also Proposition 2.2.9 for a related result). These examples also show that, in general, there is no relation between the claims $T \in \mathcal{A}_{\|\cdot\|}(X,Y)$ and $T^* \in \mathcal{A}_{\rm nu}(Y^*, X^*)$, although the implications hold if the involved spaces satisfy certain conditions (see Proposition 2.2.11, items (i) and (ii)). In Proposition 2.2.11.(iii), we saw that if a Banach space X is reflexive then $T \in \mathcal{A}_{\rm nu}(X)$ if and only if $T^* \in \mathcal{A}_{\rm nu}(X^*)$, and a related (but not analogous) result for c_0 can be found in Proposition 2.2.12. However, we do not know if we can remove reflexivity in the original statement in general. In fact, the following remains unknown.

Open Question 1. *Let X be a Banach space. If $T \in \mathcal{A}_{\rm nu}(X)$, is it true that $T^* \in \mathcal{A}_{\rm nu}(X^*)$? If $T^* \in \mathcal{A}_{\rm nu}(X^*)$, is it true that $T \in \mathcal{A}_{\rm nu}(X)$?*

In Section 2.3, we obtained a complete characterization of all the diagonal operators that belong to $\mathcal{A}_{\|\cdot\|}(X,X)$ ($X = c_0$ or ℓ_p, $1 \leqslant p \leqslant \infty$), to $\mathcal{A}_{\rm nu}(X)$ ($X = c_0$ or ℓ_p, $1 \leqslant p < \infty$), to $\mathcal{A}_{\|\cdot\|}(c_0, \ell_p)$ ($1 < p < \infty$) and to $\mathcal{A}_{\|\cdot\|}(\ell_p, c_0)$ ($1 < p < \infty$). In particular, it was shown in Corollary 2.3.8 that every canonical projection P_N belongs to both $\mathcal{A}_{\|\cdot\|}(X,X)$

and $\mathcal{A}_{\mathrm{nu}}(X)$ when $X = c_0$ or ℓ_p $(1 \leqslant p \leqslant \infty)$. Finally, in Section 2.4, relations between the sets $\mathcal{A}_{\|\cdot\|}(W, Z)$ and $\mathcal{A}_{\mathrm{nu}}(W \oplus Z)$ were studied for some particular types of direct sums. Positive results were obtained for some cases in Propositions 2.4.1 and 2.4.4 for the $\oplus_1$ and $\oplus_\infty$ sums, respectively, and some remarks were made to discuss the sharpness of the conditions in the statements and to study the $\oplus_p$ sums' scenario.

6.2 Chapter 3

The contents of Chapter 3 were extracted from the published paper [59], where we introduced and studied a version of the Bishop-Phelps-Bollobás property for numerical radius in the setting of compact operators (abbreviated BPBp-nu for compact operators). Natural adaptations of the existing proofs for the BPBp-nu provided us a first list of spaces that satisfy the BPBp-nu for compact operators: finite-dimensional spaces, $c_0(\Gamma)$ and $\ell_1(\Gamma)$ for any index set Γ, and $L_1(\mu)$ spaces for any measure μ. By adapting the concepts and results from [87] and [89], we also got in Proposition 3.2.8 that, actually, all $L_p(\mu)$ spaces have the BPBp-nu for compact operators for all $1 \leqslant p < \infty$ and for any measure μ.

In [34, Proposition 4.3], it was shown that if a Banach space has the BPBp-nu for compact operators, then some of its absolute projections also have the property. A natural question is whether or not the property can also be carried from projections of a space into the space itself. Inspired by that question and by [39, Lemma 2.1], where a similar claim was achieved for the norm instead of the numerical radius, we obtained a technical tool that allows us to carry the BPBp-nu for compact operators from some spaces to others (see Lemma 3.3.1). In particular, we showed in Proposition 3.3.2 that the property can be carried from some projections of a space into the space itself if some conditions are satisfied. These technical tools were used to show, for instance, that all isometric preduals of ℓ_1 have the BPBp-nu for compact operators (see Corollary 3.3.6).

In Section 3.4 we presented a series of topological tools with which we obtained a strong approximation property for $C_0(L)$ spaces and their duals for any locally compact Hausdorff space L (see Theorem 3.4.5). As a consequence, we showed that all $C_0(L)$ spaces have the BPBp-nu for compact operators (and therefore, so do all $C(K)$ spaces with K compact Hausdorff, and all $L_\infty(\mu)$ spaces with μ any measure). Note that in the general non-compact setting, only some particular cases of $C(K)$ spaces are known to have the BPBp-nu in the real case, and the general case and the complex setting remain open.

Open Question 2 ([13, Section 4.3, Question (a)])**.** *Let K be any compact Hausdorff space. Is it true that $C(K)$ has the BPBp-nu?*

In fact, we do not know if the BPBp-nu implies the BPBp-nu for compact operators or viceversa. It was shown in [39] that there exist pairs of Banach spaces with the BPBp for compact operators but without the BPBp, but in the setting of numerical radius, this remains an open question.

Open Question 3. *Let X be a Banach space. If X has the BPBp-nu, does X have the BPBp-nu for compact operators? If X has the BPBp-nu for compact operators, does X have the BPBp-nu?*

6.3 Chapter 4

The contents of Chapter 4 were extracted from the published paper [43], where norm-attainment notions were introduced and studied for projective tensors and for nuclear operators. First, we obtained characterizations of the existence of norm-attaining projective tensors and nuclear operators in terms of the existence of many norm-attaining operators and bilinear forms (see Theorems 4.2.1 and 4.2.2). With the help of those results, we showed that there indeed exist projective tensors and nuclear operators that attain their respective norms, and in fact, every projective tensor in $X^*\widehat{\otimes}_\pi Y$ and every nuclear operator in $\mathcal{N}(X,Y)$ attain their respective norms if X and Y are finite-dimensional, if $X = Y$ is a complex Hilbert space, or if $X = c_0$ (see Propositions 4.2.5, 4.2.6, and 4.2.8). However, we also showed that there exist projective tensors and nuclear operators that do not attain their respective norms, even if one of the involved spaces is finite-dimensional (see Proposition 4.2.10 and its consequences).

In Section 4.3, we sought for density results for norm-attainment of projective tensors and nuclear operators using two different approaches. First, we were able to obtain a positive result of density whenever the $\mathbf{L}_{o,o,\mathcal{B}}$ for bilinear holds at the respective involved Banach spaces (see Propositions 4.3.3 and 4.3.5 and its consequences). However, the $\mathbf{L}_{o,o,\mathcal{B}}$ is a very restrictive property (both Banach spaces must be reflexive, and even in the reflexive scenario, many spaces do not satisfy the property), so we went for a different approach that allowed us to reduce the problem to the finite-dimensional scenario. We were able to obtain positive density results if the domain space has the metric π-property and the range space either has the metric π-property or is uniformly convex (see Theorems 4.3.8 and 4.3.9 and their consequences). A wide list of spaces satisfying the metric π-property was provided in Example 4.3.12. Further results on the density of norm-attaining projective tensors have also been obtained in [41, Section 4] and [105].

All these positive density results made us wonder whether or not the density of norm-attaining projective tensors and nuclear operators always holds. We obtained in Theorem 4.4.1 that this is not the case for projective tensors. However, if one tries to mimic the proof of Theorem 4.4.1 (and its lemmas) for the nuclear operator case, then

$$(\ker Q)^{\perp} \neq \overline{(\ker Q)^{\perp} \cap F(Y, X^{**})}^{w^*}$$

needs to be one the hypothesis (which we cannot guarantee to be true in general). We do not know if there is a version of Theorem 4.4.1 for nuclear operators.

Open Question 4 ([43, Question 6.2])**.** *Are there Banach spaces X, Y so that $\mathrm{NA}_{\mathcal{N}}(X, Y)$ is not dense in $\mathcal{N}(X, Y)$?*

Note that it is not known up to now whether every finite-rank operator can be approximated by norm-attainin operators. However, the analogous claim for projective tensors is false, as we showed in Proposition 4.4.5. Recall that we have shown that if H is a complex Hilbert space, then every tensor in $H\widehat{\otimes}_\pi H$ attains its projective norm (see Proposition 4.2.8) and that the set $\mathrm{NA}_\pi(L_p(\mu)\widehat{\otimes}_\pi L_q(\nu))$ is dense in $L_p(\mu)\widehat{\otimes}_\pi L_p(\nu)$ for $1 < p, q < \infty$ and measures μ and ν (see Example 4.3.12.(b)). However, we do not know what happens in general when both spaces are reflexive spaces.

Open Question 5 ([43, Question 6.1])**.** *Let X, Y be reflexive Banach spaces. Is it true that the set of all norm-attaining tensors is dense in $X\widehat{\otimes}_\pi Y$?*

Let us point out, however, that in [41, Corollary 4.6], it is shown that if both spaces are reflexive (in fact, if they have the Radon-Nikodým property) and one of their duals has the approximation property, then the set of norm-attaining tensors is dense, which provides a partial answer to the previous question. A related result for nuclear operators was also given in [41, Theorem 4.5].

Finally, we say that a Banach space X has *property quasi-α* if, for an index set Γ, there are $A = \{x_\gamma \in S_X : \gamma \in \Gamma\}$, $A^* = \{x_\gamma^* \in S_{X^*} : \gamma \in \Gamma\}$, and $\lambda : A \longrightarrow \mathbb{R}$ such that $x_\gamma^*(x_\gamma) = 1$ for every $\gamma \in \Gamma$; $|x_\gamma^*(x_\eta)| \leqslant \lambda(x_\gamma) < 1$ for $\gamma \neq \eta$; and for every $e \in \operatorname{Ext}(B_{X^{**}})$, there is a subset $A_e \subseteq A$ and a scalar t with $|t| = 1$ such that $te \in \overline{J_X(A_e)}^{w^*}$ and $r_e = \sup\{\lambda(x) : x \in A_e\} < 1$, where J_X is the canonical embedding on X^{**} (see [35]). Notice that property quasi-α is weaker than property α introduced by W. Schachermayer in [109]. We have proved that $\operatorname{NA}_\pi(\ell_1 \widehat{\otimes}_\pi Y) = \ell_1 \widehat{\otimes}_\pi Y$ for every Banach space Y (see Proposition 4.2.6). Consequently, using Proposition 4.2.10, we get that

$$\overline{\operatorname{NA}_{\mathcal{B}}(\ell_1 \times Y), \mathbb{K}}^{\|\cdot\|_{\mathcal{B}}} = \mathcal{B}(\ell_1 \times Y, \mathbb{K})$$

for every Banach space Y. Note that this is a particular case of [35, Theorem 2.17], where it is shown that if X is a Banach space satisfying property quasi-α, then for every Banach space Y,

$$\overline{\operatorname{NA}_{\mathcal{B}}(X \times Y), \mathbb{K}}^{\|\cdot\|_{\mathcal{B}}} = \mathcal{B}(X \times Y, \mathbb{K}).$$

It seems natural to wonder the following.

Open Question 6 ([43, Question 6.3])**.** *Let X be a Banach space with property α (or quasi-α). Is it true that* $\operatorname{NA}_\pi(X \widehat{\otimes}_\pi Y) = X \widehat{\otimes}_\pi Y$ *holds for every Banach space Y?*

6.4 Chapter 5

The contents of Chapter 5 were extracted from the published paper [84] and the preprint paper [49], where we studied the spaciability of the set $\operatorname{SNA}(M)$ of strongly norm-attaining Lipschitz functions on an infinite complete pointed metric space M. For functionals, Rmoutil showed in 2017 that the set of norm-attaining functionals on a Banach space may not contain any 2-dimensional linear space (see [104]). However, in our setting, we were able to show that if a metric space M has at least 2^n distinct points for some $n \in \mathbb{N}$ (in particular, if M is infinite), then $\mathrm{SNA}(M)$ always contains a linear space isometric to ℓ_1^n (see Theorem 5.2.7). This arose the question of what other Banach spaces may be found in $\mathrm{SNA}(M)$. In Proposition 5.3.1 we showed that any Banach space Y is contained in $\mathrm{SNA}(B_{Y*})$, so every Banach space can be formed in $\mathrm{SNA}(M)$ if one chooses M suitably.

A natural question now was the inverse scenario: if a Banach space Y is contained in $\mathrm{SNA}(M)$, how small can M be chosen? It turns out that the claim "Y has separable dual" and "M is separable" are equivalent, as we showed in Theorem 5.3.3. Therefore, separable metric spaces present some restrictions on what Banach spaces we can form in $\mathrm{SNA}(M)$. Even more restrictions were obtained for small metric spaces: if M is σ-compact, then every space contained in $\mathrm{SNA}(M)$ must be separable and isomorphically polyhedral (see Theorem 5.3.7).

A metric space M is said to have the *small ball property* if for every $\varepsilon > 0$, it is possible to write M as a union of a sequence $(B(x_n, r_n))_n$ of closed balls such that $r_n < \varepsilon$ for all n and $r_n \xrightarrow{n\to\infty} 0$. It is known that σ-precompact spaces have the small ball property but the converse is not true in general (see [16, Theorem 5.2]). We do not know if Theorem 5.3.7 can be extended to spaces with the small ball property.

Open Question 7 ([84, Question 3])**.** *Let M be a pointed metric space with the small ball property. Is it true that all subspaces of* $\mathrm{SNA}(M)$ *are separable and isomorphically polyhedral?*

Regarding positive results for infinite-dimensional Banach spaces, we showed that if M is a metric space that contains $[0, 1]$ isometrically, then $\mathrm{SNA}(M)$ contains c_0 isometrically (see Example 5.3.8 and Proposition

5.3.9). In fact, this claim is also true for may more spaces, such as all spaces with an infinite amount of non-isolated points, but even in the spaces without that property that we were able to study, we always obtained c_0 in $\mathrm{SNA}(M)$ isomorphically. This arose the following natural questions.

Question 8 ([84, Question 1]). *Is it true that for every infinite complete pointed metric space M the corresponding* $\mathrm{SNA}(M)$ *contains infinite-dimensional closed (or at least non-closed) linear subspaces?*

Question 9 ([84, Question 2]). *Is it true that for every infinite complete pointed metric space M the corresponding* $\mathrm{SNA}(M)$ *contains an isomorphic copy of c_0?*

In the very recent work [15], Avilés, Martínez-Cervantes, Rueda Zoca, and Tradacete, by means of an elegant case distinction, and with the help of Ramsey's theorem, were able to solve in the positive both of these questions. They, however, left the following as an open question.

Question 10 ([15, Remark 3.6]). *If M is an infinite (complete) pointed metric space, then does* $\mathrm{SNA}(M)$ *contain c_0 isometrically?*

In Theorems 5.5.1 and 5.5.4, we provided a definitive negative answer to that question: there exist infinite complete pointed metric spaces M such that $\mathrm{SNA}(M)$ does not contain c_0 isometrically. The isometric containment holds, however, whenever M is not uniformly discrete (in particular, if M is compact, for instance), as we showed in Theorem 5.5.2). Finally, in the non-separable setting, with the help of Lemma 5.4.8 and Proposition 5.6.1, we showed in Theorem 5.6.2 that if $\mathrm{dens}(M') = \Gamma$ for some infinite cardinal Γ (where dens is the density character and M' is the set of non-isolated points of M), then there is a linear subspace of $\mathrm{SNA}(M)$ that is isometrically isomorphic to $c_0(\Gamma)$.

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Glossary

$\mathrm{Lip}_0(M)$ (space of Lipschitz functions $f : M \to \mathbb{R}$ with $f(0) = 0$, endowed with the Lipschitz norm), 230
$\mathbf{L}_o$ property, 97
$\mathbf{L}_{o,o}$ property, 208
$\mathbf{L}_{o,o,\mathcal{B}}$ property ($\mathbf{L}_{o,o}$ property for bilinear mappings), 208
$L(X, Y)$ (linear mappings from X to Y), 90
$\mathcal{L}(X, Y)$ (operators from X to Y), 90
M' (set of cluster points of M), 251
$\mathrm{NA}_{\mathcal{B}}(X \times Y, Z)$ (norm-attaining elements of $\mathcal{B}(X \times Y, Z)$), 91
$\mathrm{NA}_{\mathcal{N}}(X, Y)$ (norm-attaining nuclear operators from X to Y), 196
$\mathrm{NA}_{\pi}(X\widehat{\otimes}_{\pi}Y)$ (norm-attaining projective tensors from $X\widehat{\otimes}_{\pi}Y$), 196
$\mathrm{NA}(X, Y)$ (norm-attaining operators from X to Y), 91
$\mathrm{NRA}(X)$ (numerical radius attaining operators on X), 92
$\mathbb{N}$ (set of natural numbers), 90
$n(X)$ (numerical index of X), 92
$n'(X)$ (second numerical index of X), 92
$n_{\mathcal{K}}(X)$ (compact numerical index of X), 155
$n'_{\mathcal{K}}(X)$ (second compact numerical index of X), 161
$\nu(T)$ (numerical radius of T), 92
$\mathcal{N}(X, Y)$ (nuclear operators from X to Y), 194
$\Pi(X)$ (set of states of X), 92
$\mathbb{R}$ (set of real numbers), 90
R-net or R-separated set, 90
$R(x)$ (separation radius of x), 250
$\mathrm{SNA}(M)$ (set of strongly norm-attaining Lipschitz functions on M), 230
σ-precompact metric space , 245
$\mathrm{span}(A)$ (vector space spanned by the elements of A), 121
$S(c, R)$ (sphere of center c and radius R), 90
S_X (unit sphere of X), 90
T^* (adjoint operator of T), 90

$V(T)$ (numerical range of T), 92
$V \otimes W$ (tensor product of V and W), 93
$X\widehat{\otimes}_\varepsilon Y$ (injective tensor product of X and Y), 195
$X\widehat{\otimes}_\pi Y$ (projective tensor product of X and Y), 193
X^* (topological dual of X), 90
X^{**} (bidual of X), 90
$\mathbb{Z}$ (set of integers), 90
$\mathcal{Z}_{\mathcal{K}}(X)$ (skew-hermitian compact operators on X), 161
$\mathcal{Z}(X)$ (skew-hermitian operators on X), 92
card(A) (cardinality of A), 272

Absolute norm, 163
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Bilinear norm ($\|\cdot\|_{\mathcal{B}}$), 195
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Bishop-Phelps-Bollobás property (BPBp), 97
Bishop-Phelps-Bollobás property for numerical radius (BPBp-nu), 98
Bishop-Phelps-Bollobás property for numerical radius for compact operators (BPBp-nu for compact operators), 157

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Cardinal, 272
Classical Banach function spaces ($C(K,Y)$, $C(K)$, $C_0(L,Y)$, $C_0(L)$, $L_p(\mu,Y)$, $L_p(\mu)$), 94
Classical Banach sequence spaces (c_0, ℓ_p, $c_0(\Gamma,\mathbb{K})$, $c_0(\Gamma)$, $\ell_1(\Gamma,\mathbb{K})$, $\ell_1(\Gamma)$), 93
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Property quasi-α, 282

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Small ball property, 284
Strong operator topology (SOT), 224

Uniformly convex, modulus of convexity of X (δ_X), 110
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Uniformly smooth, modulus of smoothness of X (ρ_X), 112

Weak Bishop-Phelps-Bollobás property for numerical radius (weak BPBp-nu), 98
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